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On the explicit solutions of the elliptic Calogero system

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Let $q_1, q_2, \ldots, q_N$ be the coordinates of N particles on the circle, interacting with the integrable potential $\Sigma_{j<k}^N \wp(q_j - q_k)$, where $\wp$ is the Weierstrass elliptic function. We show that every symmetric elliptic function in $q_1, q_2, \ldots, q_N$ is a meromorphic function in time. We give explicit formulas for these functions in terms of genus $N-1$ theta functions. © *1999 American Institute of Physics.* [S0022-2488(99)01512-1]

I. INTRODUCTION

The elliptic Calogero system,[1]

$$\frac{d^2}{dt^2} q_i = -\sum_{j \neq i} \wp'(q_i - q_j), \quad i = 1,2,\ldots,N \tag{1.1}$$

is a canonical Hamiltonian system, describing the motion of N particles on the circle $S^1 = \mathbb{R}/\omega\mathbf{Z}$, $\omega \in \mathbb{R}$, with Hamiltonian (energy)

$$H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \sum_{j<k}^{N} \wp(q_j - q_k), \tag{1.2}$$

where $\wp(q) = \wp(q|\omega, \omega')$ is the Weierstrass elliptic function

$$\wp(q|\omega,\omega') = \sum_{m,n \in \mathbf{Z}} (q + m\omega + n\omega')^{-2}, \quad \omega'/\omega \notin \mathbb{R}. \tag{1.3}$$

Denote by Γ_1 the elliptic curve $\mathbb{C}/\{2\omega\mathbf{Z} + 2\omega'\mathbf{Z}\}$ with period lattice generated by 2ω and $2\omega'$. The Hamiltonian H is invariant under the obvious action of the permutation group $\mathcal{S}_n$, so the phase space of the compexified system is the cotangent bundle $T^*(S^N\Gamma_1)$ of the Nth symmetric product $S^N\Gamma_1$.

It is known that this system has two Lax representations (Refs. 1, 2, also see Ref. 3 for details). The Lax operator L defines N integrals of motion $I_k(p,q) = k^{-1}\mathrm{tr}(L^k), k = 1,\ldots,N$. It was proved in Ref. 4 that these integrals are in involution and hence this system is completely integrable in the Jacobi–Liouville sense.[5,6]

The Krichever Lax pair has a spectral parameter. This means that the equations of motion of the system under consideration are equivalent to the matrix equation

$$i\dot{L}(\lambda) = [L(\lambda), M(\lambda)], \tag{1.4}$$

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where $L(\lambda)=L(p,q;\lambda)$ and $M(\lambda)=M(p,q;\lambda)$ are two matrices of order N,

$$\{L(\lambda)\}_{jk}=p_j\,\delta_{jk}+i\,(1-\delta_{jk})\,\Phi(q_j-q_k,\lambda); \tag{1.5}$$

$$\{M(\lambda)\}_{jk}=\delta_{jk}\left(\sum_{l\neq j}\wp(q_j-q_l)-\wp(\lambda)\right)+(1-\delta_{jk})\,\Phi'(q_j-q_k,\lambda); \tag{1.6}$$

$$\Phi(q,\lambda)=\frac{\sigma(q-\lambda)}{\sigma(q)\,\sigma(\lambda)}\exp(\zeta(\lambda)\,q); \tag{1.7}$$

$$\sigma(q)=q\prod_{m,n}{}'\left(1-\frac{q}{\omega_{mn}}\right)\exp\left[\frac{q}{\omega_{mn}}+\frac{1}{2}\left(\frac{q}{\omega_{mn}}\right)^2\right], \tag{1.8}$$

$$\zeta(q)=\frac{\sigma'(q)}{\sigma(q)},\quad \omega_{mn}=m\omega+n\omega'.$$

As it was shown by Krichever,[2] the equations of motion may be ''linearized'' on the Jacobian of the spectral curve

$$\Gamma^N=\{(\lambda,\mu):f(\lambda,\mu)\equiv\det(L(\lambda)-\mu I)=0\}. \tag{1.9}$$

Namely, let

$$\theta(\mathbf{z}|B)=\sum_{\mathbf{N}\in\mathbf{Z}^N}e^{\pi i\langle\mathbf{N},B\mathbf{N}\rangle+2\pi i\langle\mathbf{N},\mathbf{z}\rangle},\quad \mathbf{z}\in\mathbb{C}^N \tag{1.10}$$

be the Riemann theta function with period matrix B, where

$$B=(B_{ij}),\quad B=B^t,\quad \operatorname{Im}B>0,\quad \langle x,y\rangle=\sum_j x_jy_j,\quad i,j=1,\ldots,N.$$

It has been shown by Krichever[2] that, if B is the period matrix of the curve Γ^N, then for suitable constant vectors $U,V,W\in\mathbb{C}^N$ and for a fixed parameter $t\in\mathbb{C}$, the equation

$$\theta(Uq+Vt+W)=0,\quad q\in\mathbb{C} \tag{1.11}$$

has exactly N solutions $q=q_j(t)$ on the Jacobian Jac (Γ^N) of the curve Γ^N. The functions $q_j(t)$ provide solutions of the elliptic Calogero system (1.1). The equation (1.11) for these solutions is, however, not explicit and seems to be not well understood.

The aim of the present paper is to give ''the effectivization'' of these formulas based on the projection method by Olshanetsky and Perelomov[7,8] of explicit integration of the equations of motion in the rational and the trigonometric cases, as well as on the algebro-geometric approach of Krichever.[9,2]

II. EXPLICIT SOLUTIONS

Let Γ_N be a genus N Riemann surface which is an N-sheeted covering of an elliptic curve Γ_1,

$$\Gamma_N\xrightarrow{\pi}\Gamma_1. \tag{2.1}$$

It follows from a theorem of Weierstrass (see, for example, Refs. 10, 11, 12, and 13, Theorem 7.4) that the period matrix of the curve Γ_N in a suitable basis has the form (I,B), where $I=\operatorname{diag}(1,1,\ldots,1)$, and

$$B=\begin{pmatrix} \frac{\tau}{N} & \frac{k}{N} & 0 & \dots & 0 \\ \frac{k}{N} & b_{22} & b_{23} & \dots & b_{2N} \\ 0 & b_{32} & b_{33} & \dots & b_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{N2} & b_{N3} & \dots & b_{NN} \end{pmatrix} \tag{2.2}$$

for a suitable positive integer k. Consider the Riemann theta function $\theta(x,\mathbf{t})=\theta(x,\mathbf{t}|B)$, where $\mathbf{t}=(t_1,t_2,\dots,t_{N-1})$, $(x,\mathbf{t})\in\mathbb{C}^N$. We have

$$\theta(x+1,\mathbf{t})=\theta(x,\mathbf{t}),\theta(x+\tau,\mathbf{t})=e^{-2\pi iNx-\pi iN\tau}\theta(x,\mathbf{t}),i=\sqrt{-1} \tag{2.3}$$

and therefore for any fixed $\mathbf{t}$ the function $\theta(x,\mathbf{t})$ is an elliptic theta function of order N.[14] In particular it has exactly N zeros on $\Gamma_1=\mathbb{C}/\{\mathbf{Z}+\tau\mathbf{Z}\}$ which we denote by $x_i(\mathbf{t})$, $i=1,2,\dots,N$.

Lemma 2.1: *The following identity holds:*

$$\frac{\partial^2}{\partial x^2}\log\ \theta(x,\mathbf{t}|B)=\sum_{i=1}^{N}\wp(x-x_i(\mathbf{t})|\tau)+N\frac{\theta_1'''(0)}{3\theta_1'(0)},$$

where[15]

$$\theta_1(x|\tau)=\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(x|\tau).$$

Proof: The relations

$$\theta_1(x+1)=-\theta_1(x),\theta_1(x+\tau)=-e^{-2\pi ix-\pi i\tau}\theta_1(x) \tag{2.4}$$

compared to (2.3) imply that

$$\left(\frac{\theta(x,\mathbf{t})}{\prod_{i=1}^{N}\theta_1(x-x_i(\mathbf{t}))}\right)^2 \tag{2.5}$$

is a meromorphic function in x on Γ_1 which has no poles, and hence it is a constant (in x). It follows that

$$\frac{\partial^2}{\partial x^2}\log\frac{\theta(x,\mathbf{t})}{\prod_{i=1}^{N}\theta_1(x-x_i(\mathbf{t}))}\equiv 0.$$

Finally we use that

$$\wp(x)=-\frac{\partial^2}{\partial x^2}\log\sigma(x),\qquad \theta_1(x)=c\ \exp(\eta x^2)\ \sigma(x), \tag{2.6}$$

where

$$\eta=-\frac{\theta_1'''(0)}{6\theta_1'(0)}$$

and c is a suitable constant.[15] □

Theorem 2.2: *The Krichever curve Γ^N is an N-sheeted covering of an elliptic curve $\Gamma_1 = \mathbb{C}/\{\mathbf{Z}+\tau\mathbf{Z}\}$. There exists a canonical homology basis and a normalized basis of holomorphic one-forms on Γ^N, such that the corresponding period matrix of Γ_N takes the form (I,B), where $I=\mathrm{diag}(1,1,\ldots,1)$, and*

$$B=\begin{pmatrix} \frac{\tau}{N} & \frac{1}{N} & 0 & \ldots & 0 \\ \frac{1}{N} & b_{22} & b_{23} & \ldots & b_{2N} \\ 0 & b_{32} & b_{33} & \ldots & b_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{N2} & b_{N3} & \ldots & b_{NN} \end{pmatrix}. \tag{2.7}$$

In the same basis the vectors U and V in (1.11) read

$$U=(1,0,\ldots,0), V=(0,V_2,\ldots,V_N). \tag{2.8}$$

A direct proof (without using the Weierstrass theorem) of the above Theorem will be given in the last section. From now on we make the convention that $2\omega=1$ so the period lattice of Γ_1 is

$$\mathbf{Z}+\tau\mathbf{Z}, \quad \tau=2\omega'/2\omega=2\omega'.$$

Corollary 2.3: The symmetric functions

$$f_k(t)=\sum_{i=1}^{N} \wp^{(k)}(q_i(t))$$

are meromorphic in t. Explicit formulas for them are obtained from Lemma 2.1,

$$f_0(t)=\frac{\partial^2}{\partial x^2}\log\,\theta(x,\mathbf{t})|_{x=0}-N\frac{\theta_1'''(0)}{3\theta_1'(0)},$$

$$f_k(t)=(-1)^k\frac{\partial^{k+2}}{\partial x^{k+2}}\log\,\theta(x,\mathbf{t})|_{x=0},\quad k>0,$$

where

$$\mathbf{t}=(V_2t+W_2,V_3t+W_3,\ldots,V_Nt+W_N).$$

Our next construction is motivated by Refs. 7, 8, and 2. Let us define the function

$$F(x,t)=\prod_{j=1}^{N}\frac{\sigma(x-q_j(t))}{\sigma(x)\sigma(q_j(t))}=[\theta_1'(0)]^{-N}]\prod_{j=1}^{N}\frac{\theta_1(x-q_j(t))}{\theta_1(x)\theta_1(q_j(t))},\quad \sum_{j=1}^{N}q_j(t)=0, \tag{2.9}$$

where

$$q_j(t),\quad t\in\mathbb{C},\quad j=1,2,\ldots,N,$$

is a solution of the elliptic Calogero system.

Lemma 2.4: $F(x,t)$ is a meromorphic function in x on Γ_1 and meromorphic function in t on $\mathbb{C}$, explicitly given by

$$F(x,t)=[-\theta_1'(0)]^{-N}]\frac{\theta(Ux+Vt+W)}{\theta_1(x)^N\theta(Vt+W)}. \tag{2.10}$$

Proof: We already noted that the function (2.5) is a constant in x, and hence

$$\frac{\theta(x,\mathbf{t})}{\Pi_{i=1}^N\theta_1(x-x_i(\mathbf{t}))}\equiv\frac{\theta(0,\mathbf{t})}{\Pi_{i=1}^N\theta_1(-x_i(\mathbf{t}))}.$$

This combined with (2.8) gives

$$\frac{\Pi_{i=1}^N\theta_1(x-q_i(t))}{\Pi_{i=1}^N\theta_1(q_i(t))}=(-1)^N\frac{\theta(Ux+Vt+W)}{\theta(Vt+W)}.$$

□

The expansion of $F(x,t)$ on the basis of first order theta functions in x defines $(N-1)$ meromorphic functions in the variables $q_1,\ldots,q_N$ which are also meromorphic functions in t with only simple poles. Hence we can take them as new ''good'' variables. The expansion of $F(x,t)$ can be obtained by making use of the addition formulas for elliptic functions. In the case $N=2$, we have the following ''addition formula''[15]

$$F(x,t)=-\frac{\sigma(x-q)\,\sigma(x+q)}{\sigma^2(x)\sigma^2(q)}=\wp(x)-\wp(q), \tag{2.11}$$

which generalizes for arbitrary N in the following way

Lemma 2.5: For any $\mathbf{q}=(q_1,q_2,\ldots,q_N),x$, *such that* $\Sigma q_j=0$ *define*

$$F(x,\mathbf{q})=\prod_{j=1}^N\frac{\sigma(x-q_j)}{\sigma(x)\sigma(q_j)}, \tag{2.12}$$

$$\Delta(\mathbf{q})=(N-1)!\det\begin{vmatrix}1&\wp(q_1)&\wp'(q_1)&\ldots&\wp^{(N-3)}(q_1)\\1&\wp(q_2)&\wp'(q_2)&\ldots&\wp^{(N-3)}(q_2)\\\ldots&\ldots&\ldots&\ldots&\ldots\\1&\wp(q_{N-1})&\wp'(q_{N-1})&\ldots&\wp^{(N-3)}(q_{N-1})\end{vmatrix}. \tag{2.13}$$

The following identity holds:

$$F(x,\mathbf{q})\Delta(\mathbf{q})\equiv\det\begin{vmatrix}1&\wp(x)&\wp'(x)&\ldots&\wp^{(N-2)}(x)\\1&\wp(q_1)&\wp'(q_1)&\ldots&\wp^{(N-2)}(q_1)\\\ldots&\ldots&\ldots&\ldots&\ldots\\1&\wp(q_{N-1})&\wp'(q_{N-1})&\ldots&\wp^{(N-2)}(q_{N-1})\end{vmatrix}. \tag{2.14}$$

Remark: The substitution $x=q_N$ in (2.14) gives the following addition formula for the Weierstrass $\wp$-function:

$$\det\begin{vmatrix}1&\wp(q_1)&\wp'(q_1)&\ldots&\wp^{(N-2)}(q_1)\\1&\wp(q_2)&\wp'(q_2)&\ldots&\wp^{(N-2)}(q_2)\\\ldots&\ldots&\ldots&\ldots&\ldots\\1&\wp(q_N)&\wp'(q_N)&\ldots&\wp^{(N-2)}(q_N)\end{vmatrix}\equiv0. \tag{2.15}$$

Proof: For fixed $\mathbf{q}=(q_1,q_2,\ldots,q_N)$ the functions in the left and right-hand side of the identity (2.14) are meromorphic in x on the elliptic curve Γ_1. Both of them have a pole of order N at $x=0$ and simple zeros at $x=q_1,\ldots,q_{N-1}$. It follows that their ratio is a first order elliptic function, and hence a constant in x. To compute this constant we use that $\sigma(x)=x+\ldots$, $\wp(x)=1/x^2+\ldots$, and then compare the Laurent series of the two functions in a neighborhood of $x=0$. □

Note finally that if for fixed $\mathbf{q}$ and $\tilde{\mathbf{q}}$ holds $F(x,\mathbf{q})\equiv F(x,\tilde{\mathbf{q}})$, then up to a permutation $\mathbf{q}=\tilde{\mathbf{q}}$. Therefore there is a one-to-one correspondence between the coefficients of $\wp^k(x)$ in the expansion of $F(x,\mathbf{q})$, and the points of the $(N-1)$th symmetric power of the elliptic curve $\Gamma_1\setminus\{0\}$. In particular every meromorphic function on this symmetric power is a rational function in the above coefficients. This implies the following:

Corollary 2.6: Let $f(x)$ be a meromorphic function on the elliptic curve Γ_1, and let S be a symmetric rational function in $N-1$ variables. If $q_1(t),q_2(t),\ldots,q_N(t)$, $\Sigma q_i\equiv 0$ is a solution of the elliptic Calogero system, then $S(f(q_1(t),f(q_2(t)),\ldots,f_{N-1}(q_{N-1}(t)))$ is a meromorphic function in t.

The further analysis of the explicit formulas for the solutions of the elliptic Calogero system can be based on Lemma 2.4, Lemma 2.5, and the identity

$$F(x,t)\equiv F(x,\mathbf{q}(t)).$$

Consider the seemingly trivial case of two particles $(N=2)$. Let us give first an explanation of the Krichever formula (1.11) for the solutions $q_1(t)=-q_2(t)$. Put $q_1-q_2=q$ and $p_1=-p_2=p$. The Hamiltonian H becomes $H(p,q)=p^2+\wp(q)$, and the reduced Hamiltonian system is

$$\frac{d}{dt}q=2p,\quad \frac{d}{dt}p=-\wp'(q),\quad (q,p)\in T^*\Gamma_1. \tag{2.16}$$

The Lax matrix L is

$$L(\lambda)=\begin{pmatrix} p & i\Phi(q,\lambda) \\ i\Phi(-q,\lambda) & -p \end{pmatrix}$$

and the corresponding spectral polynomial

$$\det(L(\lambda)-\mu I)=\mu^2-p^2+\Phi(q,\lambda)\Phi(-q,\lambda)=\mu^2-p^2+\wp(\lambda)-\wp(q)=\mu^2+\wp(\lambda)-H(p,q)$$

defines a spectral curve

$$\Gamma_2=\{(\mu,\lambda):\mu^2+\wp(\lambda)=h\}.$$

Suppose that $h=H(p,q)$ is fixed in such a way, that the meromorphic function $\wp(\lambda)-h$ has two distinct zeros on Γ_1. The spectral curve Γ_2 is a double ramified covering over the elliptic curve Γ_1 with projection map $\pi:\Gamma_2\to\Gamma_1:(\mu,\lambda)\to\lambda$. It follows that Γ_2 is a genus two curve with holomorphic differentials

$$\omega_1=d\lambda,\qquad \omega_2=\frac{d\lambda}{\mu}.$$

On the other hand Γ_2 is identified to the orbit

$$\{(p,q)\in T^*\Gamma_1=:H(p,q)=h\}$$

under the map

$$(p,q)\to(\mu,\lambda).$$

Consider further the embedding of the orbit Γ_2 into its Jacobian variety $\mathrm{Jac}(\Gamma_2)$

$$\Gamma_2 \to \mathrm{Jac}(\Gamma_2): P \to \left(\int_{P_0}^{P} d\lambda, \int_{P_0}^{P} \frac{d\lambda}{\mu} \right). \tag{2.17}$$

By the Riemann theorem,[16] the curve $\Gamma_2 \subset \mathrm{Jac}(\Gamma_2)$ defines a divisor which coincides, up to addition of a constant, with the Riemann theta divisor $\Theta \subset \mathrm{Jac}(\Gamma_2)$ on the Jacobian variety $\mathrm{Jac}(\Gamma_2)$.

Let $(p(t),q(t))$ be a solution of the elliptic Calogero system, with initial condition $(p(t_0),q(t_0)) = P_0$. Taking into consideration that

$$\frac{d\lambda}{\mu} = 2\,dt, \quad d\lambda = dq, \ (\lambda,\mu) \in \Gamma_2,$$

formula (2.17) takes the form

$$T^*\Gamma_1 = \mathbb{C} \times \Gamma_1 \ni (p(t),q(t)) \to (2t - 2t_0, q(t) - q(t_0)) \in \mathrm{Jac}(\Gamma_2). \tag{2.18}$$

It follows that there exist constant vectors $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{C}^2$ such that

$$\theta(\mathbf{a}q(t) + \mathbf{b}t + \mathbf{c}) \equiv 0. \tag{2.19}$$

Of course these constants depend on the choice of symplectic homology basis and the choice of normalized basis of holomorphic one-forms. Namely, let a,b be two loops on Γ_1, such that $\pi^{-1}(a) = \{a_1,a_2\}$, $\pi^{-1}(b) = \{b_1,b_2\}$, where a_i,b_j represent an integer symplectic homology basis on Γ_2: $a_i \circ b_j = \delta_{ij}$, $a_i \circ a_j = 0$, $b_i \circ b_j = 0$. Then,

$$\int_{a_1} d\lambda = \int_{a_2} d\lambda, \quad \int_{b_1} d\lambda = \int_{b_2} d\lambda,$$

$$\int_{a_1} \frac{d\lambda}{\mu} = -\int_{a_2} \frac{d\lambda}{\mu}, \quad \int_{b_1} \frac{d\lambda}{\mu} = -\int_{b_2} \frac{d\lambda}{\mu}.$$

If we define a new symplectic basis

$$\tilde{a}_1 = a_1 + a_2, \tilde{a}_2 = b_1 - b_2, \tilde{b}_1 = b_1, \tilde{b}_2 = a_2$$

and normalize the two holomorphic one-forms as

$$d\lambda \to \frac{d\lambda}{\int_{\tilde{a}} \pi^* d\lambda} = \frac{d\lambda}{2\int_a d\lambda}, \quad \frac{d\lambda}{\mu} \to \frac{d\lambda/\mu}{\int_{\tilde{a}_2} d\lambda/\mu},$$

then the period matrix of Γ_2 takes the form

$$\begin{pmatrix} 1 & 0 & \tau_1/2 & 1/2 \\ 0 & 1 & 1/2 & \tau_2/2 \end{pmatrix},$$

where

$$\tau_1 = \frac{\int_b d\lambda}{\int_a d\lambda}, \tau_2 = \frac{\int_{a_2} d\lambda/\mu}{\int_{b_1} d\lambda/\mu}.$$

This, together with (2.18) implies that

$$\mathbf{a}=\left(\frac{1}{\int_{\tilde{a}_1} d\lambda},0\right)=\left(\frac{1}{2\int_{a} d\lambda},0\right),\mathbf{b}=\left(0,\frac{1}{\int_{b_1} d\lambda/\mu}\right).$$

Finally the vector **c** is arbitrary and plays the role of initial condition. The function $F(x,t)$ defined in (2.9) takes the form

$$F(x,t)=-\frac{\sigma(x-q(t))\,\sigma(x+q(t))}{\sigma^2(x)\sigma^2(q(t))} \tag{2.20}$$

and hence[15,17]

$$F(x,t)=\wp(x)-\wp(t). \tag{2.21}$$

So the elliptic function $\wp(q|\omega,\omega')$, and also

$$\mathrm{sn}^2(q,k)\sim\frac{\theta_1^2(q|k)}{\theta_4^2(q|k)},\quad \mathrm{cn}^2(q,k)\sim\frac{\theta_2^2(q|k)}{\theta_4^2(q|k)},\quad \mathrm{dn}^2(q,k)\sim\frac{\theta_3^2(q|k)}{\theta_4^2(q|k)} \tag{2.22}$$

are ''good'' variables (in the sense that they are meromorphic in t). The equation of motion for them takes a very simple form. We get

$$\mathrm{sn}^2(q,k)=1-a^2+a^2\,\mathrm{sn}^2(\gamma t,\tilde{k}), \tag{2.23}$$

where

$$a^2=\frac{h-1}{h},\quad \gamma=2(h-k^2),\quad \tilde{k}^2=\frac{h-1}{h-k^2}\,k^2. \tag{2.24}$$

One can easily show that the even functions $\mathrm{cn}\,(q,k)$ and $\mathrm{dn}\,(q,k)$ (but not $\mathrm{sn}\,(q,k)$) are ''good'' variables and we get as in[3]

$$\mathrm{cn}\,(q,k)=\alpha\,\mathrm{cn}\,(\gamma t,\tilde{k}), \tag{2.25}$$

$$\mathrm{dn}\,(q,k)=\beta\,\mathrm{dn}\,(\gamma t,\tilde{k}),\; b=(k/\tilde{k})a. \tag{2.26}$$

III. REDUCTION OF THETA FUNCTIONS

The reduction theory was elaborated by Weierstrass (see, for example, Ref. 10) and Poincaré.[11,12] Consider first the case $N=2$. The Riemann theta function associated with the Riemann matrix (2.7) has the form,

$$\theta(z_1,z_2)=\sum_{n_i,n_j}\exp\{i\pi[B_{ij}n_in_j+2n_jz_j]\},\quad i,j=1,2, \tag{3.1}$$

where

$$B_{11}=\tau_1/2,\quad B_{22}=\tau_2/2,\quad B_{12}=B_{21}=1/2.$$

A straightforward computation gives

$$\theta(z_1,z_2)=\sum_{n_1,n_2}\exp\left\{i\pi\left[\tau_1\frac{n_1^2}{2}+n_1n_2+\tau_2\frac{n_2^2}{2}+2n_1z_1+2n_2z_2\right]\right\}$$
$$=\sum_{k_1,n_2\in\mathbf{Z}}\exp\{i\pi[2\tau_1k_1^2+4k_1z_1]\}\exp\left\{i\pi\left[\tau_2\frac{n_2^2}{2}+2n_2z_2\right]\right\}$$
$$+\sum_{k_1,n_2\in\mathbf{Z}}\exp\left\{i\pi\left[2\tau_1\left(k_1+\frac{1}{2}\right)^2+4\left(k_1+\frac{1}{2}\right)z_1\right]\right\}\exp\left\{i\pi\left[\tau_2\frac{n_2^2}{2}+2\left(n_2+\frac{1}{2}\right)z_2\right]\right\}$$
$$=\theta_3(2z_1|2\tau_1)\ \theta_3\left(z_2\left|\frac{\tau_2}{2}\right.\right)+\theta_2(2z_1|2\tau_1)\ \theta_4\left(z_2\left|\frac{\tau_2}{2}\right.\right),$$

where $\theta_1,\theta_2,\theta_3$, and θ_4 are defined by formulas,

$$\theta_1(z|\tau)=\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z|\tau)=2q^{1/4}\sum_{n=1}^{\infty}(-1)^nq^{n(n+1)}\sin[(2n+1)\pi z];\tag{3.2}$$

$$\theta_2(z|\tau)=\theta\begin{bmatrix}1/2\\0\end{bmatrix}(z|\tau)=2q^{1/4}\sum_{n=1}^{\infty}q^{n(n+1)}\cos[(2n+1)\pi z];\tag{3.3}$$

$$\theta_3(z|\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(z|\tau)=1+2\sum_{n=1}^{\infty}q^{n^2}\cos(2\pi nz);\ q=\exp(i\pi\tau);\tag{3.4}$$

$$\theta_4(z|\tau)=\theta\begin{bmatrix}0\\1/2\end{bmatrix}(z,\tau)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2\pi nz).\tag{3.5}$$

So in this case, the equation $\theta(z_1,z_2)=0$ is equivalent either to

$$A\ \mathrm{dn}(2z_1|4\tau_1)\ \mathrm{dn}(z_2|\tau_2)+\mathrm{cn}(2z_1|4\tau_1)=0,\tag{3.6}$$

or to

$$A\ \mathrm{dn}(2z_2|4\tau_2)\ \mathrm{dn}(z_1|\tau_1)+\mathrm{cn}(2z_2|4\tau_2)=0,\tag{3.7}$$

where

$$A=\frac{\theta_3(0|4\tau_1)\ \theta_3(0|\tau_2)}{\theta_2(0|4\tau_1)\ \theta_4(0|\tau_2)}\tag{3.8}$$

or

$$\mathrm{dn}(z_1|\tau_1)=B\ \mathrm{dn}(2iz_2+K|\tilde{\tau}_2).\tag{3.9}$$

Let us give also a more symmetric form of the theta divisor for this case,

$$\mathrm{dn}(2z_1,k_1)\ \mathrm{dn}(2z_2,k_2)+\mathrm{dn}(2z_1,k_1)\ \mathrm{cn}(2z_2,k_2)+\mathrm{cn}(2z_1,k_1)\ \mathrm{dn}(2z_2,k_2)$$
$$-\mathrm{cn}(2z_1,k_1)\ \mathrm{cn}(2z_2,k_2)=0.\tag{3.10}$$

Using the constraint $\theta(\mathbf{a}x+\mathbf{b}t+\mathbf{c})=0$ and taking $z_1=q$, $z_2=(1/2)K+i\gamma t$, we get once again (2.25) and (2.26).

Consider now the case of arbitrary N. Let $\theta(z_1,z_2,\ldots,z_N|B)$ be the Riemann theta function with period matrix as in Theorem 2.2. In a quite similar way we get

$$\theta(z_1,z_2,\ldots,z_N)=\sum_{j=0}^{N-1}\theta_j(z_1)\,\Theta_j(z_2,\ldots,z_N),\tag{3.11}$$

where

$$\theta_j(z_1)=\theta\begin{bmatrix} j/N \\ 0 \end{bmatrix}(Nz_1|N^2\tau_1),\tag{3.12}$$

$$\Theta_j(z_2,\ldots,z_N)=\Theta\begin{bmatrix} 0 & 0 & \cdots & 0 \\ j/N & 0 & \cdots & 0 \end{bmatrix}(z_2,\ldots,z_N|\hat{B}).\tag{3.13}$$

In the above formula $\hat{B}$ is the right lower $(N-1)\times(N-1)$ minor of B (2.7), and the theta functions with fractional characteristics are defined, for example, in Refs. 19,18,14,13. A reduction formula similar to (3.11), but containing N^2 terms, can be found in Ref. 13, Corollary 7.3.

IV. GEOMETRY OF THE SPECTRAL CURVE

In this section we prove Theorem 2.2.

Let Γ_N be a genus N Riemann surface which is an N-sheeted covering of an elliptic curve Γ_1

$$\Gamma_N \xrightarrow{\pi} \Gamma_1.\tag{4.1}$$

Choose two loops a,b which generate the fundamental group $\pi_1(\Gamma_1,P)$, $P\in\Gamma_1$, and denote $\check{\Gamma}_1=\Gamma_1\setminus\{a\cup b\}$. Let us suppose for simplicity that the ramification points of the projection map π are distinct. Connect further these ramification points by non-intersecting arcs $\gamma_i\subset\check{\Gamma}_1$. The set $\pi^{-1}(\check{\Gamma}_1\setminus\cup_i\gamma_i)$ is a disjoint union of N "sheets." To reconstruct the topological covering (4.1) we have to indicate how the opposite borders of the cuts γ_i are glued, as well how the opposite borders of the (preimages of the) cuts a and b respectively are glued together. Thus there is only a finite number of topologically different coverings (4.1). It may be shown that the Krichever curve (1.9) is of genus at most N, and for generic (p_i,q_i) its genus is exactly N. The projection map π (4.1) is defined then by $\pi(\mu,\lambda)=\lambda$. From now on we shall always assume that (p_i,q_i) are generic. In the case when Γ_N is the genus N Krichever spectral curve (1.9), and Γ_1 is the elliptic curve with half periods ω,ω', the covering (4.1) has a number of special properties.

To prove (2.7) we shall need the following:

Proposition 4.1: Let Γ_N be the Krichever curve (1.9). *There exist loops $a,b\in\pi_1(\Gamma_1,P)$ such that, if $\check{\Gamma}_1=\Gamma_1\setminus\{a\cup b\}$, $\partial\check{\Gamma}_1=a\circ b\circ a^{-1}\circ b^{-1}$, then (i) $\pi^{-1}(\check{\Gamma}_1)$ is connected; (ii) $\pi^{-1}(\partial\check{\Gamma}_1)$ has exactly N connected components.*

On its hand the above proposition implies the following:

Proposition 4.2: There exists loops $a,b\in\pi_1(\Gamma_1,P)$, $P\in\Gamma_1$, such that

$$\pi^{-1}(a)=\{a_1,a_2,\ldots,a_N\},\pi^{-1}(b)=\{b_1,b_2,\ldots,b_N\},$$

where a_i,b_i represent a symplectic homology basis of $H_1(\Gamma_N,\mathbf{Z})$, $a_i\circ b_j=\delta_{ij}$.

Proof of (2.7) assuming Proposition 4.2: Let $d\lambda$ be the holomorphic one-form on Γ_1. Then the pullback $\pi^*d\lambda$ of $d\lambda$ is a holomorphic one-form on Γ_N and we have

$$\int_{a_i}\pi^*d\lambda=\int_a d\lambda,\int_{b_i}\pi^*d\lambda=\int_b d\lambda.$$

Choose the following new integer homology basis of Γ_N:

$$\tilde{a}_1=a_1+a_2+\ldots a_N,\quad \tilde{b}_1=b_1,$$

$$\tilde{a}_2 = Nb_1 - b_1 - b_2 - \ldots - b_N, \quad \tilde{b}_2 = a_2,$$

and

$$\tilde{a}_i = b_i - b_1, \quad \tilde{b}_i = a_2 - a_i, \quad i = 3, \ldots, N.$$

This is also a symplectic basis of $H_1(\Gamma_N, \mathbf{Z})$, as

$$\sum_{i=1}^{N} \tilde{a}_i \wedge \tilde{b}_i = \sum_{i=1}^{N} a_i \wedge b_i .$$

Let $\omega_1, \omega_2, \ldots, \omega_N$ be a basis of holomorphic one-forms on Γ_N, such that

$$\omega_1 = \frac{d\lambda}{\int_{\tilde{a}_1} d\lambda}, \ \int_{\tilde{a}_i} \omega_j = \delta_{ij} .$$

Then $B = (\int_{\tilde{b}_j} \omega_i)_{i,j}^{N,N}$ is a symmetric matrix with positive definite imaginary part, such that

$$\int_{\tilde{b}_1} \omega_1 = \frac{\tau}{N}, \ \int_{\tilde{b}_2} \omega_1 = \frac{1}{N}, \ \int_{\tilde{b}_i} \omega_1 = 0, \ \ i \geqslant 3$$

which completes the proof of 2.7.

Proof of Proposition 4.1: First of all let us note that if the claim holds for some Krichever curve, then it holds for any Krichever curve. Indeed, the space of all such curves is parameterized by $\mathbb{C}^{N-1}$ (the first integrals of the integrable Hamiltonian system (1.4)) and hence it is connected. Let us fix a generic point (p_i, q_i), $i = 1, 2, \ldots, N$. It is enough to prove now our proposition for at least one pair of half-periods ω, ω', for example for $|\omega|, |\omega'| \sim \infty$.

Let us represent $\check{\Gamma}_1 \subset \mathbb{C} = \mathbb{P}^1 \setminus \infty$ as the interior of the period parallelogram formed by 2ω and $2\omega'$. When $|\omega| \to \infty$, $|\omega'| \to \infty$, the boundary. $\partial \check{\Gamma}_1 = a \circ b \circ a^{-1} \circ b^{-1}$ tends to $\infty \in \mathbb{P}^1$, and $\check{\Gamma}_1$ tends to $\check{\Gamma}_1^{\infty} = \mathbb{C}$. In a similar way we define the ''limit'' curve $\check{\Gamma}_N^{\infty}$ which is explicitly described in the following way. When $|\omega| \to \infty$, $|\omega'| \to \infty$, then on any compact set the Weierstrass functions $\sigma(q), \zeta(q), \wp(q)$ tend to $q, 1/q, 1/q^2$ respectively, and hence the function $\Phi(q, \lambda)$ tends to

$$\frac{q - \lambda}{q\lambda} \exp(q/\lambda).$$

Denote the corresponding ''limit'' Lax matrix (1.5) by $L^{\infty}(\lambda)$. The curve $\check{\Gamma}_N^{\infty}$ is the affine curve

$$\{(\lambda, \mu) : \det(L^{\infty}(\lambda) - \mu I_N) = 0\}$$

completed with N distinct points corresponding to $\lambda = 0$. The last holds true if and only if the ramification points of the projection map π (4.1) tend to some values different from $\lambda = 0$ (it is easy to check that this is a generic condition on (p_i, q_i)). We shall also suppose that these values are different from $\lambda = \infty$ (another generic condition). Under these restrictions one may prove (as in Ref. 2) that $\check{\Gamma}_N^{\infty}$ is a Riemann sphere, with N punctures (the preimages of $\lambda = \infty$). We obtain thus a map $\pi: \mathbb{P}^1 \to \mathbb{P}^1$ with $2N - 2$ ramification points different from $\lambda = 0, \infty$. The fact that $\pi^{-1}(\mathbb{C})$ is connected implies the part (i) of the proposition, and the fact that $\pi^{-1}(\infty)$ is a disjoint union of N points implies (ii).

Proof of Proposition 4.2: Let us represent $\check{\Gamma}_N$ by a graph with N vertices. A vertex corresponds to a sheet (see the beginning of this section), an edge connects two vertices if and only if the corresponding sheets have a common ramification point. Proposition 4.1 (i) implies that the graph is connected, and (ii) that each sheet contains an even number of ramification points. As the total number of ramification points is $2N-2$ and each point belongs to exactly two sheets, then in addition the graph of $\check{\Gamma}_N$ is simply connected.

Consider now the punctured curve

$$\tilde{\Gamma}_1 = \check{\Gamma}_1 \setminus \cup_i R_i,$$

where $R_i, i=1,\ldots,2N-2$ are the ramification points of π. The fundamental group $\pi_1(\tilde{\Gamma}_1,P)$ has a natural representation in the permutation group S_n. Namely, when a point $Q\in\Gamma_1$ makes one turn along a loop $a\in\pi_1(\tilde{\Gamma}_1,P)$, the set $\pi^{-1}(P)=\cup_{i=1}^N P_i$ is transformed to itself. If the loops a and b induce the identity permutation, then $\pi^{-1}(a)$, $\pi^{-1}(b)$ are disjoint unions of N loops with obvious intersections, which implies Proposition 4.2. If not, we shall modify a and b.

Let $c\in\pi_1(\tilde{\Gamma}_1,P)$ be a loop which makes one turn around some ramification point of π. Then c induces a permutation which exchanges the two sheets containing the ramification point. As the graph of $\check{\Gamma}_N$ is connected then all such transpositions generate the permutation group S_n. Thus for suitable c the loop $a\circ c$ induces the identity permutation. It remains to substitute $a\to a\circ c$ and to note that $a=a\circ c$ in $\pi_1(\Gamma_1,P)$.

Proof of (2.8) (compare to Ref. 13, Theorem 7.14): Let $0\in\Gamma_1$ be the pole of $\wp(z)$. We denote

$$\pi^{-1}(0)=\{\infty_1,\infty_2,\ldots,\infty_N\},\quad \infty_i\in\Gamma_N.$$

In a neighborhood of each point ∞_i on the Krichever curve $\{(\lambda,\mu):f(\lambda,\mu)=0\}$ the meromorphic function μ has the following Laurent expansion:[9]

$$\mu=-\frac{1}{\lambda}+O(1),\quad i=1,2,\ldots,N-1,$$

$$\mu=\frac{N-1}{\lambda}+O(1).$$

It follows that if

$$\omega_j=f_j(P)d\lambda,\quad P=(\lambda,\mu)\in\Gamma_N$$

is a differential of first kind (i.e., holomorphic) on Γ_N, then $\mu\omega_j$ is a differential of third kind with simple poles at ∞_i. The sum of the residues of $\mu\omega_j$ is equal to

$$\sum_{i=1}^{N-1} f_j(\infty_i)-(N-1)f_j(\infty_N)=0. \tag{4.2}$$

Let Ω be a differential of second kind on Γ_N with a single pole at ∞_N. Such is for example the differential

$$\frac{\mu^2-\wp(\lambda)}{\frac{\partial f}{\partial\mu}(\lambda,\mu)}d\lambda.$$

If moreover Ω is normalized as

$$\int_{\tilde{a}_i}\Omega=0,$$

then it is well known that the vector V is collinear to

$$\left(\int_{\tilde b_1}\Omega,\int_{\tilde b_2}\Omega,\ldots,\int_{\tilde b_N}\Omega\right)$$

(see, for example, Ref. 13). Equivalently, if we apply the reciprocity law to the differentials of second and first kind Ω,ω_i, we get that V is colinear to

$$(f_1(\infty_N),f_2(\infty_N),\ldots,f_N(\infty_N)).$$

On the other hand

$$\tilde a_1=a_1+a_2+\ldots+a_N=\pi^{-1}(a)$$

and hence

$$\int_{\tilde a_1}\omega_i=\sum_{k=1}^N\int_a f_i(\lambda,\mu_k)d\lambda,$$

where $(\lambda,\mu_k)\in\Gamma_N$ are the N preimages of $\lambda\in\Gamma_1$. It is clear that $\Sigma_{k=1}^N f_i(\lambda,\mu_k)$ is a single-valued function on Γ_1. As ω_i is a holomorphic differential on Γ_N and $d\lambda$ is the holomorphic differential on Γ_1, then $\Sigma_{k=1}^N f_i(\lambda,\mu_k)$ is a holomorphic function on Γ_1 and hence a constant. As ω_i is a normalized basis of holomorphic forms, then $\int_{\tilde a_1}\omega_i=0$ for $i\geqslant 2$, and hence

$$\sum_{k=1}^N\int_a f_i(\lambda,\mu_k)d\lambda=\sum_{k=1}^N f_i(\lambda,\mu_k)\int_a d\lambda\equiv 0,\quad (\lambda,\mu)\in\Gamma_N,\quad i\geqslant 2.$$

Therefore,

$$\sum_{k=1}^N f_i(\lambda,\mu_k)\equiv 0,\quad i\geqslant 2,$$

which combined with (4.2) implies that

$$f_i(\infty_N)=0,\quad i\geqslant 2$$

and hence the vector V is colinear to $(1,0,\ldots,0)$. In fact V is equal to this vector, because $q_i(t)\in\Gamma_1=\mathbb{C}/\{\mathbf{Z}+\tau\mathbf{Z}\}$. Finally, we may always suppose that $U=(0,U_2,\ldots,U_N)$. Indeed the Calogero system (1.1) is invariant under the translation

$$q_i\to q_i-V_1t.$$

□

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