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EXPLICIT SOLUTIONS OF THE GORJATCHEV-TCHAPLYGIN TOP

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In this paper we study the equations of motion of a rigid body around a fixed point in the case of Gorjatchev-Tchaplygin [1-3]. The explicite formulae (3) of the solutions as functions of complex time in terms of hyperelliptic theta functions are our main result.

According to Ziglin [10] there are only three cases of complete (analytic) integrability of the motion of a rigid body around fixed point — the well known cases of Euler, Lagrange and Kowalewski. The Euler top and the Lagrange top can be integrated in terms of elliptic quadratures after comparatively simple procedures while the Kowalewski top is much more complicated. It was studied first by Kowalewski [4] who integrated the problem in terms of hyperelliptic theta functions and gave explicite formulae of the solutions as functions of complex time. It is known that in the other important case of complete integrability on a given hypersurface — the so called Gorjatchev-Tchaplygin top, the problem can be integrated in principle in terms of hyperelliptic theta functions [2,3]. The purpose of this paper is to give simple expressions in contrast to the formulas eventually given by the standard procedure, i. e. in all expressions the numerators and the denominators shall be relatively prime. To prove that our formulae (3) have this property we are studying the asymptotic expansions of the generic solutions.

After linear change ot variables we obtain the following equivalent system of complex differential equations describing the Gorjatchev-Tchaplygin top

$$\begin{aligned}
&\dot{m}_1=3m_2\,.\,m_3\\
&\dot{m}_2=-3m_1\,.\,m_3-2\gamma_3\\
&\dot{m}_3=2\gamma_2\\
&\dot{\gamma}_1=4m_3\gamma_2-m_2\,.\,\gamma_3\\
&\dot{\gamma}_2=m_1\,.\,\gamma_3-4m_3\,.\,\gamma_1\\
&\dot{\gamma}_3=m_2\,.\,\gamma_1-m_1\,.\gamma_2.
\end{aligned}\tag{1}$$

System (1) possesses three first integrals

$$H_1=(m_1^2+m_2^2)/4+m_3^2-\gamma_1\,;$$
$$H_2=m_1\,.\,\gamma_1+m_2\,.\,\gamma_2+m_3\,.\,\gamma_3\,;$$
$$H_3=\gamma_1^2+\gamma_2^2+\gamma_3^2.$$

As observed by Tchaplygin [2] on the hypersurface $\{H_2=0\}\subset C^6$ (1) has fourth first integral

$$H_4=m_3\,.\,(m_1^2+m_2^2)+2m_1\,.\,\gamma_3.$$

We denote

$$U(u)=u^3-c_1u-c_4/4=\prod_{i=1}^{3}(u-\nu_i)$$

$$U_1^2(u)=U(u)-\sqrt{c_3}\,.\,u=\prod_{i=1}^{3}(u-\alpha_i)$$

$$U_2^2(u)=-U(u)-\sqrt{c_3}\,.\,u=\prod_{i=1}^{3}(u-\beta_i)$$

$$\Phi(u)=U_1^2(u)\,.\,U_2^2(u),$$

where c_1, c_3, c_4 are such generic complex constants, that all roots α_i, β_i, $i=1, 2, 3$ are different. Let Γ be the hyperelliptic curve of genus two $\{y^2=\Phi(x)\}$. Further we use the notation and terminology of [8] without special references. Using the Abel mapping

$$\mu:\Gamma^{(2)}\rightarrow \mathrm{Jac}(\Gamma)$$

every symmetric function of u and v can be considered according to the Jacobi inversion theorem as meromorphic function on $\mathrm{Jac}(\Gamma)$ and hence can be expressed in terms of theta functions.

Let $\omega_1=\frac{a_1x+b_1}{\sqrt{\Phi(x)}}dx$ and $\omega_2=\frac{a_2x+b_2}{\sqrt{\Phi(x)}}dx$ is a normalized basis of $H^0(\Gamma, \Omega^1)$. We put $z=(2a_1t+t_1^0, 2a_2t+t_2^0.)$ Here t_1^0 and t_2^0 are arbitrary constants. In this way any function on $\Gamma^{(2)}$ is function of the complex time t. It is well known [2,3] that every solution of (1) on the invariant two-dimensional complex manifold

$$A=\{H_1=c_1, H_2=0, H_3=c_3, H_4=c_4\}$$

can be expressed as follows:

$$\begin{aligned}
m_1&=-i(U_1(u)\,.\,U_2(v)+U_1(v)\,.\,U_2(u))/\sqrt{c_3};\\
m_2&=i\,.\,(U_1(u)\,.\,U_1(v)-U_2(u)\,.\,U_2(v))\,/\sqrt{c_3};\\
m_3&=u+v\\
\gamma_1&=(U(u)-U(v))/(u-v);\\
\gamma_2&=(U_1(u)\,.\,(U_2(u)-U_1(v)\,.\,U_2(v))/(u-v);\\
\gamma_3&=-i(U_1(u)\,.\,U_2(v)-U_1(v)\,.\,U_2(u))/(u-v).
\end{aligned}\tag{2}$$

Note that m_1, m_2 and γ_3 are multivalued on $\Gamma^{(2)}$ while m_1^2, m_2^2, γ_3^2 and $m_1.\,\gamma_3$ are now single-valued.

As usually let $\theta(z)$ be the Riemann theta function, $\varkappa$ be the Riemann constant, $\theta[p, q](z)$ be theta function of first order with characteristics $[p, q]$, $p, q\in R^g$ [5,11]. For every $x\in\Gamma$ we denote $\theta_x(z)=\theta[p, q](z)$, where $p, q\in R^2$, $0\leq p_i<1$, $0\leq q_i<1$ are such vectors that $\mu(x)-\varkappa=-\sum_{i=1}^{2}(q_i\,.\,\lambda_i+p_i\,.\,\lambda_{i+2})$ where λ_i, λ_{i+2}, $i=1, 2$ form a basis of the period lattice Λ, corresponding to the normalized period matrix Ω. In other words $\theta_x(z)$ is the only first order theta function such that

$$(\theta_x(z))=(\theta(z-\mu(x)+\varkappa)).$$

By analogy we define the function $\theta_{\Sigma x_j}(z)$ as first order theta function satisfing the equality

$$\sum_j(\mu(x_j)-\varkappa)=-\sum_{i=1}^{2}(q_i.\lambda_i+p_i.\lambda_{i+2}).$$

For the sake of brevity further we shall write θ_x instead of $\theta_x(z)$. Then the following theorem holds:

Theorem. Every solution of (1) can be expressed in the following way:

$$m_1(t)=\left(k_1\prod_{i=1}^{3}\theta_{\alpha_i}+k_2.\prod_{i=1}^{3}\theta_{\beta_i}+k_3.\prod_{i=1}^{3}\theta_{\delta_i}\right)/\theta^{3/2}_{\infty+}.\theta^{3/2}_{\infty-};$$

$$m_2(t)=\left(-k_1\prod_{i=1}^{3}\theta_{\alpha_i}+k_2\prod_{i=1}^{3}\theta_{\beta_i}\right)/\theta^{3/2}_{\infty+}.\theta^{3/2}_{\infty-};$$

(3)
$$m_3(t)=-\frac{1}{2}\frac{\partial}{\partial t}\ln(\theta_{\infty+}/\theta_{\infty-})+k_4;$$

$$\gamma_1(t)=(m_3(t))^2+k_5.\theta_{0+}.\theta_{0-}/\theta_{\infty+}.\theta_{\infty-}-c_1;$$

$$\gamma_2(t)=-\frac{1}{4}\frac{\partial^2}{\partial t^2}\ln(\theta_{\infty+}/\theta_{\infty-});$$

$$\gamma_3(t)=k_6.\theta_{\alpha_1+\alpha_2-\alpha_3}/\theta^{1/2}_{\infty+}.\theta^{1/2}_{\infty-},$$

where $\delta_i=\nu_i^-+\sum_{j=1}^{3}(\nu_j^--\alpha_j)$ and $k_1, k_2, \ldots, k_6$ are fixed constants (which will not be calculated here), t_1^0, t_2^0 are arbitrary constants playing the role of initial conditions. Moreover the numerators and the denominators of the above meromorphic functions are relatively prime on Jac(Γ).

Proof. In order to compute m_1 we use the identity

$$U_1(u).U_2(v)+U(v).U_2(u)=U_1(u).U_1(v)$$
$$+U_2(u).U_2(v)-(U_2(v)-U_1(v)).(U_2(u)-U_1(u))$$

and the following formula $f(u).f(v)=c.\frac{\prod_i\theta_{\alpha_i}}{\prod_i\theta_{\beta_i}}$, where $(f)=\sum_i(\alpha_i)-(\beta_i)$ on Γ and $c=$const. (see [9] p. 178) It is enough to note that

$$((U_2(x)-U_1(x))^2)=2\sum_{i=1}^{3}(\nu_i^-)-(3\infty^+)-(3\infty^-)$$

and to take into consideration that m_1^2 is single-valued. For $m_3=u+v$ we use the corresponding formula from [6]. It is important to note the correlation

$$\frac{\partial}{\partial t}=2a_1.\frac{\partial}{\partial z_1}+2a_2.\frac{\partial}{\partial z_2}.$$

For γ_1 we use the equality $\gamma_1=m_3^2-u.v-c_1$. From (1) we have $\gamma_2=\frac{1}{2}\frac{\partial}{\partial t}m_3$. It is less trivial to calculate γ_3. For this purpose we study the asymptotic expansions of the generic solutions of (1). (compare with [7]). It turns out that γ_3^2 has poles only of first order and therefore the denominator is

$$\theta^{1/2}_{\infty+}.\theta^{1/2}_{\infty-}.$$

On the other hand there are no square roots in the numerator and that is why it is a first order theta function. As γ_3^2 is single-valued this determines the characteristics up to a semiperiod. But $m_1 . \gamma_3$ is also single-valued. This additional condition fixes the semiperiods.

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