Ergod. Th. & Dynam. Sys. (2000), **20**, 1671–1686
Printed in the United Kingdom

Second-order analysis in polynomially perturbed reversible quadratic Hamiltonian systems

LUBOMIR GAVRILOV† and ILIYA D. ILIEV‡

† *Laboratoire Emile Picard, CNRS UMR 5580, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France*
‡ *Institute of Mathematics, Bulgarian Academy of Sciences, PO Box 373, 1090 Sofia, Bulgaria*

(*Received* 20 *August* 1998 *and accepted in revised form* 10 *February* 1999)

Abstract. We study degree n polynomial perturbations of quadratic reversible Hamiltonian vector fields with one center and one saddle point. It was recently proved that if the first Poincaré–Pontryagin integral is not identically zero, then the exact upper bound for the number of limit cycles on the finite plane is $n-1$. In the present paper we prove that if the first Poincaré–Pontryagin function is identically zero, but the second is not, then the exact upper bound for the number of limit cycles on the finite plane is $2(n-1)$. In the case when the perturbation is quadratic ($n=2$) we obtain a complete result—there is a neighborhood of the initial Hamiltonian vector field in the space of all quadratic vector fields, in which any vector field has at most two limit cycles.

1. *Introduction*

To study the limit cycles in small polynomial perturbations of Hamiltonian vector fields in a plane, the inspection of higher-order derivatives of the first return mapping is necessary in the two following cases:

(a) when the Hamiltonian vector field is degenerate in some sense (e.g. has a symmetry),
(b) when the degree of the perturbation is greater than the degree of the original Hamiltonian system.

The reason is that in both of these cases, the second variation $M_2(h)$ of the displacement function

$$d(h,\varepsilon)=\varepsilon M_1(h)+\varepsilon^2 M_2(h)+\varepsilon^3 M_3(h)+\cdots$$

has more isolated zeros and, respectively, produces more limit cycles than the first one, etc. The order of the Poincaré–Pontryagin function $M_k(h)$ (also called Melnikov integral), giving the possible maximum number of zeros, is known only in the quadratic case and in the symmetric cubic case (when the perturbed field possesses central symmetry). Thus, the cyclicity under arbitrary quadratic perturbations of the period annulus of a reversible quadratic Hamiltonian vector field is determined by the second Poincaré–Pontryagin integral, except for the Hamiltonian triangle, whose cyclicity is determined by the third-order variation [**14, 22**]. In general, the order of the Poincaré–Pontryagin integral which generates a module of Abelian integrals of a maximal possible dimension is unknown, and its determination appears to be a difficult task involving the solution of the corresponding center-focus problem and viewing the structure of the related center manifold. On the one hand, the calculation of the higher-order Poincaré–Pontryagin integrals depends on the relative cohomology decomposition of polynomial one-forms which, in the presence of symmetry, includes (as a rule) not only polynomials but also some elementary functions such as $\log x$, $\operatorname{atan} x$ or even non-elementary functions. One can expect that all the functions in such a decomposition should take the form of quasi-polynomials (that is polynomials of x, y, H and certain transcendental functions which have elementary functions as their derivatives). The presence of such functions in the relative cohomology decompositions is a reflection of the specific geometry of the ovals from the corresponding period annulus. The above discussion on the different structure of the decompositions suggests that each particular class needs to be considered separately.

In the present paper we determine the exact upper bound for the number of zeros of the second-order Poincaré–Pontryagin integral $M_2(h)$ related to small nth-degree polynomial perturbations

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x, y, \varepsilon), \\ \dot{y} &= -H_x + \varepsilon g(x, y, \varepsilon), \end{aligned} \tag{1}$$

of a Hamiltonian vector field X_H

$$\begin{aligned} \dot{x} &= H_y, \\ \dot{y} &= -H_x, \end{aligned} \tag{2}$$

corresponding to a reversible cubic Hamiltonian H with just one saddle point and one center. The family $\mathcal{H}$ of such Hamiltonians is a codimension-one set in the two-dimensional space of all cubic Hamiltonians possessing a center. We establish that the number of isolated zeros of $M_2(h)$ in a suitable complex domain $\mathcal{D}$ does not exceed $2n-2$, which in particular yields for the quadratic case $n=2$ that any small quadratic perturbation of X_H, $H \in \mathcal{H}$ can produce no more than two limit cycles. Using the notion of a Chebyshev system, this means that the $(2n-1)$-dimensional space of second-order Poincaré–Pontryagin functions corresponding to nth-degree polynomial perturbations of X_H, $H \in \mathcal{H}$, forms a Chebyshev system for any n. We note that according to a recent result in [**7**], the first-order Poincaré–Pontryagin functions $M_1(h)$ also belong to a Chebyshev system (of dimension n).

Consider the quadratic Hamiltonian vector field X_H where

$$H = \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + axy^2, \quad a \in (-\tfrac{1}{2}, 0).$$

Our main results are the following.

THEOREM 1. *Let $K \subset \mathbb{R}^2$ be a fixed compact domain. If $M_1(h) \not\equiv 0$, then* (1) *can have in K at most $n-1$ limit cycles for ε small enough. If $M_1(h) \equiv 0$ but $M_2(h) \not\equiv 0$, then* (1) *can have at most $2(n-1)$ limit cycles in K. Both bounds are exact.*

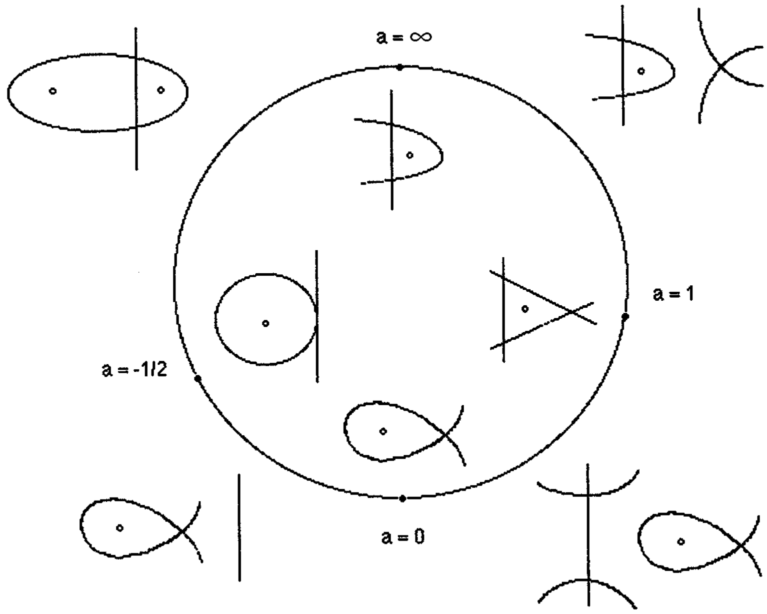

FIGURE 1. Separatrix level curves of the reversible Hamiltonian (13). The corresponding separatrix cycles are: (i) an elliptic segment ($-\infty < a < -\frac{1}{2}$); (ii) a saddle-loop ($-\frac{1}{2} < a < 1$); (iii) a hyperbolic segment ($1 < a < \infty$); (iv) a homoclinic loop through a degenerate saddle ($a = -\frac{1}{2}$); (v) a triangle ($a = 1$); and (vi) a parabolic segment ($a = \infty$).

THEOREM 2. *There is a neighborhood $\mathcal{U}$ of X_H in the space of all quadratic vector fields, such that any $X \in \mathcal{U}$ has at most two limit cycles.*

We recall that when $a = 0$ (the Bogdanov–Takens case), the analogue of Theorem 1 follows from the results of Petrov [**18, 19**], Mardešić [**17**] and Li and Zhang [**16**]. It should also be noticed that the Chebyshev property does not hold for some of the reversible cases [**11**]. Hence, one cannot expect that Theorem 1, as stated above, will also take place for the remaining reversible Hamiltonians which correspond to values $a \in \mathbb{R} \setminus (-\frac{1}{2}, 0)$, see Figure 1.

The paper is organized as follows. In the next section we obtain the cohomology decomposition formulae of polynomial one-forms related to the cubic Hamiltonians. We use them in §3 to derive, by the Françoise recursive procedure [**3**], an appropriate formula for the second variation $M_2(h)$ of the Poincaré return map. In §4 we estimate, following [**7**], the zeros of M_2 and the limit cycles of (1) provided $M_2(h) \not\equiv 0$. Then the result from Theorem 2 is a consequence of the fact that in the quadratic case $n = 2$, the second variation of the Poincaré map in the considered case suffices to determine the limit cycles in the whole plane [**14**].

2. *The relative cohomology decomposition of polynomial one-forms*

In this section we describe the decompositions of polynomial one-forms related to cubic Hamiltonians. First we recall the normal form for all cubic Hamiltonians having a center.

LEMMA 1. [**10**] *Any cubic Hamiltonian $H(x, y)$ having a critical point of a center type at the origin can be put via affine changes of variables into a normal form*

$$H(x, y) = \tfrac{1}{2}(x^2 + y^2) - \tfrac{1}{3}x^3 + axy^2 + \tfrac{1}{3}by^3 \tag{3}$$

where the parameters a, b are taken from the set

$$\Omega = \{-\tfrac{1}{2} \le a \le 1, 0 \le b \le (1-a)(1+2a)^{1/2}\}.$$

The closed ovals around the center at the origin are defined for Hamiltonian values $h \in \Sigma = (0, \frac{1}{6})$. The generic Hamiltonians are presented by the internal points of Ω and the non-generic (reversible) ones—by the points on its boundary.

As in [**10, 11**], introduce the following basic one-forms

$$\omega_X = xy\,dx, \quad \omega_Y = \tfrac{1}{2}y^2\,dx, \quad \omega_L = x^2y\,dx, \quad \omega_M = y\,dx$$

and the corresponding integrals

$$X = \int_{\delta(h)} \omega_X, \quad Y = \int_{\delta(h)} \omega_Y, \quad L = \int_{\delta(h)} \omega_L, \quad M = \int_{\delta(h)} \omega_M, \quad h \in \Sigma, \tag{4}$$

where $\delta(h)$ is the oval contained in the level set $\{H = h\}$. We shall consider polynomial one-forms of degree n,

$$\omega = g(x, y)\,dx - f(x, y)\,dy = \sum_{i+j\le n} b_{ij}x^iy^j\,dx - \sum_{i+j\le n} a_{ij}x^iy^j\,dy. \tag{5}$$

Below, $[r]$ denotes the entire part of r. Our first result in this section is the following.

PROPOSITION 1. *Assume that $a(b^2 - 4a^3) \ne 0$. Then any polynomial one-form ω of degree n can be decomposed into*

$$\omega = dQ(x, y) + q(x, y)\,dH + \xi(H)\omega_X + \eta(H)\omega_Y + \lambda(H)\omega_L + \mu(H)\omega_M \tag{6}$$

where $Q(x, y)$ and $q(x, y)$ are polynomials of degrees $n + 1$ and $n - 2$ respectively and $\xi(h), \eta(h), \lambda(h), \mu(h)$ are polynomials of degrees $\deg\xi = \deg\eta = [(n-2)/3]$, $\deg\lambda = [(n-3)/3]$ *and* $\deg\mu = [(n-1)/3]$.

Proof. The above proposition already follows from [**5**]. Indeed, consider the Petrov $\mathbb{R}[h]$ module $\mathcal{P}_H$. Recall that this is the quotient vector space formed by polynomial one-forms $\omega = P\,dx + Q\,dy$, modulo one-forms $dA + B\,dH$ where A, B are polynomials. $\mathcal{P}_H$ is a module over the ring of polynomials $\mathbb{C}[h]$, under the multiplication $R(h) \cdot \omega = R(H)\omega$. As the monomials $1, x, y, x^2$ form a base of the quotient vector space $\mathbb{R}[x, y]/\langle H_x, H_y\rangle$, then the monomial one-forms $\omega_X, \omega_Y, \omega_L, \omega_M$, where

$$d\omega_X = x\,dy \wedge dx, \quad d\omega_Y = y\,dy \wedge dx, \quad d\omega_L = x^2\,dy \wedge dx, \quad d\omega_M = dy \wedge dx$$

generate the free rank-four module $\mathcal{P}_H$. Thus in the decomposition (6) the real polynomials ξ, η, λ, μ are unique, and their degrees satisfy

$$3\deg\xi + 2 \le n, \quad 3\deg\eta + 2 \le n, \quad 3\deg\lambda + 3 \le n, \quad 3\deg\mu + 1 \le n.$$

The polynomials $Q(x, y)$ and $q(x, y)$ are not unique, but they can be chosen in the following way. If $\tilde{q}$, $\tilde{H}$ denote the highest-order homogeneous parts of q and H, then $d\tilde{q} \wedge d\tilde{H} \equiv 0$ implies that $\tilde{q}$ is a polynomial of $\tilde{H}$. Therefore, after an addition of an appropriate polynomial in H to q, we may always suppose that $d\tilde{q} \wedge d\tilde{H} \not\equiv 0$. In particular we get $\deg q \le n-2$, and hence $\deg Q \le n+1$. □

PROPOSITION 2. *Assume that H is generic, $a(b^2-4a^3) \ne 0$ and the polynomial one-form* (5) *satisfies $\int_{\delta(h)} \omega \equiv 0$ in Σ. Then ω is decomposed into*

$$\omega = dQ(x, y) + q(x, y)\, dH \tag{7}$$

where $Q(x, y)$, $q(x, y)$ are as in (6).

Proof. This follows from [**5**, Proposition 3.2]. □

Proposition 2 is no longer true for the non-generic Hamiltonians. Below we formulate the analogue of Proposition 2 concerning the non-generic case. Let us point out that except for the parabolic segment (given by $a = \frac{1}{2}$, $b = \frac{1}{\sqrt{2}}$), all non-generic Hamiltonians can be obtained from (3) by setting $b = 0$ and $a \in \mathbb{R}$, cf. [**10**, Figure 1] and Figure 1 above. The parabolic segment then corresponds to $a = \infty$. However, for $a \in \mathbb{R} \setminus [-\frac{1}{2}, 1]$, the closed orbits around the center at the origin exist for Hamiltonian values in $\Sigma = (0, h_l)$, $h_l = (3a+1)/24a^3$ instead of $\Sigma = (0, \frac{1}{6})$ (h_l is the level corresponding to the invariant line). We also note that the parabolic segment and the Bogdanov–Takens Hamiltonian ($a = b = 0$ in (3)) were not considered in Proposition 1 because $b^2 - 4a^3 = 0$ for them. The Hamiltonian triangle (given by $a = 1$, $b = 0$ in (3)) is another very specific case and the proposition below does not hold for it as well.

PROPOSITION 3. *Assume that H is a non-generic Hamiltonian presented in (3) by $b = 0$, $a \in \mathbb{R} \setminus (\{0\} \cup \{1\})$. Then any polynomial one-form (5) is decomposed into*

$$\begin{aligned} \omega = {} & d[Q(x, y) + \eta_1(H)\ln(1+2ax)] + [q(x, y) - \eta_1'(H)\ln(1+2ax)]\, dH \\ & + \xi(H)\omega_X + \lambda(H)\omega_L + \mu(H)\omega_M \end{aligned} \tag{8}$$

where $\eta_1(H) = (2a)^{-1}(H - h_l)\eta(H)$ and all other functions are the same as in (6)*. If, in addition, $\int_{\delta(h)} \omega \equiv 0$ in Σ, then ω is decomposed into*

$$\omega = d[Q(x, y) + \eta_1(H)\ln(1+2ax)] + [q(x, y) - \eta_1'(H)\ln(1+2ax)]\, dH. \tag{9}$$

Proof. Considering the decomposition (6), we observe that if $b = 0$, then by symmetry, ω_Y can be expressed as

$$\begin{aligned} \omega_Y &= d\left[\frac{1}{18a}x^3 - \frac{1+3a}{24a^2}x^2 + \frac{1+3a}{24a^3}x + \frac{1}{2a}\left(H - \frac{1+3a}{24a^3}\right)\ln(1+2ax)\right] \\ &\quad - \frac{1}{2a}\ln(1+2ax)\, dH \\ &= dF(x, H) - \frac{1}{2a}\ln(1+2ax)\, dH. \end{aligned}$$

Using this identity, we can express $\eta(H)\,\omega_Y$ in the form $dz - z_H dH$ where $z(x, H) = F(x, H)\eta(H)$. Replacing this in (6) we immediately obtain the representation in (8). If $\int_{\delta(h)} \omega \equiv 0$, then (8) implies

$$\xi(h)X(h) + \lambda(h)L(h) + \mu(h)M(h) \equiv 0.$$

Using [**7**, Proposition 8], we conclude that $\xi(h) = \lambda(h) = \mu(h) \equiv 0$. □

Remark. (1) When $a = 1$, $b = 0$ (the Hamiltonian triangle case), there are three axes of symmetry in the Hamiltonian vector field. For this case (8) should be replaced with

$$\begin{aligned}\omega = d[&Q(x, y) + \xi_1(H)\ln(1 - x + y\sqrt{3}) + \eta_1(H)\ln(1 + 2x)]\\ &+ [q(x, y) - \xi_1'(H)\ln(1 - x + y\sqrt{3}) - \eta_1'(H)\ln(1 + 2x)]\,dH\\ &+ \lambda(H)\omega_L + \mu(H)\omega_M \end{aligned} \tag{10}$$

where $\xi_1(H) = -\frac{1}{\sqrt{3}}(H - \frac{1}{6})\xi(H)$, $\eta_1(H) = \frac{1}{2}(H - \frac{1}{6})[\eta(H) - \frac{1}{\sqrt{3}}\xi(H)]$, and Q, q, etc. are as above.

(2) In the Bogdanov–Takens case $a = b = 0$ the decomposition takes the form (cf. [**13**])

$$\omega = dQ(x, y, H) + q(x, y, H)\,dH + \xi(H)\omega_X + \mu(H)\omega_M \tag{11}$$

where $Q(x, y, H)$ and $q(x, y, H)$ are weighted polynomials of degrees $n + 1$ and $n - 1$, respectively (the weight attached to x, y is one whilst the weight attached to H is two), and $\deg\xi = [(n - 2)/2]$, $\deg\mu = [(n - 1)/2]$. We are not going to derive (10) and (11) in detail here.

3. *The first return map*

We consider polynomial perturbations of the Hamiltonian vector field X_H,

$$\begin{aligned}\dot{x} &= H_y + \varepsilon f(x, y, \varepsilon),\\ \dot{y} &= -H_x + \varepsilon g(x, y, \varepsilon),\end{aligned} \tag{12}$$

where H is the non-generic cubic Hamiltonian

$$H = \tfrac{1}{2}(x^2 + y^2) - \tfrac{1}{3}x^3 + axy^2, \quad a \neq 0, 1. \tag{13}$$

In (12), f and g are polynomials in x, y of degrees at most n with coefficients depending analytically on the small parameter ε. In a more general context, assume that we have a Hamiltonian vector field $dH = 0$ with a center at the origin which is surrounded by continual set of periodic orbits $\delta(h)$, $h \in \Sigma$ (the period annulus). Using the energy level $H = h$ as a parameter, $h \in \Sigma$, we can express the first return mapping of (12) in terms of h and ε. The corresponding displacement function $d(h, \varepsilon) = \mathcal{P}(h, \varepsilon) - h$ has a representation as a power series in ε,

$$d(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \cdots \tag{14}$$

which is convergent for small ε. The zeros in Σ of the first non-vanishing Poincaré–Pontryagin function $M_k(h)$ in (14) determine the limit cycles in (12) emerging from the period annulus. Moreover, if the period annulus is surrounded by a homoclinic loop through a non-degenerate saddle, then the Poincaré–Pontryagin functions can be used to determine the total number of limit cycles produced from the center itself, the period annulus around it and the homoclinic loop.

To obtain $M_k(h)$ in an appropriate form, we write system (12) as a Pfaff equation

$$dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots = 0 \tag{15}$$

where $\omega_j = g_j(x, y)\,dx - f_j(x, y)\,dy$ with $\deg f_j \le n$, $\deg g_j \le n$. Then

$$M_1(h) = \int_{\delta(h)} \omega_1$$

and, by (8), in our particular case (13), this yields

$$M_1(h) = \xi(h)X(h) + \lambda(h)L(h) + \mu(h)M(h). \tag{16}$$

We next recall in brief the Françoise's recursive procedure [**3**] for a calculation of the higher-order Poincaré–Pontryagin functions $M_k(h)$ related to a perturbation (12) of any Hamiltonian vector field having a period annulus (cf. [**3, 12, 21**]).

PROPOSITION 4. (Françoise's recursion formula [**3**]) *Assume that for some* $k \ge 2$ *one has* $M_1(h) = \cdots = M_{k-1}(h) \equiv 0$ *in* (14). *Then*

$$M_k(h) = \int_{\delta(h)} \Omega_k \tag{17}$$

where

$$\Omega_1 = \omega_1, \quad \Omega_m = \omega_m + \sum_{i+j=m} r_i\omega_j, \quad 2 \le m \le k, \tag{18}$$

and the functions r_i, $1 \le i \le k-1$ *are determined successively from the representations* $\Omega_i = dR_i + r_i\,dH$.

Proof. The proof is by induction on k, cf. [**12, 21**]. The fact that the module of Abelian integrals is free (see e.g. [**7**]) yields that

$$\int_{\delta(h)} \Omega_i = 0 \Leftrightarrow \Omega_i = dR_i + r_i\,dH$$

with certain functions R_i, r_i that are analytic in a neighborhood of $\delta(h)$ (in fact the proof uses only the fact that they are uniformly Lipschitz continuous there). We multiply (15) with $1 + \varepsilon r_1 + \cdots + \varepsilon^k r_k$ and then rearrange the monomials in the resulting expression to obtain

$$\begin{aligned} &dH + \varepsilon(r_1\,dH - \omega_1) + \varepsilon^2(r_2\,dH - r_1\omega_1 - \omega_2) + \cdots \\ &\quad + \varepsilon^k(r_k\,dH - r_{k-1}\omega_1 - \cdots - r_1\omega_{k-1} - \omega_k) = O(\varepsilon^{k+1}). \end{aligned}$$

By (18), this is equivalent to

$$dH - (\varepsilon\,dR_1 + \varepsilon^2\,dR_2 + \cdots + \varepsilon^{k-1}\,dR_{k-1}) + \varepsilon^k(r_k\,dH - \Omega_k) = O(\varepsilon^{k+1}).$$

We integrate the last equation along the phase curve γ that was used to define the first return map. Taking into account the fact that

$$\int_\gamma dH = d(h,\varepsilon), \quad \int_\gamma (\varepsilon\, dR_1 + \varepsilon^2\, dR_2 + \cdots + \varepsilon^{k-1}\, dR_{k-1}) = O(\varepsilon^{k+1})$$

(the last estimate follows from the fact that $d(h,\varepsilon) = O(\varepsilon^k)$ and from the Lipschitz condition, cf. [**12**]), we obtain

$$d(h,\varepsilon) = \varepsilon^k \int_\gamma (\Omega_k - r_k\, dH) + O(\varepsilon^{k+1}) = \varepsilon^k \int_{\delta(h)} \Omega_k + O(\varepsilon^{k+1}). \qquad \square$$

Applying Françoise's procedure to determine $M_2(h)$ for our particular case (13), we point out that the argument presented above also holds in a neighborhood of the saddle loop contained in $\{H = \frac{1}{6}\}$, because the functions R_1 and r_1 are Lipschitz continuous near the loop, as can be seen from (8). Recall that (13) has a saddle loop for $a \in (-\frac{1}{2}, 1)$ only.

Introduce the following rational one-form ω_R and the related integral,

$$\omega_R = \frac{y\,dx}{1+2ax}, \quad R(h) = \int_{\delta(h)} \omega_R, \quad h \in \Sigma. \tag{19}$$

PROPOSITION 5. *The second-order Poincaré–Pontryagin function for (12) and (13) can be expressed in the form*

$$M_2(h) = \xi(h)X(h) + \varrho(h)R(h) + \mu(h)M(h) \tag{20}$$

where ξ, ϱ, μ *are polynomials of degrees* $[(2n-4)/3]$, $[(2n-2)/3]$ *and* $[(2n-3)/3]$, *respectively.*

Proof. We have to integrate the one-form $\Omega_2 = \omega_2 + r_1\omega_1$. From (9) we obtain (the calculations below are performed modulo one-forms $dP + p\,dH$)

$$\begin{aligned} r_1\omega_1 &= [q - \eta_1'(H)\ln(1+2ax)]d[Q + \eta_1(H)\ln(1+2ax)] \\ &= q\,dQ + Qd[\eta_1'(H)\ln(1+2ax)] + qd[\eta_1(H)\ln(1+2ax)] \\ &= q\,dQ + 2a(1+2ax)^{-1}[Q\eta_1'(H) + q\eta_1(H)]\,dx. \end{aligned}$$

We then use the definition of η_1 and the equalities

$$H - h_l = (1+2ax)H_0(x,y), \quad \deg H_0 = 2,$$
$$Q(x,y) = (1+2ax)Q_0(x,y) + Q_1(y), \quad \deg Q_0 = n, \quad \deg Q_1 = n+1$$

to obtain

$$\begin{aligned} \Omega_2 &= \omega_2 + q\,dQ + [Q_0\eta(H) + QH_0\eta'(H) + qH_0\eta(H)]\,dx + \eta(H)\frac{Q_1(y)\,dx}{1+2ax} \\ &= \bar{\Omega}_2 + \eta(H)\frac{Q_1(y)\,dx}{1+2ax} \end{aligned}$$

with $\deg \bar{\Omega}_2 = 2n-2$. From the identity

$$y^2 = \frac{2H - 2h_l}{1+2ax} + \frac{1}{3a}x^2 - \frac{1+3a}{6a^2}x + \frac{1+3a}{12a^3} \equiv \mathcal{R}(x,H)$$

we have

$$y^k \partial_x \mathcal{R}(x, H)\, dx = y^k [d\mathcal{R}(x, H) - \partial_H \mathcal{R}(x, H)\, dH] = y^k\, dy^2 - y^k \partial_H \mathcal{R}(x, H)\, dH,$$

therefore (modulo forms $dP + p\, dH$)

$$\frac{y^{k+2}\, dx}{1+2ax} = \frac{y^k \mathcal{R}\, dx}{1+2ax} = y^k \left(\frac{\mathcal{R}}{1+2ax} + \frac{\partial_x \mathcal{R}}{2a} \right) dx = y^k \left(c_0 + c_1 x + \frac{c_2}{1+2ax} \right) dx$$

with some constants c_i. This means

$$\eta(H) \frac{Q_1(y)\, dx}{1+2ax} = \tilde{\Omega}_2 + c\eta(H)\omega_R$$

where $\deg \tilde{\Omega}_2 = 2n-2$ and c is a constant. We now apply Proposition 3 to $\omega = \bar{\Omega}_2 + \tilde{\Omega}_2$ and then integrate along $\delta(h)$, which yields

$$M_2(h) = \int_{\delta(h)} \Omega_2 = \bar{\xi}(h)X(h) + \bar{\lambda}(h)L(h) + \bar{\mu}(h)M(h) + c\eta(h)R(h)$$

where $\deg \bar{\xi} = [(2n-4)/3]$, $\deg \bar{\lambda} = [(2n-5)/3]$ and $\deg \bar{\mu} = [(2n-3)/3]$. Finally, we use the identity

$$L(h) = AX(h) + B(h-h_l)R(h) + CM(h) \quad (A, B, C = \text{constants}) \tag{21}$$

to obtain (20). □

PROPOSITION 6. *The derivative $M_2'(h)$ can be expressed as*

$$M_2'(h) = \tilde{\xi}(h)X'(h) + \tilde{\varrho}(h)R'(h) + \tilde{\mu}(h)M'(h)$$

where $\tilde{\xi}$, $\tilde{\varrho}$ and $\tilde{\mu}$ have the same degrees as ξ, ϱ and μ, respectively.

Proof. Denote $\mathbf{I} = \mathrm{col}(R, M, X)$. Then $\mathbf{I}(h)$ satisfies a Picard–Fuchs system of the form

$$\mathbf{I} = (\mathbf{A}h + \mathbf{B})\mathbf{I}' \tag{22}$$

and the validity of the result in the proposition depends on whether the matrix **A** is lower-triangular. To check this, we can use the system (1.5) derived in [**15**]. We need to perform an affine change of variables there, taking $x = 1 + 2a\bar{x}$, $y = \bar{y}/\sqrt{2}$. Then the integrals in [**15**] become, the modulo is an inessential factor, $(J_{-1}, J_0, J_1) = (R, M, 2aX + M)$. After elementary algebraic manipulations, we obtain a system in the form of (22), with

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ -3(a+1)/4 & 3/2 & 0 \\ (9a+5)(1-a)/16a & (a-1)/8a & 1 \end{pmatrix}.$$

Using this, the proof follows by differentiation of (20). □

Consider the linear space $\mathcal{P}^n$ of polynomial one-forms ω of degree n,

$$\omega = \omega(\varepsilon) = \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \quad \omega_j = g_j(x, y)\, dx - f_j(x, y)\, dy.$$

Let $\gamma(\varepsilon)$ be the trajectory of the vector field $dH = \omega$ beginning at point $(x_0, 0)$ and ending at $(x_1, 0)$ where $H(x_0, 0) = h \in \Sigma$ and x_1 is determined by the first return mapping. Let $\gamma(0) = \delta(h)$. Define the space of integrals

$$\mathcal{M}_1^n = \left\{ M_1^\omega(h) : M_1^\omega(h) = \frac{d}{d\varepsilon} \int_{\gamma(\varepsilon)} \omega(\varepsilon) \Big|_{\varepsilon=0}, h \in \Sigma, \omega \in \mathcal{P}^n \right\}.$$

Clearly, $M_1^\omega(h) = \int_{\delta(h)} \omega_1$. In other words, $\mathcal{M}_1^n$ is the linear space of the (first-order) Poincaré–Pontryagin functions corresponding to an nth degree polynomially perturbed Hamiltonian vector field $dH = 0$. Let $\mathcal{I}_1 : \mathcal{P}^n \to \mathcal{M}_1^n$ be the linear mapping defined by $\omega \to M_1^\omega(h)$. Then $\mathcal{I}_1$ is an isomorphism of modules (see [**5, 7**] where $\mathcal{I}_1$ is studied in detail). This implies that $\dim \mathcal{M}_1^n = n$ and

$$\mathcal{M}_1^n = \operatorname{Span}\left\{ h^k X(h), k \le \left[\frac{n-2}{3}\right], h^k L(h), k \le \left[\frac{n-3}{3}\right], h^k M(h), k \le \left[\frac{n-1}{3}\right] \right\}.$$

Next, define the space

$$\mathcal{M}_2^n = \left\{ M_2^\omega(h) : M_2^\omega(h) = \frac{1}{2}\frac{d^2}{d\varepsilon^2} \int_{\gamma(\varepsilon)} \omega(\varepsilon) \Big|_{\varepsilon=0}, h \in \Sigma, \omega \in \operatorname{Ker} \mathcal{I}_1 \right\}.$$

By Proposition 4, $M_2^\omega(h) = \int_{\delta(h)} \omega_2 + r_1\omega_1$ and therefore $\mathcal{M}_2^n$ is the space of the second-order Poincaré–Pontryagin functions corresponding to the nth degree polynomially perturbed Hamiltonian vector field $dH = 0$ by one-forms from the linear subspace $\operatorname{Ker} \mathcal{I}_1$.

PROPOSITION 7. *We have*

(i) $$\mathcal{M}_2^n = \operatorname{Span}\left\{ h^k X(h), k \le \left[\frac{2n-4}{3}\right], h^k R(h), k \le \left[\frac{2n-2}{3}\right], h^k M(h), k \le \left[\frac{2n-3}{3}\right] \right\}.$$

(ii) $\dim \mathcal{M}_2^n = 2n - 1$.

Proof. Denote by $\mathcal{L}$ the linear span in the right-hand side. By Proposition 5, we only have to prove that the (nonlinear) mapping $\mathcal{I}_2 : \operatorname{Ker} \mathcal{I}_1 \to \mathcal{L}$ defined by $\omega \to M_2^\omega(h)$ is onto. Take the following one-form $\omega \in \mathcal{P}^n$: $\omega = \varepsilon\omega_1 + \varepsilon^2\omega_2$ where ω_1 has the form (9) with

$$Q = (\xi_1(H) - \xi_1(h_l))xy + (\lambda_1(H) - \lambda_1(h_l))x^2y + (\mu_1(H) - \mu_1(h_l))y + \mu_0 y,$$
$$q = x^{j+1}H^{(n-3-j)/3}, \quad n \equiv j \pmod 3, \quad \eta_1(H) = \frac{1}{2a}(H - h_l),$$

and $\omega_2 = \xi_2(H)\omega_X + \lambda_2(H)\omega_L + \mu_2(H)\omega_M$. In the above formulae, ξ_k etc. are polynomials of degrees respectively $\deg \xi_k = [(n-k)/3]$, $\deg \lambda_k = [(n-k-1)/3]$, $\deg \mu_k = [(n-k+1)/3]$ and μ_0 is a constant. Then clearly $\omega \in \operatorname{Ker} \mathcal{I}_1$ and one can easily check (which we do next) that given any element $I(h) \in \mathcal{L}$, then an appropriate choice of the coefficients in ω exists so that $M_2^\omega(h) = I(h)$. To see this, we use the formulae derived in proving Proposition 5 above. For our particular perturbation, one obtains (as before, the calculations are modulo forms $dP + p\,dH$)

$$\begin{aligned}\omega_2 + r_1\omega_1 &= \omega_2 + \bar{\omega}_1 + \mu_0\omega_R \\ &\quad - (j+1)H^{(n-j-3)/3}[\xi_1(H)x^{j+1} + \lambda_1(H)x^{2+j} + \mu_1(H)x^j]y\,dx\end{aligned}$$

where $\deg \bar{\omega}_1 = n$. For $j = 0$, this yields immediately

$$\omega_2 + r_1\omega_1 = \bar{\omega}_1 + \mu_0\omega_R + (\xi_2 - H^{(n-3)/3}\xi_1)\omega_X + (\lambda_2 - H^{(n-3)/3}\lambda_1)\omega_L \\ + (\mu_2 - H^{(n-3)/3}\mu_1)\omega_M$$

which together with (21) yields (20) with all the $2n-1$ coefficients in ξ, ϱ, μ independently free. For $j = 1$, one obtains

$$\omega_2 + r_1\omega_1 = \bar{\omega}_1 + \mu_0\omega_R + (\xi_2 - 2H^{(n-4)/3}\mu_1)\omega_X + (\lambda_2 - 2H^{(n-4)/3}\xi_1)\omega_L \\ + (\mu_2 - 2H^{(n-4)/3}\lambda_1 x^3)\omega_M. \tag{23}$$

We can then use the identity

$$(1 + 2ax)^k \partial_x \mathcal{R}(x, H) y\, dx = (1 + 2ax)^k y[d\mathcal{R}(x, H) - \partial_H \mathcal{R}(x, H)\, dH] \\ = (1 + 2ax)^k y\, dy^2 = -\tfrac{4}{3}ak(1 + 2ax)^{k-1}\mathcal{R}(x, H) y\, dx$$

(with $k = 2$) to obtain the representation formula

$$x^3\omega_M = -\frac{3}{10}(H - h_l)\omega_M + \frac{21a + 1}{40a^2}\omega_X + \frac{21a - 9}{20a}\omega_L.$$

Replacing this expression in (23) and using (21), we obtain formula (20) with all the coefficients independent. Finally, if $j = 2$, one obtains

$$\omega_2 + r_1\omega_1 = \bar{\omega}_1 + \mu_0\omega_R + (\xi_2 - 3H^{(n-5)/3}\lambda_1 x^3)\omega_X + (\lambda_2 - 3H^{(n-5)/3}\mu_1)\omega_L \\ + (\mu_2 - 3H^{(n-5)/3}\xi_1 x^3)\omega_M$$

and the proof is completed in a similar way. □

4. *Non-oscillation of Abelian integrals*

Let

$$X_H = H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y}$$

be a reversible quadratic Hamiltonian vector field with one center and one saddle point. In the normal form (13), any such Hamiltonian is present by $a \in (-\frac{1}{2}, 0]$. In what follows, we will assume that $a \neq 0$. Recall that the case $a = 0$ is the Bogdanov–Takens case.

Consider the following real vector space

$$\mathcal{A}_n = \{J(h) : J(h) = \xi(h)X'(h) + \varrho(h)R'(h) + \mu(h)M'(h)\} \tag{24}$$

where ξ, ϱ, μ are polynomials of degree $[(2n - 4)/3]$, $[(2n - 2)/3]$ and $[(2n - 3)/3]$, respectively, and the Abelian integrals $X(h)$, $R(h)$, $M(h)$ are defined in (4), (19).

THEOREM 3. *The vector space $\mathcal{A}_n$ has the Chebyshev property in the complex domain $\mathcal{D} = \mathbb{C}\backslash[\frac{1}{6}, \infty)$ (see Figure 2). This means that any function $J(h) \in \mathcal{A}_n$ can have at most*

$$\dim \mathcal{A}_n - 1 = \deg \xi + \deg \varrho + \deg \mu + 2 = 2n - 2$$

zeros in $\mathcal{D}$.

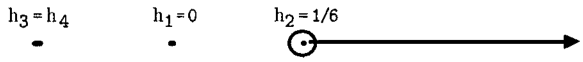

FIGURE 2. The complex domain $\mathcal{D} = \mathbb{C}\backslash[h_2, \infty)$.

The proof repeats the arguments of [**6**]. To find a bound for the number of the zeros of $J(h) \in \mathcal{A}_n$ in $\mathcal{D}$ we shall evaluate the increment of the argument of $F(h) = J(h)/M'(h)$ along the boundary of $\mathcal{D}$. We recall that ω'_M is a holomorphic form and hence $M'(h)$ does not vanish [**9**]. Denote by $F^+(h)$ ($F^-(h)$) the analytic continuation of $F(h)$ on $[\frac{1}{6}, \infty)$, along a path contained in the half-plane $\operatorname{Im} h > 0$ ($\operatorname{Im} h < 0$). The Picard–Lefschetz formula implies

$$\operatorname{Im} F^{\pm}(h) = \pm \frac{\xi(h) W_{\delta_1,\delta_2}(\omega'_X, \omega'_M) + \varrho(h) W_{\delta_1,\delta_2}(\omega'_R, \omega'_M)}{|M'(h)|^2} \tag{25}$$

where δ_1, δ_2 are the cycles vanishing at $h_1 = 0$, $h_2 = \frac{1}{6}$ and

$$W_{\delta_1,\delta_2}(\omega_1, \omega_2) = \det \begin{pmatrix} \int_{\delta_1} \omega_1 & \int_{\delta_1} \omega_2 \\ \int_{\delta_2} \omega_1 & \int_{\delta_2} \omega_2 \end{pmatrix}.$$

An elementary computation shows that on a fixed level set $\{H = h\}$ the following identity holds

$$-2\,dy = 4a(h - h_l)\omega'_R + \frac{1}{3a}\omega'_Z + \frac{1 + 3a}{6a^2}\omega'_M \tag{26}$$

where

$$h_l = (3a + 1)/24a^3 \quad \text{and} \quad \omega_Z = (3a - 1)\omega_X - 4a\omega_L.$$

As the one-form ω'_Z is of the second kind [**11**] then ω'_R is of second kind too (that is to say it has no residues). The Picard–Lefschetz formula implies that $W_{\delta_1,\delta_2}(\omega'_R, \omega'_M)$ is single-valued in h on the complex plane $\mathbb{C}$, and (26) implies that it has a single pole at $h = h_l$. Moreover for $|h| \approx \infty$ the asymptotic estimates

$$|M'(h)| \approx |h|^{-1/3}, \quad |R'(h)| \lessapprox |h|^{-2/3}$$

imply that $(h - h_l)W_{\delta_1,\delta_2}(\omega'_R, \omega'_M)$ is bounded in h. It follows that it is a (non-zero) constant. Furthermore, the reciprocity law for meromorphic differentials of the first and third kind [**6**] implies

$$W_{\delta_1,\delta_2}(\omega'_X, \omega'_M) = 2\pi\sqrt{-1}\,\operatorname{Res}_{P_+} \omega'_X \int_{P_-}^{P_+} \omega'_M \tag{27}$$

where the path of integration from P_- to P_+ in the integral above is contained in Γ_h cut along the loops $\delta_1(h)$ and $\delta_2(h)$ as shown on Figure 3. Note that $\operatorname{Res}_{P_+} \omega'_X$ is a purely imaginary constant in h, and $\int_{P_-}^{P_+} \omega'_M$ is also imaginary. We finally obtain the fact that on the interval $[\frac{1}{6}, \infty)$ the following holds:

$$W_{\delta_1,\delta_2}(\omega', \omega'_M) = \tilde{\xi}(h) \int_{P_-}^{P_+} \omega'_M + \frac{\tilde{\rho}(h)}{h - h_l} \tag{28}$$

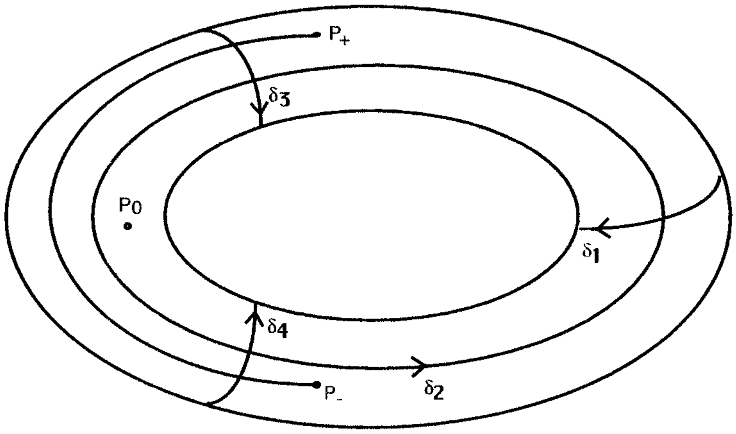

FIGURE 3. The vanishing cycles on the elliptic curve Γ_h, $-\frac{1}{2} < a < 0$.

where $\tilde{\tilde{\xi}}(h)$, $\tilde{\varrho}(h)$ are real polynomials of the same degree as ξ and ϱ, respectively. Denote by $\mathcal{B}_n$ the vector space of functions (28), continued analytically to holomorphic functions in the larger domain $\mathbb{C}\backslash(-\infty, h_1]$. Obviously

$$\dim \mathcal{B}_n = \dim \xi + \dim \varrho + 2.$$

LEMMA 2. *The space of functions $\mathcal{B}_n$ is Chebyshev in the complex domain* $\mathbb{C}\backslash(-\infty, h_1]$.

Proof. For $|h| \approx \infty$ we have that

$$\left|\tilde{\tilde{\xi}}(h) \int_{P_-}^{P_+} \omega'_M + \frac{\tilde{\rho}(h)}{h - h_l}\right|$$

grows no faster than $|h|^{\deg \varrho - 1}$ and its imaginary part on $(-\infty, h_l)$ equals

$$\pm\tilde{\tilde{\xi}}(h) \int_{\delta(h)} \omega'_M, \quad \delta(h) \in H_1(H^{-1}(h), \mathbb{Z}).$$

As the one-form ω'_M is holomorphic on the elliptic curve Γ_h then the integral $\int_{\delta(h)} \omega'_M$ cannot vanish. The argument principle implies that the number of the zeros minus one (because of the pole at $h = h_l$) of every function of the form (28) is less than or equal to $\deg(\tilde{\tilde{\xi}}) + \deg(\tilde{\varrho}) - 1 + 1 = \dim \mathcal{B}_n - 2$, in the domain $\mathbb{C}\backslash(-\infty, h_1)$. □

Proof of Theorem 3. Let R be a large enough constant and r be a small enough constant. Denote by $\mathcal{D}'$ the set obtained by removing the small disk $\{|h - h_2| < r\}$ from $\mathcal{D} \cap \{|h| < R\}$. To estimate the number of zeros of the Abelian integral $J(h)$ in $\mathcal{D}'$ (and hence in $\mathcal{D}$) we shall evaluate the increment of the argument of the function $F(h)$ along the boundary of $\mathcal{D}'$.

Along the circle $\{|h| = R\}$ we have

$$|F(h)| \lessapprox |h|^{\deg \mu}$$

and on the interval (h_2, ∞) the imaginary part of $F(h)$ has at most $\dim \xi + \dim \varrho + 1$ zeros (Lemma 2). The change of the argument of $F(h)$, when h makes one turn along the circle $\{|h - h_2| = r\}$ is close to zero (or negative, see [**6**]). This yields that the increment of the argument of $F(h)$ along the boundary of $\mathcal{D}'$ is less than or equal to

$$2\pi(\deg \mu + \deg \xi + \deg \varrho + 1 + 1) \leq 2\pi(\deg \mathcal{A}_n - 1)$$

and hence $F(h)$ can have at most $\dim \mathcal{A}_n - 1$ zeros in $\mathcal{D}'$. □

To count limit cycles in a neighborhood of the homoclinic trajectory we shall need a stronger version of Theorem 3, which establishes the Chebyshev property of the space $\mathcal{A}_n$ in a domain which 'includes' the point $h_2 = \frac{1}{6}$.

Recall the following definition of 'multiplicity' at $h = h_2$ (see [**6, 8, 17**]).

Definition 1. We shall say that a function $J(h) \in \mathcal{A}_n$ has a zero of multiplicity k at $h_2 \in \partial\mathcal{D}$, provided that in a neighborhood of $h = h_2$ in $\mathcal{D}$ either the estimate

$$|J(h)| \approx |(h - h_2)^{k/2} \log(h - h_2)|, \quad \text{where } k \text{ is even}; \tag{29}$$

or the estimate

$$|J(h)| \approx |(h - h_2)^{(k-1)/2}|, \quad \text{where } k \text{ is odd}; \tag{30}$$

holds.

The next theorem establishes the generalized Chebyshev property of the space $\mathcal{A}_n$.

THEOREM 4. *If the Abelian integral $J(h) \in \mathcal{A}_n$ has a zero of multiplicity k at $h_2 = \frac{1}{6}$, then it has at most* $\dim \mathcal{A}_n - k - 1$ *zeros in $\mathcal{D}$. This bound is exact.*

The proof of this theorem is the same as of Theorem 3 and therefore will be omitted. The reader may consult [**6**] for the details.

Proof of Theorem 1. If $M_1(h) \not\equiv 0$ then Theorem 1 is proved in [**7**, Theorem 3]. Suppose that $M_1(h) \equiv 0$, but $M_2(h) \not\equiv 0$. If ε is sufficiently small, then the compact domain K contains only two critical points of the vector field X_ε defined by (12). When $\varepsilon \to 0$, a limit cycle of X_ε tends either to a periodic solution of X_H, or to the origin $(0, 0)$, or to the homoclinic loop $\{H = \frac{1}{6}\}$. According to the Poincaré–Pontryagin criterion the number of limit cycles of X_ε which tend to a periodic solution of X_H is less than or equal to the number of zeros of the Abelian integral $M_2(h)$ on the open interval $(0, \frac{1}{6})$, which equals the number of the zeros of $J(h) = M_2'(h)$ (as $M_2(0) = 0$). The number of limit cycles which tend to the origin is less than or equal to the order of $M_2(h)$ at $h = 0$ minus one, and hence equals the order of $J(h)$ at $h = 0$. Finally, to evaluate the number of limit cycles which tend to the homoclinic trajectory of X_H, we shall use Roussarie's theorem [**20**]. In our case it can be stated in the following form (see [**17**]):

> Suppose that k limit cycles tend to the homoclinic trajectory as $\varepsilon \to 0$. Then the Abelian integral $J(h)$ has a zero of multiplicity at least k at $h = h_2$.

Theorem 4 implies Theorem 1. □

Proof of Theorem 2. If in (14) all $M_k(h)$ vanish, then the perturbation is integrable and hence has no limit cycles. So let $M_k(h)$ be the first non-vanishing function in (14). Then either $k = 1$ or $k \geq 2$ and $M_k(h)$ has the same structure as $M_2(h)$ [**14**]. Moreover, the unperturbed field X_H corresponding to (13) has no degenerate critical point at infinity, provided $a \in (-\frac{1}{2}, 0)$. Therefore no limit cycle can appear from infinity as a result of a quadratic perturbation. Thus Theorem 2 is a consequence of Theorem 1, applied for $n = 2$. □

Acknowledgements. We thank the referee for several useful remarks. Part of this work was carried out while the second author (I.I.) visited Laboratoire Emile Picard, University of Toulouse III. He is grateful for their hospitality. I.I. is supported by Grant MM-810/1998 from the NFSR of Bulgaria.

REFERENCES

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko. *Singularities of Differentiable Maps (Monographs in Mathematics, 1)*. Birkhäuser, Boston, 1985.
V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko. *Singularities of Differentiable Maps (Monographs in Mathematics, 2)*. Birkhäuser, Boston, 1988.

[2] V. I. Arnold, Yu. S. Il'yashenko. Ordinary differential equations. *Dynamical Systems (Encyclopaedia of Mathematical Sciences, 1)*. Springer, Berlin, 1988, pp. 1–148.

[3] J.-P. Françoise. Successive derivatives of a first return map, application to the study of quadratic vector fields. *Ergod. Th. & Dynam. Sys.* **16** (1996), 87–96.

[4] L. Gavrilov. Isochronicity of plane polynomial Hamiltonian systems. *Nonlinearity* **10** (1997), 433–448.

[5] L. Gavrilov. Petrov modules and zeros of Abelian integrals. *Bulletin des Sciences Mathématiques* **122** (1998), 571–584.

[6] L. Gavrilov. Nonoscillation of elliptic integrals related to cubic polynomials with symmetry of order three. *Bull. London Math. Soc.* **30** (1998), 267–273.

[7] L. Gavrilov. Abelian integrals related to Morse polynomials and perturbations of plane Hamiltonian vector fields. *Annales Inst. Fourier* **49** (1999), 611–652.

[8] L. Gavrilov and E. Horozov. Limit cycles of perturbations of quadratic vector fields. *J. Math. Pures Appl.* **72** (1993), 213–238.

[9] P. A. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley, 1978.

[10] E. Horozov and I. D. Iliev. On the number of limit cycles in perturbations of quadratic Hamiltonian systems. *Proc. London Math. Soc.* **69** (1994), 198–224.

[11] E. Horozov and I. D. Iliev. Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians. *Nonlinearity* **11** (1998), 1521–1537.

[12] I. D. Iliev. On second order bifurcations of limit cycles. *J. London Math. Soc.* **58** (1998), 353–366.

[13] I. D. Iliev. On the limit cycles available from polynomial perturbations of the Bogda-nov-Takens Hamiltonian. *Israel J. Math.* **115** (2000), 269–284.

[14] I. D. Iliev. Perturbations of quadratic centers. *Bull. Sci. Math.* **122** (1998), 107–161.

[15] I. D. Iliev. Higher-order Melnikov functions for degenerate cubic Hamiltonians. *Adv. Differential Equations* **1** (1996), 689–708.

[16] Bao-yi Li and Zhi-fen Zhang. A note on a result of G. S. Petrov about the weakened 16th Hilbert problem. *J. Math. Anal. Appl.* **190** (1995), 489–516.

[17] P. Mardešić. The number of limit cycles of polynomial deformations of a Hamiltonian vector field. *Ergod. Th. & Dynam. Sys.* **10** (1990), 523–529.

[18] G. S. Petrov. On the number of zeros of complete elliptic integrals. *Funct. Anal. Appl.* **18** (1984), 73–74.

[19] G. S. Petrov. Elliptic integrals and their nonoscillation. *Funct. Anal. Appl.* **20** (1986), 46–49.

[20] R. Roussarie. On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields. *Bol. Soc. Bras. Math.* **17**(2) (1986), 67–101.

[21] R. Roussarie. *Bifurcation of Planar Vector Fields and Hilbert's Sixteenth Problem (Progress in Mathematics, 164)*. Birkhäuser, Basel, 1998.

[22] H. Żołądek. Quadratic systems with center and their perturbations. *J. Differential Equations* **109** (1994), 223–273.