Composito Mathematica **123:** 167–184, 2000. 167

The Real Period Function of A_3 Singularity and Perturbations of the Spherical Pendulum

LUBOMIR GAVRILOV and OLIVIER VIVOLO
Laboratoire Emile Picard, UMR 5580, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex, France. e-mail: {gavrilov, vivolo}@picard.ups-tlse.fr

(Received: 26 September 1997; in final form: 22 October 1998)

Abstract. We prove that the Hessian matrix of the real period function $\psi(\lambda)$ associated with the real versal deformation $f_\lambda(x) = \pm x^4 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ of a singularity of type A_3, is nondegenerate, provided that $\lambda \in \mathbb{R}^3$ does not belong to the discriminant set of the singularity. We explain the relation between this result and the perturbations of the spherical pendulum.

Mathematics Subject Classifications (2000): 33E05, 70H05, 58F05.

Key words: elliptic integral, KAM theory, spherical pendulum.

1. Introduction

Let $H(x, y)$ be a fixed real polynomial, and let $R(x, y)$ be a real polynomial of degree at most n. The study of real zeros of the Abelian integral

$$I(h) = \int\int_{H \leqslant h} R(x, y)\mathrm{d}x\,\mathrm{d}y \qquad (1)$$

is closely related to the 'weakened 16th Hilbert problem' [1, p. 313], [4,11]. It is proved that, in many cases, the vector space of Abelian integrals of the form (1) has the following non-oscillation property: the number of the zeros of $I(h)$ is less than the dimension of the vector space.

Consider a *real* polynomial versal deformation $f_\lambda(x, y)$ of the real polynomial $f_0(x, y)$ with an isolated critical point of multiplicity μ. Let $\gamma(\lambda) \in H_1(\Gamma_\lambda, \mathbb{Z})$ be a continuous family of *real* vanishing cycles in the fibres $\Gamma_\lambda = \{(x, y) \in \mathbb{C}^2 : f_\lambda(x, y) = 0\}$ of the central Milnor fibration associated to f_0 [3]. A key role in the understanding of the non-oscillation property of Abelian integrals is played by the following *real period function* $\psi(\lambda) = \int_{\gamma(\lambda)} y\,\mathrm{d}x, \quad \lambda \in \mathbb{R}^\mu$. Indeed the integral (1) can be represented as a linear combination of $(\partial/\partial\lambda_i)\psi(\lambda), i = 1, 2, \ldots, \mu$, whose coefficients are, for almost all f_0, polynomial functions in λ [9]. In this paper we consider one of the simplest nontrivial cases $f_0(x, y) = y^2 \pm x^4$. The versal deformation of f_0 is $f_\lambda(x, y) = y^2 \pm x^4 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$, $\mu = 3$, and the fibres Γ_λ are affine elliptic curves with two removed points at 'infinity'. Denote the real part of the complex discriminant set of f_λ by $\Sigma^{\mathbb{R}}$. The main result of the paper is that the map

$$\lambda \to \left(\frac{\partial}{\partial\lambda_0}\psi(\lambda), \frac{\partial}{\partial\lambda_1}\psi(\lambda), \frac{\partial}{\partial\lambda_2}\psi(\lambda)\right) = \frac{1}{2}\left(\int_{\gamma(\lambda)} \frac{\mathrm{d}x}{y}, \int_{\gamma(\lambda)} \frac{x\,\mathrm{d}x}{y}, \int_{\gamma(\lambda)} \frac{x^2\,\mathrm{d}x}{y}\right),$$
$$\lambda \in \mathbb{R}^3 \backslash \Sigma^{\mathbb{R}} \tag{2}$$

is nondegenerate. This implies obviously the following claims

– the real level surfaces of the functions

$$I_0(\lambda) = \int_{\gamma(\lambda)} \frac{\mathrm{d}x}{y}, \qquad I_1(\lambda) = \int_{\gamma(\lambda)} \frac{x\,\mathrm{d}x}{y}, \qquad I_2(\lambda) = \int_{\gamma(\lambda)} \frac{x^2\,\mathrm{d}x}{y}$$

are smooth and intersect transversally at any point $\lambda \in \mathbb{R}^3\backslash\Sigma^{\mathbb{R}}$

– for any fixed real constants $(a_0, a_1, a_2) \neq (0, 0, 0)$ the level surface

$$\{\lambda \in \mathbb{R}^3\backslash\Sigma^{\mathbb{R}} : a_0 I_0(\lambda) + a_1 I_1(\lambda) + a_2 I_2(\lambda) = \text{const.}\}$$

is smooth.

We prove a similar assertion for the restriction of the real period function to the planes $\{\lambda_2 = \text{const}\}$.

The proof is based on the observation that the nondegeneracy of the Hessian matrix of the period function depends only on the isomorphism class of the curve Γ_λ and not on the curve itself. Using this we put Γ_λ into a canonical form $\{y^2 + (x^2+1)^2 + t = 0\}$ and then study the corresponding real complete elliptic integrals as functions in t in a complex domain.

The motivation of our results came from mechanics. The period function $\psi(\lambda)$ can be viewed as an 'action variable' of a completely integrable Hamiltonian system, and the nondegeneracy of the map (2) is equivalent to the nondegeneracy of the frequency map of an appropriate integrable system. The latter is, according to the Kolmogorov–Arnold–Moser theory, a typical condition which ensures the survival of most of the invariant Liouville tori after a small Hamiltonian perturbation of the integrable system [2, App. 8]. The question of nondegeneracy of the frequency map of the usual spherical pendulum was raised by Duistermaat and completely answered by Horozov [10] (see also Cushman and Bates [6, pp.182–186]). It turns out that this map is nondegenerate for all regular values of the energy-momentum map. The proof of this (quite mysterious) theorem is based on the direct study of Picard–Fuchs equations satisfied by Abelian integrals and Picard–Lefschetz theory. Our results were obtained in an attempt to understand Horozov's theorem. In Section 4 we prove that the nondegeneracy of the frequency map of the spherical pendulum is equivalent to the nondegeneracy of the Hessian matrix of the real period function $\psi(\lambda)|_{\lambda_2=\text{const.}}$. This gives a new proof of Horozov's theorem. As a by-product we also obtain a simpler expression for the action variables of the spherical pendulum (Proposition 4.1).

2. Statement of the Result

Consider a germ of a real analytic function $f: \mathbb{C}, 0 \to \mathbb{C}, 0$ with a singularity at the origin of type $A_3^{\pm}$ [2]. Put f in a normal form $f(x) = \pm x^4$ and choose a versal deformation $f_\lambda(x) = \pm x^4 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ of $f(x)$. Denote by $\Sigma \subset \mathbb{C}^3$ the subset of those $\lambda \in \mathbb{C}^3$ for which the affine algebraic curve $\Gamma_\lambda = \{(x, y) \in \mathbb{C} : y^2 + f_\lambda(x) = 0\}$ is singular. Consider the central global Milnor fibration with base $\mathbb{C}^3 \backslash \Sigma$ and fibers the smooth affine algebraic curves Γ_λ. When $\lambda \in \mathbb{R}^3$ the curve Γ_λ carries a natural real structure (anti-holomorphic involution)

$$\sigma : \Gamma_\lambda \to \Gamma_\lambda : (x, y) \to (\bar{x}, \bar{y}) . \tag{3}$$

From now on we restrict our attention to the *real* Milnor fibration $\Gamma_\lambda \to \mathbb{R}^3 \backslash \Sigma^{\mathbb{R}}$, where $\Sigma^{\mathbb{R}}$ is the real part of Σ. We note that $\mathbb{R}^3 \backslash \Sigma^{\mathbb{R}}$ has three connected components $\Lambda_0, \Lambda_1, \Lambda_2$ in which the polynomial $f_\lambda(x)$ has 0, 2 and 4 real roots, respectively. The components Λ_1, Λ_2 are simply connected, while $\pi_1(\Lambda_0) = \mathbb{Z}$.

Let $\gamma(\lambda) \in H_1(\Gamma_\lambda, \mathbb{Z})$, $\lambda \in \Lambda_i$, $i = 0, 1, 2$, be a continuous family of real vanishing cycles. This means that $\gamma(\lambda)$ is a locally constant section of the homology Milnor bundle, $H_1(\Gamma_\lambda, \mathbb{Z}) \to \lambda, \quad \lambda \in \Lambda_i$, which vanishes at a single Morse critical point of $y^2 + f_\lambda(x)$ as λ tends to $\lambda_0 \in \Sigma^{\mathbb{R}}$ along an appropriate analytic curve. The reality condition on the cycle $\gamma(\lambda)$ means that $\sigma_* \gamma(\lambda) = \gamma(\lambda) \in H_1(\Gamma_\lambda, \mathbb{Z})$ where σ is the anti-holomorphic involution (3). Note that there are continuous families of nonreal vanishing cycles, as well continuous families of real cycles which are not vanishing.

DEFINITION 1. The function $\psi(\lambda) = \int_{\gamma(\lambda)} y \, dx, \quad \lambda \in \Lambda_i \subset \mathbb{R}^3$ is called a real period function of the singularity of type A_3 provided that $\gamma(\lambda) \in H_1(\Gamma_\lambda, \mathbb{Z})$, $\lambda \in \Lambda_i$, is a continuous family of real vanishing cycles.

In the Hamiltonian mechanics the period function $\psi(\lambda)$ is called action variable of the one degree of freedom Hamiltonian system $(d^2/dt^2)x = (d/dx) f_\lambda(x)$ [1, p. 281]. The purpose of the present article is to prove the following

THEOREM 2.1. *Let $\psi(\lambda)$, $\lambda \in \mathbb{R}^3 \backslash \Sigma^{\mathbb{R}}$, be a real period function of the singularity of type A_3. Then*

$$\det \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial \lambda_0^2} & \dfrac{\partial^2 \psi}{\partial \lambda_0 \partial \lambda_1} & \dfrac{\partial^2 \psi}{\partial \lambda_0 \partial \lambda_2} \\ \dfrac{\partial^2 \psi}{\partial \lambda_1 \partial \lambda_0} & \dfrac{\partial^2 \psi}{\partial \lambda_1^2} & \dfrac{\partial^2 \psi}{\partial \lambda_1 \partial \lambda_2} \\ \dfrac{\partial^2 \psi}{\partial \lambda_2 \partial \lambda_0} & \dfrac{\partial^2 \psi}{\partial \lambda_2 \partial \lambda_1} & \dfrac{\partial^2 \psi}{\partial \lambda_2^2} \end{pmatrix} \neq 0 . \tag{4}$$

If in addition $\lambda \in \Lambda_0$, then

$$\det \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial \lambda_0^2} & \dfrac{\partial^2 \psi}{\partial \lambda_0 \partial \lambda_1} \\ \dfrac{\partial^2 \psi}{\partial \lambda_1 \partial \lambda_0} & \dfrac{\partial^2 \psi}{\partial \lambda_1^2} \end{pmatrix} \neq 0 \tag{5}$$

Remark. For $\lambda \in \Lambda_0$ the locally constant section $\gamma(\lambda)$ of the homology Milnor bundle may be multivalued. In this case the period function $\psi(\lambda)$ will be multivalued too. It worth noting, however, that the representation of $\pi_1(\Lambda_0)$ on $H_1(\bar{\Gamma}_\lambda, \mathbb{Z})$, where $\bar{\Gamma}_\lambda$ is the compactified curve, is trivial [13]. This, together with the fact that the meromorphic differential 1-forms

$$\frac{\partial^2 \psi}{\partial \lambda_i \partial \lambda_j} = -\frac{1}{4} \frac{x^{i+j} \mathrm{d}x}{y^3}, \quad 0 \leqslant i + j \leqslant 4$$

have no residues on the compact curve $\bar{\Gamma}_\lambda$, implies that the derivatives $\partial^2 \psi / \partial \lambda_i \partial \lambda_j$ are single-valued functions even in the real domain Λ_0.

3. Proof of Theorem 2.1

The proof will consist in two steps. First we show that it is enough to prove the inequality (4) in the case when $f_\lambda(x)$ has the 'canonical' form $f_t(x) = (x^2 \pm 1)^2 + t$ for suitable negative real constant t, and $\gamma(t)$ is an appropriate continuous family of vanishing cycles on the curve $\Gamma_t = \{y^2 + (x^2 + 1) + t = 0\}$, $t \leqslant 0$. At the second step we study, following Petrov [12], the properties of the Abelian integrals

$$\int_{\gamma(t)} \frac{x^i \mathrm{d}x}{y^3}, \quad 0 \leqslant i \leqslant 4$$

as functions in $t \in \mathbb{C}$.

LEMMA 3.1. *Let $P(x) = x^4 + \ldots$ be a real polynomial of degree four. There always exist real constants a, b, c, d, $ad - bc \neq 0$, such that*

$$(cx + d)^4 P\left(\frac{ax + b}{cx + d}\right) = (x^2 \pm 1)^2 + t \tag{6}$$

for some real nonpositive constant t.

Remark. If $P(x)$ is a degree four complex polynomial, then there always exist complex numbers a, b, c, d, such that (6) holds true for some *complex* number t.

Proof. We have to consider three cases:

– Suppose that $P(x)$ has two pairs of complex conjugate roots. The circles perpendicular to the real axe $\mathrm{Im}(x) = 0$ (including the vertical lines $\mathrm{Re}(x) = \mathrm{const}$) are mapped one to the other by real Möbius transformations

$$S : \mathbb{C} \to \mathbb{C} : x \to \frac{ax+b}{cx+d}, a, b, c, d \in \mathbb{R}, ad - bc \neq 0 .$$

Let S be the Möbius transformation which maps the circle through the roots of $P(x)$ to the vertical line $\mathrm{Re}(x) = 0$. Then

$$(cx+d)^4 P\left(\frac{ax+b}{cx+d}\right) = a_0x^4 + a_2x^2 + a_4,$$

where $a_0 = c^4 P(a/c) > 0$. An obvious linear change of the variable x puts $a_0x^4 + a_2x^2 + a_4$ in the form (6).

– If $P(x)$ has a pair of complex conjugate roots, then we may suppose that these roots are $x_1 = i, x_2 = -i, x_3, x_4 \in \mathbb{R}$. The isotropy group of the point $i \in \mathbb{C}$ consists of real Möbius transformations S_θ of the form

$$S_\theta(x) = \frac{x\cos(\theta) + \sin(\theta)}{-x\sin(\theta) + \cos(\theta)} .$$

If $x_3 + x_4 = 0$, then $P(x) = x^4 + a_2x^2 + a_4$. If $x_3 + x_4 \neq 0$, then $S_\theta(x_3) + S_\theta(x_4) = 0$, where

$$\mathrm{cotg}(2\theta) = \frac{x_3x_4 - 1}{x_3 + x_4} .$$

It follows that

$$(-x\sin(\theta) + \cos(\theta))^2 P\left(\frac{x\cos(\theta) + \sin(\theta)}{-x\sin(\theta) + \cos(\theta)}\right) = a_0x^4 + a_2x^2 + a_4 ,$$

where $a_0a_4 < 0$. If $a_0 > 0$ we proceed as in the preceding case. If $a_4 > 0$ we change the variable $x \to 1/x$.

– Suppose at last that $P(x)$ has four distinct real roots. We may always suppose that $x_1 = 1, x_2 = -1, |x_3|, |x_4| < 1$. The subgroup of real Möbius transformations, such that $S(1) = 1, S(-1) = -1$ consists of maps

$$S_{a,b}(x) = \frac{ax+b}{bx+a} , a^2 - b^2 \neq 0.$$

The identity $S_{a,b}(x_3) + S_{a,b}(x_4) = 0$ is equivalent to

$$a^2 + 2ab\frac{1 + x_3x_4}{x_3 + x_4} + b^2 = 0 .$$

As

$$\left(\frac{1+x_3x_4}{x_3+x_4}\right)^2 - 1 = \frac{(x_3^2-1)(x_4^2-1)}{(x_3+x_4)^2} > 0$$

then there exist real a, b, such that

$$(bx+a)^4 P\left(\frac{ax+b}{bx+a}\right) = a_0x^4 + a_2x^2 + a_4$$

where $a_0 = b^4P(a/b)$ and $a_4 = a^4P(b/a)$. As $P(x) > 0$ for $|x| > 1$ then either a_0 or a_4 is positive.

Note finally that in the three cases above the polynomial $(y \pm 1)^2 + t$ has real roots which implies $t \leqslant 0$. This completes the proof of Lemma 3.1. □

Let $P(x)$ be an arbitrary polynomial of degree four. Consider the complete Abelian integrals $w_i = \int_\gamma x^i \mathrm{d}x/y^3$, associated to the algebraic curve

$$\Gamma = \{(x, y) \in \mathbb{C}^2 : y^2 + P(x) = 0\}$$

where $\gamma \in H_1(\Gamma, \mathbb{Z})$ and put

$$D(\Gamma) = \det\begin{pmatrix} w_0 & w_1 & w_2 \\ w_1 & w_2 & w_3 \\ w_2 & w_3 & w_4 \end{pmatrix}. \tag{7}$$

The bi-rational change of variables

$$x \to \frac{ax+b}{cx+d}, \qquad y \to \frac{y}{(cx+d)^2}, \qquad a, b, c, d = \text{const}, \quad ad - bc \neq 0 \tag{8}$$

induces a bi-holomorphic map from the curve Γ to

$$\tilde{\Gamma} = \{(x, y) \in \mathbb{C}^2 : y^2 + \tilde{P}(x) = 0\},$$

where

$$\tilde{P}(x) = (cx+d)^4 P\left(\frac{ax+b}{cx+d}\right).$$

Let $\tilde{\gamma}$ be the image of the cycle γ under this bi-holomorphic map. Denote as above $\tilde{w}_i = \int_{\tilde{\gamma}}(x^i \mathrm{d}x/y^3)$ and

$$D(\tilde{\Gamma}) = \det\begin{pmatrix} \tilde{w}_0 & \tilde{w}_1 & \tilde{w}_2 \\ \tilde{w}_1 & \tilde{w}_2 & \tilde{w}_3 \\ \tilde{w}_2 & \tilde{w}_3 & \tilde{w}_4 \end{pmatrix}. \tag{9}$$

LEMMA 3.2. *The determinants $D(\Gamma)$ and $D(\tilde{\Gamma})$ are related in the following way*

$$D(\Gamma) = (ad - bc)^9 D(\tilde{\Gamma}). \tag{10}$$

Remark. The above Lemma implies that the nonvanishing property of the (real) determinant $D(\Gamma)$ depends only on the isomorphism class of the curve Γ, and not on Γ itself (compare to Horozov [10, Fig. 3]).

Proof. The change of variables (8) transforms the differential one form $x^i \mathrm{d}x/y^3$, $0 \leqslant i \leqslant 4$ to

$$(ad - bc)(ax + b)^i (cx + d)^{4-i} \frac{\mathrm{d}x}{y^3},$$

which explains the factor $(ad - bc)^3$ in (10). Consider (as suggested by the referee) the following homogeneous matrix

$$M(x, \xi) = \begin{pmatrix} \xi^4 & x\xi^3 & x^2\xi^2 \\ x\xi^3 & x^2\xi^2 & x^3\xi \\ x^2\xi^2 & x^3\xi & x^4 \end{pmatrix} = \begin{pmatrix} \xi^2 \\ x\xi \\ x^2 \end{pmatrix} \cdot \begin{pmatrix} \xi^2 & x\xi & x^2 \end{pmatrix}.$$

We have

$$M(ax + b\xi, cx + d\xi) = B \cdot M(x, \xi) \cdot B^*, \ \ B \in \mathrm{GL}(3, \mathbb{C}), \tag{11}$$

where

$$B \begin{pmatrix} \xi^2 \\ x\xi \\ x^2 \end{pmatrix} = \begin{pmatrix} (cx + d\xi)^2 \\ (ax + b\xi)(cx + d\xi) \\ (ax + b\xi)^2 \end{pmatrix},$$

and B^* is the transposed matrix B. The last identity is the natural representation of $\mathrm{GL}(2, \mathbb{C})$ on the space of homogeneous polynomials of degree two in two variables x, ξ. The map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \det(B)$$

is then a Lie group homomorphism from $\mathrm{GL}(2, \mathbb{C})$ to $\mathbb{C}^*$ and, hence, $\det(B)$ can be only a factor of $ad - bc$. It is easy to check that $\det(B) = (ad - bc)^3$. Finally (11) implies

$$\begin{pmatrix} \tilde{w}_0 & \tilde{w}_1 & \tilde{w}_2 \\ \tilde{w}_1 & \tilde{w}_2 & \tilde{w}_3 \\ \tilde{w}_2 & \tilde{w}_3 & \tilde{w}_4 \end{pmatrix} = (ad - bc)^3 B \begin{pmatrix} w_0 & w_1 & w_2 \\ w_1 & w_2 & w_3 \\ w_2 & w_3 & w_4 \end{pmatrix} B^*$$

which completes the proof of Lemma 3.2. □

Using (10) and Lemma 3.1 we see that it is enough to prove Theorem 2.1 in the case when Γ_λ has the 'canonical' form

$$\Gamma_t = \{\pm y^2 + x^4 \pm 2x^2 + 1 + t = 0\}, \ \ t \leqslant 0.$$

The continuous family of cycles $\gamma(\lambda)$ is transformed to a continuous family of cycles

$\gamma(t) \in H_1(\Gamma_t, \mathbb{Z})$, which vanish as $t \in \mathbb{R}$ tends to -1, or 0. We can make further the following simplifications. If $\Gamma_t = \{-y^2 + x^4 \pm 2x^2 + 1 + t = 0\}$, then we substitute $y \to \sqrt{-1}y$. If $\Gamma_t = \{y^2 + x^4 - 2x^2 + 1 + t = 0\}$, then we substitute $x \to \sqrt{-1}x$. A real cycle on the initial curve is transformed to a cycle invariant under one of the following four anti-holomorphic involutions

$$\sigma_i : \Gamma_t \to \Gamma_t : (x, y) \to (\pm\bar{x}, \pm\bar{y})$$

on the curve $\Gamma_t = \{y^2 + (x^2 + 1) + t = 0\}$, $t \leqslant 0$.

Consider now the three continuous families of σ_i–invariant (for appropriate σ_i) vanishing cycles $\gamma(t)$, $t \in (-\infty, 0)$, and $\delta_{1,2}(t)$, $t \in (-1, 0)$, as it is shown in Figure 1. Lemma 3.1 and Lemma 10 imply the following:

COROLLARY 3.3. *To prove the inequality (4) it is enough to show that*

$$\det\begin{pmatrix} \int_{\gamma(t)} \frac{\mathrm{d}x}{y^3} & \int_{\gamma(t)} \frac{x\mathrm{d}x}{y^3} & \int_{\gamma(t)} \frac{x^2\mathrm{d}x}{y^3} \\ \int_{\gamma(t)} \frac{x\mathrm{d}x}{y^3} & \int_{\gamma(t)} \frac{x^2\mathrm{d}x}{y^3} & \int_{\gamma(t)} \frac{x^3\mathrm{d}x}{y^3} \\ \int_{\gamma(t)} \frac{x^2\mathrm{d}x}{y^3} & \int_{\gamma(t)} \frac{x^3\mathrm{d}x}{y^3} & \int_{\gamma(t)} \frac{x^4\mathrm{d}x}{y^3} \end{pmatrix} \neq 0, \ \textit{for} -\infty < t < -1, \ -1 < t < 0 \tag{12}$$

and

$$\det\begin{pmatrix} \int_{\delta_1(t)} \frac{\mathrm{d}x}{y^3} & \int_{\delta_1(t)} \frac{x\mathrm{d}x}{y^3} & \int_{\delta_1(t)} \frac{x^2\mathrm{d}x}{y^3} \\ \int_{\delta_1(t)} \frac{x\mathrm{d}x}{y^3} & \int_{\delta_1(t)} \frac{x^2\mathrm{d}x}{y^3} & \int_{\delta_1(t)} \frac{x^3\mathrm{d}x}{y^3} \\ \int_{\delta_1(t)} \frac{x^2\mathrm{d}x}{y^3} & \int_{\delta_1(t)} \frac{x^3\mathrm{d}x}{y^3} & \int_{\delta_1(t)} \frac{x^4\mathrm{d}x}{y^3} \end{pmatrix} \neq 0, \ \textit{for} \ -1 < t < 0\,. \tag{13}$$

where $\gamma(t)$, $\delta_1(t)$ are the cycles on the curve $\Gamma_t = \{y^2 + (x^2 + 1) + t = 0\}$, shown in Figure 1.

The next step is to study the properties of the Abelian integrals, which we summarize in the following

PROPOSITION 3.4. *If the orientation of $\delta_1(t)$, $\delta_2(t)$ is fixed in such a way that $\sqrt{-1}\int_{\delta_{1,2}(t)} \mathrm{d}x/y > 0$ then for every $t \in]-1, 0[$ and any constants a, b, c holds*

$$-\sqrt{-1}\int_{\delta_{1,2}(t)} \frac{(ax^2 + bx + c)^2\mathrm{d}x}{y^3} > 0 \tag{14}$$

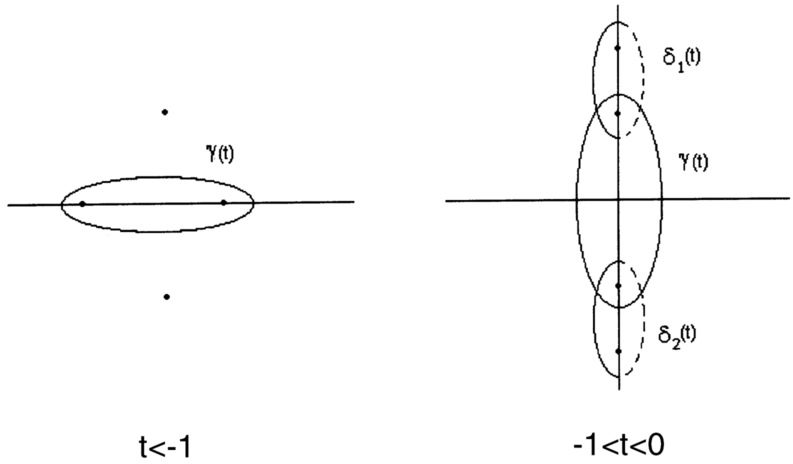

Figure 1.

and

$$\int_{\delta_{1,2}(t)} \frac{x\mathrm{d}x}{y^3} = \int_{\delta_{1,2}(t)} \frac{x^3\mathrm{d}x}{y^3} = 0\ . \tag{15}$$

If the orientation of $\gamma(t)$ *is fixed in such a way that* $\int_{\gamma(t)} \mathrm{d}x/y > 0$, *then*

$$\int_{\gamma(t)} \frac{(x^2+1)^2\mathrm{d}x}{y^3} < 0, \int_{\gamma(t)} \frac{\mathrm{d}x}{y^3} > 0, \forall t \in (-\infty, 0), \tag{16}$$

$$\int_{\gamma(t)} \frac{x^4\mathrm{d}x}{y^3} > 0,\ \forall t \in (-1, 0), \tag{17}$$

$$\int_{\gamma(t)} \frac{x^4\mathrm{d}x}{y^3} < 0,\ \forall t \in (-\infty, -1), \tag{18}$$

$$\int_{\gamma(t)} \frac{x^2\mathrm{d}x}{y^3} < 0,\ \forall t \in (-1, 0), \tag{19}$$

$$\int_{\gamma(t)} \frac{x\mathrm{d}x}{y^3} = \int_{\gamma(t)} \frac{x^3\mathrm{d}x}{y^3} = 0,\ \forall t \in (-\infty, 0). \tag{20}$$

Proof. To prove (14) we note that as for $t \in]-1, 0[$

$$\sqrt{-1}\int_{\delta_{1,2}(t)} \frac{\mathrm{d}x}{y} = \sqrt{-1}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{y} > 0 ,$$

then

$$-\sqrt{-1}\int_{\delta_{1,2}(t)} \frac{(ax^2+bx+c)^2\mathrm{d}x}{y^3} = -\sqrt{-1}\int_{-\infty}^{+\infty} \frac{(ax^2+bx+c)^2\mathrm{d}x}{y^3} > 0 .$$

(the last integral converges!). The involution $(x, y) \to (-x, -y)$ acts on $H_1(\gamma, \mathbb{Z})$ in the following way $(\gamma, \delta_1, \delta_2) \to (\gamma, \delta_2, \delta_1)$, where δ_1 and δ_2 are homologous cycles on the compactified curve $\bar{\Gamma}_t$. This implies that any anti-invariant differential 1-form without residues on $\bar{\Gamma}_t$ is co-homologous to zero which proves (15) and (20).

Put

$$w_i(t) = \int_{\gamma(t)} \frac{x^i\mathrm{d}x}{y^3}, 0 \leqslant i \leqslant 4 .$$

The Abelian integrals $w_i(t)$ are also defined for complex values of t. As $\gamma(t)$ vanishes as t tends to -1, then the Picard–Lefschetz formula implies that $w_i(t)$ are holomorphic functions in the complex domain $\mathcal{D} = \mathbb{C}\backslash[0, \infty[$ (Figure 2). We shall count, following [12], the zeros of $w_i(t)$ in this larger domain. Consider first the holomorphic function $w_0(t)$, $t \in \mathcal{D}$. Let us evaluate the increment of the argument of $w_0(t)$ along the boundary of $\mathcal{D}$ described in a positive direction (anticlockwise). For $t \in]0, \infty[$ we define $w_0^+(t)$ and $w_0^-(t)$ to be the analytic continuation of $w_0(t)$ along a path on which $\mathrm{Im}(t) > 0$ and $\mathrm{Im}(t) < 0$ respectively. Let R be a big enough and r be a small enough constants. Denote by $\mathcal{D}'$ the set obtained from $\mathcal{D} \cap \{|t| < R\}$ by removing the circle of radius r centered at $t = 0$ (Figure 2). Consider the increase of the argument of $w_0(t)$ along the boundary of $\mathcal{D}'$. For $t \in \mathcal{D}$ we have $|x| \overset{\infty}{\approx} |t|^{1/4}$, $|y| \overset{\infty}{\approx} |t|^{1/2}$ so the argument of $w_0(t)$ decreases by at least $5\pi/2$ along the circle $|t| = R$.

The Picard–Lefschetz formula implies that when t makes one turn about 0 in a small neighborhood of 0, the cycle $\gamma(t)$ is transformed in the following way

$$\gamma(t) \to \gamma(t) - (\gamma(t)\cdot\delta_1(t))\delta_1(t) - (\gamma(t)\cdot\delta_2(t))\delta_2(t) . \tag{21}$$

It follows that in a neighborhood of 0

$$\int_{\gamma(t)} y\mathrm{d}x = \frac{\log(t)}{2\pi\sqrt{-1}}\int_{\delta_1(t)} y\mathrm{d}x + \frac{\log(t)}{2\pi\sqrt{-1}}\int_{\delta_2(t)} y\mathrm{d}x + \mathrm{O}(1),$$

$$\int_{\delta_{1,2}(t)} y\mathrm{d}x = \mathrm{O}(t)$$

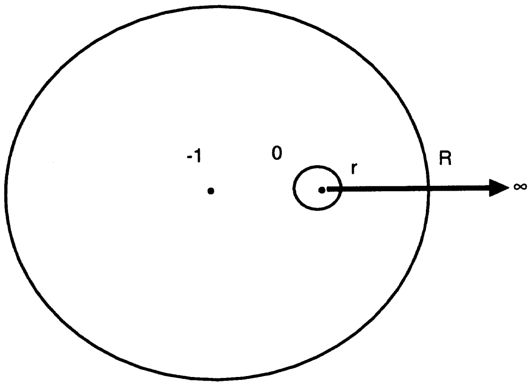

Figure 2. The region $\mathcal{D}$ and $\mathcal{D}'$.

and, hence,

$$\int_{\gamma(t)} \frac{\mathrm{d}x}{y^3} = \frac{2\sqrt{-1}}{\pi t}\left(\int_{\delta(t)} \frac{\mathrm{d}x}{y} + 0(t\log(t))\right), \tag{22}$$

where $\delta(t)$ is a cycle homologous to δ_1 or δ_2 on the compactified curve $\bar{\Gamma}_t$. It follows that along the circle $|t| = r$ the increment of the argument of $w_0(t)$ is close to 2π. Along the interval $[r, R]$ we use the Picard–Lefschetz formula (21) to get

$$w_0^+(t) - w_0^-(t) = \pm 2\sqrt{-1}\mathrm{Im} w_0^{\pm}(t) = \pm 2\int_{\delta(t)} \frac{\mathrm{d}x}{y^3}\ .$$

Using the same argument as in the proof of (14) we conclude that $\int_{\delta(t)} \mathrm{d}x/y^3 \neq 0$ on the interval $]0, \infty[$ and hence the increase of the argument of $w_0^{\pm}(t)$ along the interval $[r, R]$ is less than π. Putting the above data together yields that the increment of the argument of $w_0(t)$ along the boundary of $\mathcal{D}'$ is less than $2\pi - \pi/2$. The argument principle implies that $w_0(t)$ has no zeros in $\mathcal{D}'$ (and hence in $\mathcal{D}$). For the Abelian integral $w_4(t) + 2w_2(t) + w_0(t)$ the analogue of (22) in a neighborhood of $t = 0$ is

$$w_4(t) + 2w_2(t) + w_0(t) = \frac{\log(t)}{\pi\sqrt{-1}}\int_{\delta(t)} \frac{(x^2+1)^2\mathrm{d}x}{y^3} + \mathrm{O}(1) \tag{23}$$

as

$$\int_{\delta(t)} \frac{(x^2+1)\mathrm{d}x}{y}$$

vanishes at $t = 0$. Applying once again the argument principle we obtain that $w_4(t) + 2w_2(t) + w_0(t)$ has no zeros in $\mathcal{D}$. In a quite similar way one shows that

$w_2(t), w_4(t)$ have at most one zero in $\mathcal{D}$. Let us compute now $w_i(-1)$. We have

$$\int_{\gamma(-1)} \frac{\mathrm{d}x}{y} = \lim_{t\to -1} \int_{\gamma(t)} \frac{\mathrm{d}x}{\sqrt{-(x^2+1)^2 - t}}$$
$$= 2\pi\sqrt{-1}\,\mathrm{Res}_{x=0} \frac{\mathrm{d}x}{\sqrt{-(x^2+1)^2+1}} = \sqrt{2}\pi > 0\ .$$

In a similar way

$$w_0(-1) = 2\pi\sqrt{-1}\,\mathrm{Res}_{x=0} \frac{\mathrm{d}x}{(\sqrt{-(x^2+1)^2+1})^3} = \frac{3\pi}{4\sqrt{2}} > 0,$$

$$w_2(-1) = 2\pi\sqrt{-1}\,\mathrm{Res}_{x=0} \frac{x^2\mathrm{d}x}{(\sqrt{-(x^2+1)^2+1})^3} = -\frac{\pi}{\sqrt{2}} < 0,$$

$$w_4(-1) = 2\pi\sqrt{-1}\,\mathrm{Res}_{x=0} \frac{x^4\mathrm{d}x}{(\sqrt{-(x^2+1)^2+1})^3} = 0,$$

$$w_4(-1) + 2w_2(-1) + w_0(-1) = -\frac{5\pi}{4\sqrt{2}} < 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} w_4(t)|_{t=-1} = 3\pi\sqrt{-1}\,\mathrm{Res}_{x=0} \frac{x^4\mathrm{d}x}{(\sqrt{-(x^2+1)^2+1})^5} = \frac{3\pi}{4\sqrt{2}} > 0.$$

The above already implies (16) and (17), (18). In the interval $(-1, 0)$ the inequalities (16) and (17) imply (19).

End of the proof of Theorem 2.1. On the interval $]-\infty, -1[$ and up to multiplication by a constant factor, the determinant (12) equals to $w_0(w_2^2 - w_0 w_4)$ which is not zero, as $w_0 w_4 < 0$. On the interval $]-1, 0[$ we have $0 < w_0 + w_4 < -2w_2$ which implies

$$(w_0 + w_4)^2 < 4w_2^2 \Leftrightarrow 0 \leqslant (w_0 - w_4)^2 < 4(w_2^2 - w_0 w_4)$$

and hence $w_0(w_2^2 - w_0 w_4) > 0$.

The matrix in (13) is even definite. Indeed, the last claim is equivalent to

$$\int_{\delta_{1,2}(t)} \frac{(ax^2 + bx + c)^2 \mathrm{d}x}{y^3} \neq 0, \forall a, b, c \in \mathbb{R}$$

which holds true according to (14).

The proof of (5) goes along the same lines. Namely, the bi-rational transformation (8) puts the matrix (5) in the form

$$(da-bc)\begin{pmatrix} \int_{\gamma(\lambda)} \frac{(cx+d)^4 \mathrm{d}x}{y^3} & \int_{\gamma(\lambda)} \frac{(cx+d)^3(ax+b)\mathrm{d}x}{y^3} \\ \int_{\gamma(\lambda)} \frac{(cx+d)^3(ax+b)\mathrm{d}x}{y^3} & \int_{\gamma(\lambda)} \frac{(cx+d)^2(ax+b)^2\mathrm{d}x}{y^3} \end{pmatrix}.$$

Suppose that Γ_λ is in a canonical form. As $\lambda \in \Lambda_0$ then $-1 < t < 0$ and the vanishing cycle $\gamma(\lambda)$ becomes the vanishing cycle $\gamma(t)$ shown on Figure 1. It follows that $w_1(t) = w_3(t) = 0$ and the determinant of the above matrix is

$$(da-bc)^2\big(w_0\, w_2\, d^4 + c^4\, w_2 w_4 + c^2 d^2 (w_0 w_4 - 3\, {w_2}^2)\big)\,. \tag{24}$$

This, combined with $w_0 w_4 - w_2^2 < 0$, $w_0 > 0$, $w_4 > 0$, $w_2 < 0$ completes the proof of Theorem 2.1. □

4. Perturbations of the Spherical Pendulum

The spherical pendulum is a mechanical system which consists of a heavy particle of mass m, moving without friction on the sphere$\{x \in \mathbb{R}^3 : |x| = l\}$, under the action of the gravitational force mg. Its motion is governed by the Lagrange function $L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - mgx_3$ constrained on the sphere of radius l. After obvious rescalings L and in spherical coordinates $x_1 = \sin\phi\cos\theta, x_2 = \sin\phi\sin\theta, x_3 = \cos\phi$. L takes the form $L = \frac{1}{2}\left(\dot{\phi}^2 + \dot{\theta}^2\sin^2\phi\right) - \cos\phi$. The corresponding Hamiltonian system (H, T^*S, ω)

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \qquad \dot{p_\theta} = -\frac{\partial H}{\partial \theta},$$
$$\dot{\phi} = \frac{\partial H}{\partial p_\phi}, \qquad \dot{p_\phi} = -\frac{\partial H}{\partial \phi}.$$

lives on the cotangent bundle T^*S of the unit sphere S with its canonical simplectic structure $\omega = \mathrm{d}p_\theta \wedge \mathrm{d}\theta + \mathrm{d}p_\phi \wedge \mathrm{d}\phi$, where $p_\theta = \partial L/\partial\dot{\theta}, p_\phi = \partial L/\partial\dot{\phi}$ are the conjugate moments, and

$$H(p_\theta, \theta, p_\phi, \phi) = \frac{1}{2}{p_\phi}^2 + \frac{p_\theta^2}{2\sin^2\phi} + \cos\phi\,.$$

The variable θ is obviously ignorable and the corresponding first integral (momentum of the particle about the vertical) is $G = p_\theta = \dot{\theta}\sin^2\phi$. The momentum mapping of the spherical pendulum is

$$F : T^*S \longrightarrow \mathbb{R}^2, \qquad (\theta, \phi, p_\theta, p_\phi) \longmapsto (H, G)\,.$$

Denote by U_r the set of the regular points of the momentum mapping. The level surface $H = h, G = g$ for $(h, g) \in U_r$ is a torus $T_{h,g}$ and the action variables of the spherical pendulum [8], in a neighborhood of $T_{h,g}$ are given by

$$\begin{aligned} I_1(h, g) &= \frac{1}{2\pi} \oint_{\gamma(h,g)} (2(h - \cos\phi) - g^2 \sin^{-2}\phi)^{1/2} \mathrm{d}\phi \\ &= \oint_{\gamma(h,g)} \frac{\sqrt{2(h-u)(1-u^2) - g^2}}{1-u^2} \mathrm{d}u \end{aligned} \tag{25}$$

$I_2 = g$, where $u = \cos\phi$, and the cycle $\gamma = \gamma(h, g)$ is shown in Figure 3. Recall that for $(h, g) \in U_r$ the polynomial $2(h-u)(1-u^2) - g^2$ [14] has two real roots u_1 and u_2 ($u_1 \leqslant u_2$) on the interval $-1 \leqslant u \leqslant 1$ and one for $u > 1$ (Figure 3(a)) [14].

Consider the *real* elliptic curve Γ with affine equation

$$y^2 + x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{26}$$

and natural anti-holomorphic involution $(x, y) \to (\overline{x}, \overline{y})$, and put

$$g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^2 - 4\frac{a_1}{4}\frac{a_3}{4}, \qquad g_3 = -\det\begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}. \tag{27}$$

It is well known that the curve Γ and the real curve

$$C := \{\eta^2 = 4\xi^3 - g_2\xi - g_3\}\,. \tag{28}$$

with anti-holomorphic involution $(\xi, \eta) \to (\overline{\xi}, \overline{\eta})$ are isomorphic over $\mathbb{C}$ [7]. Although these curves are not real isomorphic, there is still a relation between their real structures. The Jacobian variety $J(\Gamma)$ of the elliptic curve Γ turns out to be real isomorphic to the curve C [15]. For that reason we call C Jacobian and write $J(\Gamma) = C$. The Jacobian $J(\Gamma)$ is complex isomorphic to Γ, but not real isomorphic (unless $\Gamma_{\mathbb{R}} \neq \emptyset$, as noted by M. Audin [5, p. 123]). The bi-rational isomorphism which identifies C and Γ over $\mathbb{C}$ is given by

$$(x, y) \longrightarrow \left(\xi = \frac{A_1}{r_0 - x} - \frac{A_2}{2}, \eta = \frac{A_1}{(r_0 - x)^2} y\right), \tag{29}$$

where r_0 is a root of $f(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ and $A_1 = r_0^3 + 3a_1r_0^2/4 + a_2r_0/2 + a_3/4$, $A_2 = r_0^2 + a_1r_0/2 + a_2/6$. Of course (29) maps the holomorphic differential $\mathrm{d}\xi/\eta$ to a multiple of dx/y. Using (29) it is easy to check that in fact $\mathrm{d}\xi/\eta = \mathrm{d}x/y$. Next we apply the above to compute the action variable I_1 (25),

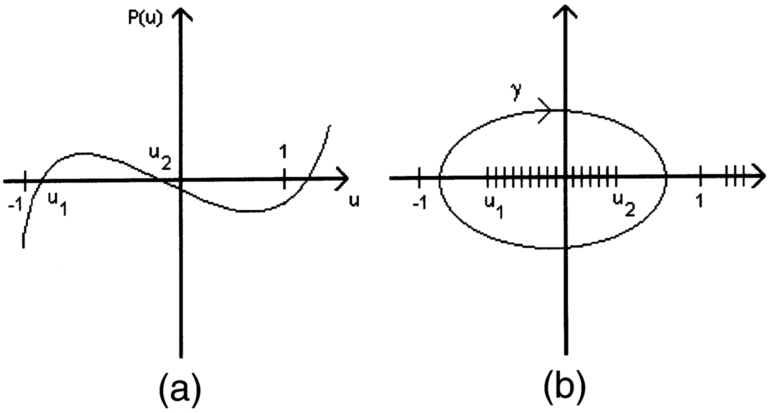

Figure 3. (a) Graph of the $P(u) = 2(h - u)(1 - u^2) - g^2$ for $(h, g) \in U_r$; (b) Projection of the cycle $\gamma(h, g)$ on the complex u-plane.

we put $a_1 = 0, a_2 = 2h, a_3 = 2g$ and $a_4 = 1$. We have

$$\frac{\partial I_1}{\partial a_2} = \frac{1}{4\pi} \oint_\gamma \frac{\mathrm{d}u}{\sqrt{2u^3 - a_2u^2 - 2u + a_2 - a_3^2/4}}$$
$$= \frac{1}{4\pi} \oint_\gamma \frac{\mathrm{d}\xi}{\eta} = \frac{1}{4\pi} \oint_\gamma \frac{\mathrm{d}x}{y} = \frac{1}{2\pi} \frac{\partial}{\partial a_2} \left(\oint_\gamma \frac{y}{x^2} \mathrm{d}x \right).$$

Then

$$I_1 = \frac{1}{2\pi} \oint_\gamma \frac{y}{x^2} \mathrm{d}x + \varphi(a_3),$$

where $\varphi(a_3)$ is a function. To compute $\varphi(a_3)$, we note that for any fixed a_3 such that the polynomial $f(x)$ has no real roots, we may continuously deform a_2 in such a way, that (a_2, a_3) lies on the discriminant of $f(x)$ and in addition the cycle $\gamma(a_2, a_3)$ vanishes. This implies

$$I_1(a_2, a_3) = 0, \quad \oint_{\gamma(a_2, a_3} \frac{y}{x^2} \mathrm{d}x = 0, \quad \varphi(a_3) = 0.$$

We obtain the following proposition:

PROPOSITION 4.1. *Consider the elliptic curve*

$$\Gamma_{h,g} = \{y^2 + f(x) = 0\}, f(x) = x^4 + 2hx^2 + 2gx + 1 .$$

Denote by $\Lambda_0 \subset \mathbb{R}^2$ *the connected component of the complement to the discriminant of* $f(x)$ *in* $\mathbb{R}^2$ *(the hatched part in Figure 4) in which* $f(x)$ *has no real roots. Then* $\Lambda_0 \equiv U_r$

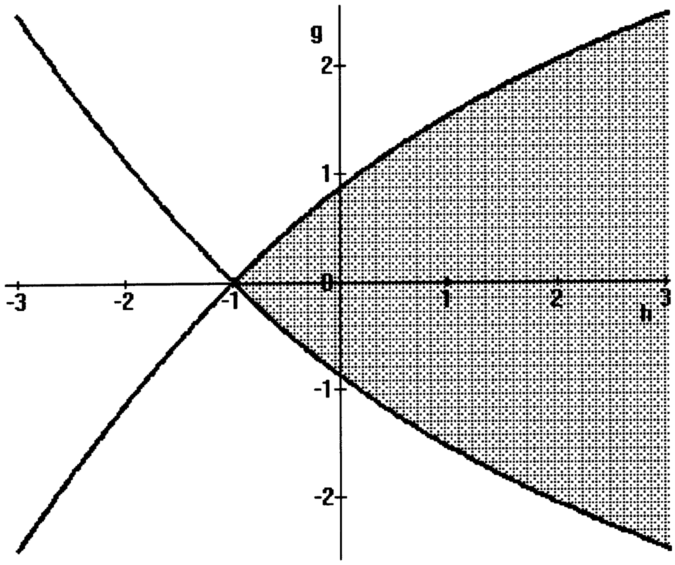

Figure 4. The discriminant locus of $f(x)$.

and for $(h, g) \in U_r$ *the action variables are also given by*

$$I_1 = \frac{1}{2\pi} \oint_{\gamma(h,g)} \frac{y}{x^2} \mathrm{d}x, \quad I_2 = g,$$

where $\gamma(h, g)$ *is a cycle on the elliptic curve* $\Gamma_{h,g}$ *as in Figure 5.*

The above Proposition combined with Theorem 2.1 implies immediately the following

THEOREM 4.2 ([10]). *Denote by* $H(I_1, I_2)$ *the Hamiltonian of the spherical pendulum in action-angle co-ordinates. For every* $(h, g) \in U_r$ *holds*

$$\det \begin{pmatrix} \dfrac{\partial^2 H}{\partial I_1^2} & \dfrac{\partial^2 H}{\partial I_1 \partial I_2} \\ \dfrac{\partial^2 H}{\partial I_1 \partial I_2} & \dfrac{\partial^2 H}{\partial I_2^2} \end{pmatrix} \neq 0. \tag{30}$$

Proof. Put

$$\psi = \oint_{\gamma(h,g)} \frac{y}{x^2} \mathrm{d}x.$$

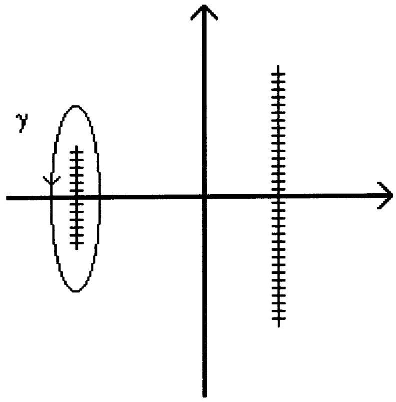

Figure 5. Projection of the cycle γ on the x-plane.

By a straightforward computation, we have

$$\frac{1}{2\pi}\left(\frac{\partial I_1}{\partial H}\right)^4 \det\begin{pmatrix} \dfrac{\partial^2 H}{\partial I_1^2} & \dfrac{\partial^2 H}{\partial I_1 \partial I_2} \\ \dfrac{\partial^2 H}{\partial I_1 \partial I_2} & \dfrac{\partial^2 H}{\partial I_2^2} \end{pmatrix} = \det\begin{pmatrix} \dfrac{\partial^2 \psi}{\partial H^2} & \dfrac{\partial^2 \psi}{\partial H \partial G} \\ \dfrac{\partial^2 \psi}{\partial H \partial G} & \dfrac{\partial^2 \psi}{\partial G^2} \end{pmatrix}$$

$$= \det\begin{pmatrix} \displaystyle\int_\gamma \frac{x^2\mathrm{d}x}{y^3} & \dfrac{x\mathrm{d}x}{y^3} \\ \dfrac{x\mathrm{d}x}{y^3} & \dfrac{\mathrm{d}x}{y^3} \end{pmatrix}$$

which does not vanish, according to (5). □

References

1. Arnold, V. I.: *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer, New York, 1988.
2. Arnold, V. I.: *Mathematical Methods of Classical Mechanics*, Springer, New York, 1978.
3. Arnold, V. I., Gusein-Zade, S. M. and Varchenko, A. N.: *Singularities of Differentiable Maps, I and II*, Birkhäuser, Basel, 1988.
4. Arnold, V. I. and Il'yashenko, Yu. S.: Ordinary differential equations, In: *Dynamical Systems I*, Encyclopaedia of Math. Sci. 1, Springer, Berlin, 1988.
5. Audin, M.: *Spinning Tops*, Cambridge Stud. Adv. Math. 51, Cambridge Univ. Press, 1996.
6. Cushman, R. and Bates, L.: *Global Aspects of Classical Integrable Systems*, Birkhäuser, Basel, 1997.

7. Erdély, A. (ed.): *Bateman Manuscript Project, Higher Transcendental Functions*, vol. II, McGraw-Hill, Englewood Cliffs, 1953.
8. Duistermaat, J. J.: On global action-angles coordinates, *Comm. Pure Appl. Math.* **32**, (1980), 687–706.
9. Gavrilov, L.: Petrov modules and zeros of Abelian integrals, *Bull. Sci. Math.* **122** (1998) 571–584.
10. Horozov, E.: Perturbations of the spherical pendulum and Abelian integrals, *J. Reine Angew. Math.* **408** (1990) 114–135
11. Il'yashenko, Yu. and Yakovenko, S. (eds.): *Concerning Hilbert's 16th Problem*, Adv. in Math. Sci. 27, Amer. Math. Soc., Providence, 1995.
12. Petrov, G. S.: Number of zeros of complete Abelian integrals, *Funct. Anal. Appl.* **18** (1984), 73–74.
13. Vivolo, O.: The monodromy of action variables of Lagrange top, Preprint 62, Laboratoire Emile Picard, University of Toulouse III, 1995, to appear in J. Geom. Physics.
14. Whittaker, E. T.: *A Treatise on the Analytical Dynamics of Particles and Bodies*, Cambridge Univ. Press, 1904.
15. Weil, A.: Remarques sur un mémoire d'Hermite, *Collected Papers*, **2**, pp. 111–116.