NONOSCILLATION OF ELLIPTIC INTEGRALS RELATED TO CUBIC POLYNOMIALS WITH SYMMETRY OF ORDER THREE

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Abstract

We study zeros of elliptic integrals $I(h) = \iint_{H \leqslant h} R(x, y)\, dx\, dy$, where $H(x, y)$ is a real cubic polynomial with a symmetry of order three, and $R(x, y)$ is a real polynomial of degree at most n. It turns out that the vector space $\mathscr{A}_n$ formed by such integrals is a Chebishev system: the number of zeros of each elliptic integral $I(h) \in \mathscr{A}_n$ is less than the dimension of the vector space $\mathscr{A}_n$.

1. *Statement of the result*

Let $H(x, y)$ be a real polynomial of degree d with an elliptic Morse critical point at the origin, $H = (x^2 + y^2)/2 + \ldots$, and let $\gamma(h) \subset \{H = h\}$ be a continuous family of compact ovals defined for $h \in \Delta =]0, a[$. If we denote by $\{H \leqslant h\}$ the interior of $\gamma(h)$, then the Hilbert–Arnold problem (see [**1**, p. 313] and [**2**]) asks for the maximal number of the zeros of the Abelian integral

$$I(h) = \iint_{H \leqslant h} R(x, y)\, dx\, dy \tag{1.1}$$

on the interval Δ, where $R(x, y)$ is a real polynomial of degree n.

This problem is far from being solved. In a series of papers [**10**], Petrov found the maximal number of zeros of $I(h)$ in the case $H = y^2 + P(x)$, where $P(x)$ is a fixed polynomial of degree at most four and having only real critical values. If H is a generic cubic polynomial with one elliptic and three hyperbolic critical points, and $\deg R \leqslant 1$, then the maximal number of zeros of $I(h)$ was found in [**6**, **8**]. It turns out that in most of the studied cases, the vector space $\mathscr{A}_n$ of Abelian integrals $I(h)$ with $\deg R \leqslant n$ forms a Chebishev system (the number of zeros of functions in $\mathscr{A}_n$ is less than the dimension of $\mathscr{A}_n$).

In the present paper we prove the above nonoscillation property in the case when $H(x, y)$ is a cubic polynomial invariant under a rotation of $\mathbb{R}^2$ through an angle $2\pi/3$. This implies that for some constant $c \neq 0$, and some linear non-homogeneous functions $l_i = l_i(x, y)$, $i = 1, 2, 3$, $H = l_1 l_2 l_3 + c$ holds. After a linear change of variables, $H(x, y)$ can always be put into the normal form

$$H = \tfrac{1}{2}(x^2 + y^2) - \tfrac{1}{3}x^3 + xy^2 = \tfrac{1}{2}|z|^2 - \tfrac{1}{6}(z^3 + \bar{z}^3), \tag{1.2}$$

where $z = x + \sqrt{-1}y$. The polynomial H has one elliptic critical point at the origin, three hyperbolic Morse critical points at $(-1/2, \pm\sqrt{3}/2)$ and $(1, 0)$, and the corresponding critical values of H are $h = 0$ and $h = 1/6$ (see Fig. 1). The maximal open interval on which the compact oval $\gamma(h)$ exists is $\Delta = (0, 1/6)$.

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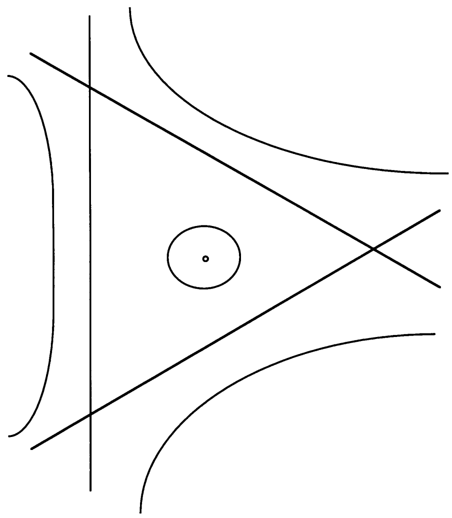

FIG. 1. Level sets $\{H = h\}$ for $h \in \mathbb{R}$

THEOREM 1.1. *Let $\mathscr{A}_n$ be the vector space formed by Abelian integrals $I(h)$ as in* (1.1), *where* $\deg R(x, y) \leqslant n$ *and $H(x, y)$ is defined by* (1.2). *Every function $I(h) \in \mathscr{A}_n$ which is not identically zero has at most* $\dim \mathscr{A}_n - 1$ *zeros on the interval* Δ. *The dimension of $\mathscr{A}_n$ is equal to* $[\frac{2}{3}n] + 1$.

2. *Proof*

We shall need first three propositions, which will be proved later in this section. To any polynomial $H \in \mathbb{R}[x, y]$ we associate, following [5], the $\mathbb{R}[h]$ module $\mathscr{P}_H$ of real polynomial one-forms with equivalence relation

$$\omega_1 \sim \omega_2 \Leftrightarrow \exists\, A, B \in \mathbb{R}[x, y] \quad \text{such that} \quad \omega_1 - \omega_2 = dA + B\, dH$$

and multiplication

$$R(h) \cdot \omega = R(H)\omega, \quad \text{for all } R(h) \in \mathbb{R}[h].$$

PROPOSITION 2.1. *The module $\mathscr{P}_H$ is free and generated by the one-forms*

$$\omega_1 = y\, dx, \quad \omega_2 = y^3 dx, \quad \omega_3 = y^2 dx, \quad \omega_4 = xy\, dx.$$

This means that if $\omega = P_{n+1}dx + Q_{n+1}dy$ is a real polynomial one-form of degree $n+1$, then there exist unique polynomials $p_i(h)$ such that in $\mathscr{P}_H$,

$$\omega \sim \sum_{i=1}^{4} p_i(h)\omega_i$$

holds, and moreover

$$3\deg(p_1) \leqslant n, \quad 3\deg(p_2) + 2 \leqslant n, \quad 3\deg(p_3) + 1 \leqslant n, \quad 3\deg(p_4) + 1 \leqslant n.$$

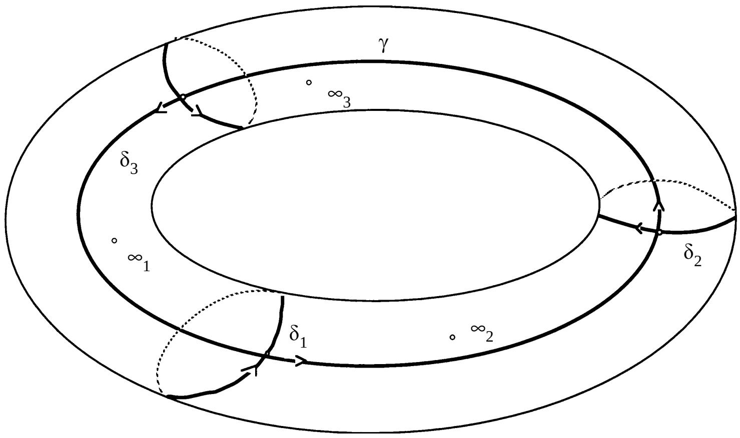

FIG. 2. The vanishing cycles $\gamma(h)$, $\delta_i(h)$ on Γ_h

The Abelian integral

$$I(h) = \iint_{H \leqslant h} R_n(x, y)\, dx\, dy = \int_{\gamma(h)} P_{n+1}(x, y)\, dx + Q_{n+1}(x, y)\, dy$$

is an analytic function in h for $|h|$ sufficiently small and, as we shall see, its analytic continuation in the complex domain $\mathscr{D} = \mathbb{C} \setminus [1/6, \infty[$ is a holomorphic function. We shall prove the nonoscillation property of $I(h)$ in this larger domain.

Let

$$\Gamma_h = \{(x, y) \in \mathbb{C}^2 : H(x, y) = h\}$$

be the complexification of the compact oval $\gamma(h)$. The oval $\gamma(h) \subset \Gamma_h$ represents an integer homology class in $H_1(\Gamma_h, \mathbb{Z})$ for $h \in \Delta$ with an orientation fixed by

$$\iint_{H \leqslant h} dx\, dy > 0.$$

As Γ_h is a genus one Riemann surface with three points removed, we have

$$\dim H_1(\Gamma_h, \mathbb{Z}) = 4,$$

and we may define continuous families of cycles $\delta_i(h) \subset H_1(\Gamma_h, \mathbb{Z})$, $h \in \Delta$, such that $\gamma(h), \delta_1(h), \delta_2(h), \delta_3(h)$ form a basis of $H_1(\Gamma_h, \mathbb{Z})$. The family $\gamma(h)$ is uniquely defined by the condition that it vanishes at the origin as $h \to 0$, $h \in \Delta$. Similarly, the continuous families of cycles $\delta_i(h)$ are uniquely defined by the condition that they vanish at the critical points $(-1/2, \pm\sqrt{3}/2)$ and $(1, 0)$ as $h \to 1/6$, $h \in \Delta$, and $(\gamma(h) \circ \delta_i(h)) = 1$ (see Fig. 2). The families $\gamma(h)$, $\delta_i(h)$ are, in fact, well defined for $h \in \mathbb{C} \setminus \{0, 1/6\}$ as locally constant sections of the homology bundle of the affine curve Γ_h. These sections, however, are not globally constant, and their monodromy is given by the Picard–Lefschetz formula. Namely, let $l \in \pi_1(\mathbb{C} \setminus \{0, 1/6\}, h_0)$, $h_0 \in \Delta$, be a loop which makes one turn around the critical value $h = 0$ anticlockwise, and is contained in the half plane $\mathrm{Re}(h) \leqslant h_0$. The corresponding monodromy transformation of $H_1(\Gamma_{h_0}, \mathbb{Z})$ is given by

$$\alpha(h_0) \to \alpha(h_0) - (\alpha(h_0) \circ \gamma(h_0))\gamma(h_0), \quad \text{for all } \alpha(h_0) \in H_1(\Gamma_{h_0}, \mathbb{Z}). \tag{2.1}$$

Similarly, let $l \in \pi_1(\mathbb{C} \setminus \{0, 1/6\}, h_0)$ be a loop which makes one turn around the critical value $h = 1/6$ anticlockwise, and is contained in the half plane $\mathrm{Re}(h) \geqslant h_0$. The corresponding monodromy transformation of $H_1(\Gamma_{h_0}, \mathbb{Z})$ is given by

$$\alpha(h_0) \to \alpha(h_0) - \sum_{i=1}^{3} (\alpha \circ \delta_i(h_0))\delta_i(h_0), \quad \text{for all } \alpha(h_0) \in H_1(\Gamma_{h_0}, \mathbb{Z}). \tag{2.2}$$

It follows that the Abelian integral (1.1) is a holomorphic function in the complex domain $\mathscr{D}$.

Proposition 2.2.

$$\int_{\gamma(h)} xy\, dx = \int_{\gamma(h)} y^2 dx \equiv 0.$$

Proposition 2.3. *The function* $\int_{\gamma(h)} y\, dx$ *has one simple zero in* $\mathscr{D}$, *at* $h = 0$. *If* $P_p(h), Q_q(h)$ *are degree* p *and* q *real polynomials, then the function*

$$F(h) = P_p(h) + Q_q(h) \frac{\int_{\gamma(h)} y^3 dx}{\int_{\gamma(h)} y\, dx}, \quad h \in \mathscr{D},$$

is holomorphic and has at most $q + \max\{p, q\} + 1$ *zeros in* $\mathscr{D}$.

Proof of Theorem 1.1, *assuming the above propositions.* By Propositions 2.1 and 2.2, we deduce that the Abelian integral $I(h)$ belongs to the vector space

$$\mathscr{A}_n = \left\{ I(h) = \int_{\gamma(h)} P_{n+1}(x, y)\, dx + Q_{n+1}(x, y)\, dy : \deg(P_{n+1}), \deg(Q_{n+1}) \leqslant n + 1 \right\}$$

if and only if it can be written in the form

$$I(h) = P(h) \int_{\gamma(h)} y\, dx + Q(h) \int_{\gamma(h)} y^3 dx, \tag{2.3}$$

where P and Q are suitable real polynomials satisfying

$$\deg(P) \leqslant \frac{n}{3}, \quad \deg(Q) \leqslant \frac{n-2}{3}. \tag{2.4}$$

Moreover, $I(h) \equiv 0$ if and only if $P(h) \equiv 0$ and $Q(h) \equiv 0$. It follows that the dimension of $\mathscr{A}_n$ equals $[\frac{n}{3}] + 1 + [\frac{n-2}{3}] + 1$. Clearly, any function $I(h) \in \mathscr{A}_n$ has a zero at $t = 0$, and the function $\int_{\gamma(h)} y\, dx$ has a zero at $h = 0$ of order one, hence $F(h) = I(h) / \int_{\gamma(h)} y\, dx$ is holomorphic and has at most $[\frac{n}{3}] + [\frac{n+1}{3}] = [\frac{2}{3}n]$ zeros in $\mathscr{D}$ (Proposition 2.3). As $\Delta \subset \mathscr{D}$ and $\int_{\gamma(h)} y\, dx \neq 0$ in Δ, this completes the proof of Theorem 1.1.

We now prove Propositions 2.1–2.3. Proposition 2.1 is a particular case of a general result proved in [**5**]. Here we sketch a direct proof. Consider the function

$$W(h) = \det\left(\int_{\delta_i} \omega_j \right), \tag{2.5}$$

where $\delta_4 = \gamma$. Standard arguments (the Picard–Lefschetz formula and growth at infinity) show that $W(h)$ is a polynomial [**3**]. This polynomial vanishes at $h = 0$, has at least a triple zero at $h = 1/6$, and grows at infinity not faster than ch^4, where c is a constant [**4**]. It follows that $W(h) = ch(h - 1/6)^3$, so the one-forms ω_i are always independent in $H^1_{DR}(\Gamma_h, \mathbb{C})$, $h \neq 0, 1/6$. As $\dim H^1_{DR}(\Gamma_h, \mathbb{C}) = \dim H_1(\Gamma_h, \mathbb{R}) = 4$, it follows that the ω_i, $i = 1, \ldots, 4$, generate a basis of $H^1_{DR}(\Gamma_h)$ for every $h \neq 0, 1/6$. Thus on each level set Γ_h, every one-form ω is, up to addition of an exact form, a linear combination of the ω_i, $i = 1, \ldots, 4$, and hence

$$\int_{\delta_i(h)} \omega = \sum_{j=1}^{4} p_j(h) \int_{\delta_i(h)} \omega_j, \quad i = 1, 2, 3, 4.$$

Using the Kramer formulae and with the same arguments as for $W(h)$, we deduce that the $p_i(h)$ are polynomials. Their degrees can be obtained from the asymptotic behaviour of the Abelian integrals $\int_{\delta_i(h)} \omega_j$ at 'infinity'. Finally, as the restriction of the one-form

$$\omega_0 = \omega - \sum_{i=1}^{4} p_i(h)\omega_i$$

on Γ_h represents the zero cohomology class in $H^1_{DR}(\Gamma_h; \mathbb{C})$ for every fixed h, we may use [**9**, Theorem 1] to conclude that ω_0 is also zero in $\mathscr{P}_H$.

Proof of Proposition 2.2. As H is invariant under rotations $z \to ze^{2\pi i/3}$,

$$\iint_{H \leqslant h} z\, dz \wedge \bar{z} = e^{2\pi i/3} \iint_{H \leqslant h} z\, dz \wedge \bar{z} = 0,$$

which implies

$$\int_{\gamma(h)} xy\, dx = \int_{\gamma(h)} y^2 dx \equiv 0.$$

Proof of Proposition 2.3. To compute the number of zeros of $S(h) = \int_{\gamma(h)} y\, dx$ in $\mathscr{D}$, we shall evaluate the increment of the argument of $S(h)$ along the boundary of $\mathscr{D}$ described in a positive direction (anticlockwise). For $h \in]1/6, \infty[$, we define $S^+(h)$ and $S^-(h)$ to be the analytic continuations of $S(h)$ along paths on which $\operatorname{Im}(h) > 0$ and $\operatorname{Im}(h) < 0$, respectively. Let R be a large enough constant, and let r be a small enough constant. Denote by $\mathscr{D}'$ the set obtained from $D \cap \{|h| < R\}$ by removing the circle of radius r centred at $h = 1/6$ (see Fig. 3).

Consider the increase of the argument of $S(h)$ along the boundary of $\mathscr{D}'$. For $h \in \mathscr{D}$, we have $|x|, |y| \overset{\infty}{\sim} |h|^{1/3}$, so the argument of $S(h)$ increases by no more than $4\pi/3$ along the circle $|h| = R$. Similarly, for $h \in \mathscr{D}$ in a neighbourhood of $1/6$, we have $S(h) \overset{1/6}{\sim} S(1/6) \neq 0$. It follows that along the circle $|h - 1/6| = r$, the increment of the argument of $S(h)$ is close to zero. Along the interval $[r, R]$, we use the Picard–Lefschetz formula (2.2) to obtain

$$S^+(h) - S^-(h) = \pm 2\sqrt{-1}\, \operatorname{Im} S^\pm(h) = \pm \sum_{i=1}^{3} \int_{\delta_i(h)} y\, dx.$$

The real function $J(h) = \sqrt{-1} \sum_i \int_{\delta_i(h)} y\, dx$ has no zeros on $]1/6, \infty[$. Indeed, as $\delta_i(h)$ vanishes as $h \to 1/6$, we have $J(1/6) = 0$. On the other hand,

$$J'(h) = \frac{d}{dh}\sqrt{-1} \sum_{i=1}^{3} \int_{\delta_i(h)} y\, dx = \sqrt{-1} \sum_{i=1}^{3} \int_{\delta_i(h)} \frac{dx}{H_y} = 3\sqrt{-1} \int_{\delta_1(h)} \frac{dx}{H_y}.$$

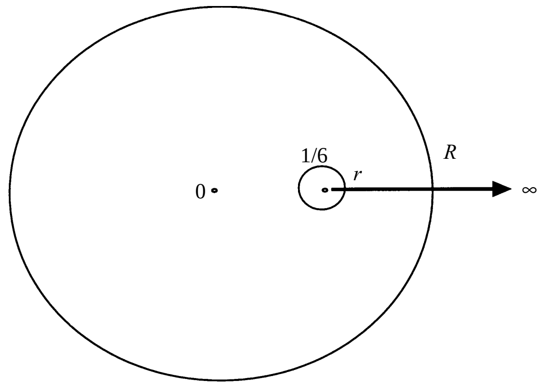

FIG. 3. The region $\mathscr{D}'$

The last identity holds because the cycles $\delta_i(h)$ are homologous on the compactified curve $\bar{\Gamma}_h$ (see Fig. 2), and dx/H_y is a holomorphic differential on $\bar{\Gamma}_h$. We have $(\gamma(h) \circ \delta_1(h)) = 1$, and a well-known theorem of Jacobi [7] says that the compact elliptic curve $\bar{\Gamma}_h$ is isomorphic to $\mathbb{C}/\Lambda$, where Λ is the lattice

$$\Lambda = \left\{ \mathbb{Z} \int_{\delta_1(h)} \frac{dx}{H_y} + \mathbb{Z} \int_{\gamma(h)} \frac{dx}{H_y} \right\}.$$

In particular, $\operatorname{rank}(\Lambda) = 2$ and hence $\int_{\delta_1(h)} dx/H_y \neq 0$. We conclude that $J'(h) \neq 0$ on $]1/6, \infty[$, and hence $J(h) = 2\sqrt{-1}\,\operatorname{Im} S^{\pm}(h) \neq 0$ on $]1/6, \infty[$. Putting the above data together yields that the increment of the argument of $S(h)$ along the boundary of $\mathscr{D}'$ is less than $2\pi + 4\pi/3$. Using the argument principle, we obtain that $S(h)$ has at most one zero in $\mathscr{D}'$ (and hence in $\mathscr{D}$). In fact, $S(h)$ has exactly one zero, as $S(0) = 0$.

To compute the zeros of $F(h)$ in $\mathscr{D}$, we proceed in the same way. As $S(h)$ has a zero of order one at $h = 0$, $F(h) = I(h)/S(h)$ is holomorphic in $\mathscr{D}$. For $h \in]1/6, \infty[$, we define $F^+(h)$ and $F^-(h)$ as above. Consider the increase of the argument of $F(h)$ along the boundary of $\mathscr{D}'$. Along the circle $|h| = R$, we have

$$|F(h)| \sim |h|^{\max\{p, q+2/3\}},$$

so the argument of $F(h)$ increases by no more than $2\pi \max\{p, q + 2/3\}$. Using the Picard–Lefschetz formula (2.2), we obtain the asymptotic estimate (for $h \in \mathscr{D}$)

$$|F(h)| \overset{1/6}{\sim} ch^k$$

or

$$|F(h)| \overset{1/6}{\sim} ch^k \log(h),$$

where c is a non-zero constant and k is a non-negative integer. In both cases the increase of the argument of $F(h)$ along the circle $|h - 1/6| = r$ in a clockwise direction is either close to zero or negative. Finally, along the interval $[r, R]$ we use the Picard–Lefschetz formula (2.2) to obtain

$$F^+(h) - F^-(h) = \pm 2\sqrt{-1}\,\operatorname{Im} F^{\pm}(h) = Q_q(h) \frac{\tilde{W}(h)}{|\int_{\gamma(h)} y\, dx|^2},$$

where

$$\tilde{W}(h) = \det \begin{pmatrix} \int_{\gamma(h)} y\,dx & \int_{\delta(h)} y\,dx \\ \int_{\gamma(h)} y^3 dx & \int_{\delta(h)} y^3 dx \end{pmatrix}, \quad \delta(h) = \delta_1(h) + \delta_2(h) + \delta_3(h).$$

The Picard–Lefschetz formulae (2.2) and (2.1) show that the function $\tilde{W}(h)$ is single-valued and hence holomorphic on $\mathbb{C}$. As $|\int_{\delta_i(h)} y^3 dx|$ and $|\int_{\delta_i(h)} y\,dx|$ grow at infinity no faster than $|h|^{4/3}$ and $|h|^{2/3}$, $|\tilde{W}(h)|$ grows no faster than $|h|^2$. It follows that $\tilde{W}(h)$ is a polynomial, and as $\tilde{W}(0) = \tilde{W}(1/6) = 0$, we conclude that $\tilde{W}(h) = ch(h - 1/6)$ for some constant c. In fact, the constant c is not equal to zero as the determinant $W(h)$ in (2.5) is not identically zero. It follows that the imaginary part of $F(h)$ has at most q zeros on the interval $]1/6, \infty[$. Putting the above data together yields that the increment of the argument of $F(h)$ along the boundary of $\mathscr{D}'$ is less than $2\pi(\max\{p, q + 2/3\} + q + 1)$. This implies that $F(h)$ has at most $q + \max\{p, q\} + 1$ zeros in $\mathscr{D}$, which completes the proof of Proposition 2.3.

Note added in proof. In the recent preprint by E. Horozov and I. Iliev, 'Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians' (Sofia University, July 1997), the authors study zeros of Abelian integrals related to arbitrary cubic polynomials. They find an upper bound $Z(n) \leqslant 5n + 20$ for the number of the zeros of an Abelian integral $I(h) \in \mathscr{A}_n$ of degree n on a maximal open interval on which the compact oval exists. They also show that for generic cubic polynomials H, the Chebishev property does not hold true.

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