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Generalized Jacobians of spectral curves and completely integrable systems

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Abstract. Consider an ordinary differential equation which has a Lax pair representation $\dot{A}(x) = [A(x), B(x)]$, where $A(x)$ is a matrix polynomial with a fixed regular leading coefficient and the matrix $B(x)$ depends only on $A(x)$. Such an equation can be considered as a completely integrable complex Hamiltonian system. We show that the generic complex invariant manifold

$$\{A(x) : \det(A(x) - yI) = P(x, y)\}$$

of this Lax pair is an affine part of a non-compact commutative algebraic group – the generalized Jacobian of the spectral curve $\{(x, y) : P(x, y) = 0\}$ with its points at "infinity" identified. Moreover, for suitable $B(x)$, the Hamiltonian vector field defined by the Lax pair on the generalized Jacobian is translation-invariant.

1 Introduction

Let M^J be the affine vector space of all complex matrix polynomials $A(x)$ in a variable x, of fixed degree d and dimension r

$$A(x) = Jx^d + A_{d-1}x^{d-1} + \ldots + A_0\,, \quad A_i \in \mathbf{gl}_r(\mathbb{C})$$

where $J \in \mathbf{gl}_r(\mathbb{C})$ is a fixed matrix. The matricial polynomial Lax equations

$$\frac{d}{dt}A(x) = \left[\frac{A^k(a)}{x-a}, A(x)\right], k \in \mathbb{N}, a \in \mathbb{C} \tag{1}$$

are well known to be Hamiltonian (with respect to several compatible Poisson structures on M^J) and completely integrable. The corresponding Hamiltonian vector fields define a complete set of commuting vector fields on the isospectral manifolds

$$M_P^J = \{A(x) \in M^J : \det(A(x) - yI_r) = P(x,y)\}\,.$$

The system (1) has an obvious symmetry group $G = \mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$ which is the subgroup of the projective group $\mathbb{P}\mathbf{GL}_r(\mathbb{C})$ formed by matrices which commute with J. The group G acts on M^J by conjugation, the action is Poisson, and the reduced Hamiltonian system is completely integrable too. As the symmetry group G acts freely and properly on the general isospectral manifold M_P^J, then M_P^J can be considered as the total space of a holomorphic principal fibre bundle ξ with base M_P^J/G, structural group G, and natural projection map

$$M_P^J \xrightarrow{\phi} M_P^J/G\,.$$

The purpose of the present article is to describe the algebraic structure of the above fibre bundle. Our main result, Theorem 2.1, implies that when the spectral curve X defined by $\{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\}$ is smooth, then M_P^J is smooth and bi-holomorphic to a Zariski open subset of the generalized Jacobian variety $J(X')$. The curve X' is singular and as a topological space it is just X with its "infinite" points $\infty_1, \infty_2, \ldots, \infty_r$ identified to a single point ∞. Thus $J(X')$ is a non-compact commutative algebraic group and it can be described as an extension of the usual Jacobian $J(X)$ by the algebraic group $G = (\mathbb{C}^*)^{s-1} \times \mathbb{C}^{r-s}$, where $s \leq r$ is the number of distinct eigenvalues of the leading term J

$$0 \to G \to J(X') \xrightarrow{\phi} J(X) \to 0\,. \tag{2}$$

As analytic spaces $J(X')$ and $J(X)$ are complex tori

$$J(X') = \mathbb{C}^{p_a}/\Lambda'\,, \quad J(X) = \mathbb{C}^{p_g}/\Lambda$$

where Λ', Λ are lattices of rank $2p_g + s - 1$ and $2p_g$ respectively, p_g is the genus of X, and $p_a = p_g + r - 1$ is the arithmetic genus of X'. The generalized Jacobian $J(X')$ can be also considered as the total space of a holomorphic principal fibre bundle with base $J(X)$, projection ϕ, and structural group G. The group G is then identified with the symmetry group $\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$ of (1), and the manifold M_P^J/G with a Zariski open subset of the usual Jacobian $J(X) = J(X')/G$. The algebraic description of the reduced invariant manifold M_P^J/G is a well known result proved by A.Beauville [4] and M.R.Adams, J.Harnad, J.Hurtubise [1] (see also M. Adler, P. van Moerbeke [2], and Sect. 8.2. of the survey [17] by A.G.Reyman and M.A. Semenov-Tian-Shansky).

The Hamiltonian structures of the differential equation (1) is briefly recalled in Sect. 3 where we show that the Hamiltonian vector fields (1) for $k \in \mathbb{N}, a \in \mathbb{C}$ define translation invariant vector fields on the generalized Jacobian $J(X')$, so the system is algebraically completely integrable.

In the case when the spectral curve X is singular, the variety $M_P^J/\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$ was studied by M.R.Adams, J.Harnad, J.Hurtubise [1] (see also Beauville [4, p.218], P. van Moerbeke and D. Mumford [13, p.112]). Note that our approach is quite the opposite in the sense that, while in [1] the singular spectral curve X is desingularized, in the present article the regular spectral curve X is singularized to a curve X'.

We conclude the paper with two applications of Theorem 2.1 (Sect. 4). We prove that even in the simplest case when X is elliptic and $G = \mathbb{C}^*$, the extension (2) is not trivial, and then describe the corresponding two degrees of freedom algebraically completely integrable system. It turns out to be the well known symmetric (Lagrange) top, and $\mathbb{C}^*$ is just the complexified group of rotations about the symmetry axis of the top. This result, proved *ad hoc* by Gavrilov and Zhivkov [9], motivated the present paper. Another classical problem related to Theorem 2.1 is to solve a system of hyperelliptic differential equations (Jacobi [10], 1846). We prove that the phase space of such a system is the generalized Jacobian $J(X')$ of a hyperelliptic curve X with two points at "infinity" identified, each orbit is a straight line isomorphic to $\mathbb{C}^*$, and the space of orbits is parameterized by the usual Jacobian $J(X)$. This gives a new proof of Jacobi's theorem.

2 Spectral curves and their Jacobians

A polynomial

$$P(x, y) = y^r + s_1(x)y^{r-1} + \ldots + s_r(x)$$

is called spectral, provided that the affine curve $\{(x, y) \in \mathbb{C}^2 : P(x, y) = 0\}$ is the spectrum of some polynomial $r \times r$ matrix $A(x)$

$$P(x, y) = \det(A(x) - y.I_r)\,.$$

In this case $\deg(s_i(x)) \leq i.d$, where d is the degree of $A(x)$

$$A(x) = A_d x^d + A_{d-1}x^{d-1} + \ldots + A_0\,, \quad A_i \in \mathbf{gl}_r(\mathbb{C})\,. \tag{3}$$

Consider the weighted projective space $\mathbb{P}^2(d) = \mathbb{C}^3\backslash\{0\}/\mathbb{C}^*$, where the $\mathbb{C}^*$-action on $\mathbb{C}^3$ is defined by

$$t \cdot (x, y, z) \to (tx, t^d y, tz)\,, \quad t \in \mathbb{C}^*\,.$$

$\mathbb{P}^2(d)$ is a compact complex surface with one singular point $\{[0,1,0]\} = \mathbb{P}^2(d)_{sing}$. The affine curve $\{(x,y) \in \mathbb{C}^2 : \det(A(x) - y.I_r) = 0\}$ is naturally embedded in $\mathbb{P}^2(d)$,

$$\mathbb{C}^2 \to \mathbb{P}^2(d) : (x,y) \mapsto [x,y,1],$$

and the condition $\deg(s_i(x)) \leq i.d$ shows that its closure X is contained in the smooth surface $\mathbb{P}^2(d)_{reg} = \mathbb{P}^2(d)\backslash\{[0,1,0]\}$. Let x be an affine coordinate on $\mathbb{P}^1$. The surface $\mathbb{P}^2(d)_{reg}$ is identified with the total space of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ with base $\mathbb{P}^1$ and projection

$$\pi : \mathbb{P}^2(d)_{reg} \to \mathbb{P}^1 : [x,y,z] \to [x,z].$$

The induced projection

$$\pi : X \to \mathbb{P}^1 \tag{4}$$

is a ramified covering of degree r, and over the affine plane $\mathbb{C}$ it is simply the first projection

$$\pi : \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\} \to \mathbb{C} : (x,y) \to x\,.$$

Definition 1 *(spectral curve of $A(x)$) We define the spectral curve X of the matrix polynomial $A(x)$ (3), to be the closure of the affine curve*

$$\{(x,y) \in \mathbb{C}^2 : \det(A(x) - y.I_r) = 0\}$$

in the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$.

From now on we fix the spectral polynomial $P(x,y)$ and suppose that the spectral curve X is *smooth* and irreducible.

We are going now to singularize the curve X. Let

$$m = \sum_{i=1}^{s} n_i P_i, P_i \in X, n_i > 0$$

be an effective divisor on X. To the pair (X,m) we associate a singular curve $X' = X_{reg} \cup \infty$, where if $S = \cup_{i=1}^{s} P_i$ is the support of m, then $X_{reg} = X - S$, and ∞ is a single point. The structure sheaf $\mathcal{O}'$ of $X' \sim (X,m)$ is defined in the following way. Let $\mathcal{O}_{X'}$ be the direct image of the structure sheaf $\mathcal{O} = \mathcal{O}_X$ under the canonical projection $X \to X'$. Then

$$\mathcal{O}'_P = \begin{cases} \mathcal{O}_P, & P \in X_{reg} \\ \mathbb{C} + i_\infty, & P = \infty \end{cases}$$

where i_∞ is the ideal of $\mathcal{O}_\infty$ formed by the functions f having a zero at P_i of order at least n_i. Thus a regular function f on X' is a regular function f on X, and such that for some $c \in \mathbb{C}$ and any i holds $v_{P_i}(f-c) \geq n_i$, where

$v_P(.)$ is the order function. If p_g is the genus of X then the arithmetic genus p_a of the singular curve X' is $p_a = p_g + \deg(m) - 1$.

Example Let $m = P^+ + P^-$ be a divisor on the Riemann surface X. Then in a neighborhood of ∞ the singularized curve X' is analytically isomorphic either to the germ of analytical curve $xy = 0$ ($P^+ \neq P^-$), or to $y^2 = x^3$ ($P^+ = P^-$).

Definition 2 *(singularized spectral curve of $A(x)$) If π is the projection (4) and $\infty = [1, 0] \in \mathbb{P}^1$ the "infinite" divisor, then the effective divisor $m = \pi^*(\infty)$ is called modulus of the spectral curve X. We have $\deg(m) = r$ and we denote by X' the singular curve associated to the regular curve X and to the modulus m.*

Remark In the Serre's book [19] any affective divisor m on a regular algebraic curve X is called modulus. Indeed, the moduli space of singularized curves $X' \sim (X, m)$, $p_g(X) = const.$, $\deg(m) = const.$, is of dimension strictly bigger than the dimension of the moduli space of regular curves X.

We shall recall now the construction of the generalized Jacobian variety $J(X')$ of a singular curve $X' \sim (X, m)$. For proofs we refer the reader to Serre [19].

A holomorphic line bundle L' on X' is described by an open covering $\{U_\alpha\}_\alpha$ of X' and transition functions $g_{\alpha\beta} \in \mathcal{O}'^*(U_\alpha \cap U_\beta)$, such that

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1,\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1\ .$$

Two line bundles L'_1, L'_2 on X' are equivalent if and only if there exist $f_\alpha \in \mathcal{O}'^*(U_\alpha)$ such that $g^1_{\alpha\beta} = (f_\alpha/f_\beta) \cdot g^2_{\alpha\beta}$. Thus the Picard group $\mathrm{Pic}(X')$ of equivalence classes of holomorphic line bundles on the singular curve X' is just $H^1(X', \mathcal{O}'^*)$. If D is a divisor on X_{reg} with local equations $\{f_\alpha\}$, then the functions $g_{\alpha\beta} = f_\alpha/f_\beta \in \mathcal{O}'^*(U_\alpha \cap U_\beta)$ define a line bundle L'_D on X', and any holomorphic line bundle on X' can be written in such a way. Two line bundles L'_{D_1}, L'_{D_2} on X' a equivalent if and only if $D_1 \overset{m}{\sim} D_2$. This means that there exists a global meromorphic function f on X, such that $(f) = D_1 - D_2$ and $v_{P_i}(f-1) \geq n_i$, $i = 1, 2 \ldots s$. Let $\mathcal{L}(D)$ be the sheaf of sections of the holomorphic line bundle L_D over the smooth curve X and suppose as before that the support of D is contained in the set of regular points X_{reg}. Then the sheaf of sections $\mathcal{L}'(D)$ of the line bundle L'_D over X' is defined as

$$\mathcal{L}'(D)_P = \begin{cases} \mathcal{O}'_\infty, & P = \infty \\ \mathcal{L}(D)_P, & P \neq \infty. \end{cases}$$

The sheaf $\mathcal{L}'(D)$is a locally free $\mathcal{O}'$ module of rank one, that is to say it is invertible. More generally, there is an one-to-one correspondence between isomorphism classes of invertible sheaves on X', and isomorphism classes of line bundles over X'. This set of isomorphism classes is a group under the operation $\otimes$, $\mathcal{L}'(D_1) \otimes \mathcal{L}'(D_2) = \mathcal{L}'(D_1 + D_2)$, called Picard group $\mathrm{Pic}(X')$ of the curve X'. Let $\mathrm{Pic}^0(X')$ be the subgroup of $\mathrm{Pic}(X')$ formed by degree zero line bundles. It is called *Jacobian* of X' and we denote $J(X') = \mathrm{Pic}^0(X')$. The Jacobian $J(X')$ of the singular algebraic curve X' has a natural structure of commutative algebraic group. As an analytic manifold we have

$$J(X') = H^0(X, \Omega^1(m))^* / H_1(X_{reg}, \mathbb{Z}) = \mathbb{C}^{p_a} / \Lambda' \;,$$

where Λ' is a rank $2p_g + s - 1$ lattice, and $\Omega^1(m)$ is the sheaf of meromorphic one-forms ω, such that $(\omega) \geq -m$. Similarly, for the usual Jacobian $J(X) = \mathrm{Pic}^0(X) \subset J(X')$, we have

$$J(X) = H^0(X, \Omega^1)^* / H_1(X, \mathbb{Z}) = \mathbb{C}^{p_g} / \Lambda \;,$$

where $\Lambda \subset \Lambda'$ is a rank $2p_g$ lattice. $J(X')$ is a non-trivial *extension* of $J(X)$ by the algebraic group $G = (\mathbb{C}^*)^{s-1} \times \mathbb{C}^{\deg(m)-s}$

$$0 \to G \to J(X') \xrightarrow{\phi} J(X) \to 0 \tag{5}$$

where $\phi(\mathcal{L}'(D)) = \mathcal{L}(D)$. This means that the sequence is exact in the usual sense and moreover the algebraic structure of G (respectively of $J(X)$) is induced (respectively quotient) of the algebraic structure of $J(X')$. Both $J(X)$ and $J(X')$ are commutative algebraic groups. Note however that $J(X')$ is non-compact. Indeed, while the topological space of $J(X)$ is $(S^1)^{2p_g}$, the one of $J(X')$ is $(S^1)^{2p_g+s-1} \times \mathbb{R}^{2\deg(m)-s-1}$. To every extension (5) we associate a holomorphic principal fibre bundle with total space $J(X')$, base $J(X)$, projection ϕ, and structural group $G = (\mathbb{C}^*)^{s-1} \times \mathbb{C}^{\deg(m)-s}$. Two extensions are equivalent if and only if the associated principal bundles are equivalent.

Let $J^{p_a}(X) = \mathrm{Pic}^{p_a}(X)$ be the variety (isomorphic to the Jacobian $J(X)$) formed by line bundles of degree $p_a = p_g + r - 1$ on X, and denote by $J^{p_a}(X') = \mathrm{Pic}^{p_a}(X')$ the variety (isomorphic to the generalized Jacobian $J(X')$) formed by line bundles of degree $p_a = p_g + r - 1$ on the singularized curve X'. Denote further by Θ the canonical Riemann theta divisor of $J^{p_a}(X)$ formed by special line bundles L_D of degree $p_a = p_g + r - 1$, that is to say $\dim H^1(X, \mathcal{L}(D)) \neq 0$. By Riemann-Roch theorem

$$\begin{aligned} \dim H^0(X, \mathcal{L}(D)) &= \deg(D) - p_g + 1 + \dim H^1(X, \mathcal{L}(D)) \\ &= r + \dim H^1(X, \mathcal{L}(D)) \end{aligned}$$

so Θ is the set of line bundles L_D with at least $r + 1$ holomorphic sections. Similarly, let $\Theta' \subset J^{p_a}(X')$ be the canonical divisor formed by degree

p_a special line bundles $L'(D)$, that is to say $\dim H^1(X', \mathcal{L}'(D)) \neq 0$. By Riemann-Roch theorem

$$\begin{aligned}\dim H^0(X', \mathcal{L}'(D)) &= \deg(D) - p_a + 1 + \dim H^1(X', \mathcal{L}'(D)) \\ &= 1 + \dim H^1(X', \mathcal{L}'(D))\end{aligned}$$

so such bundles have at least two holomorphic sections. It is easy to see that $\Theta' = \phi^{-1}(\Theta)$, where ϕ is the map induced by (5).

Let M_P be the variety of $r \times r$ polynomial matrices of degree d (3), which have a fixed spectral polynomial $P(x, y)$

$$M_P = \{A(x) : \det(A(x) - yI_r) = P(x, y)\}.$$

and let $M_P^J = M_P \cap M^J$ be the isospectral manifold formed by matrices of the form (3) with fixed leading term $A_d = J$

$$A(x) = Jx^d + A_{d-1}x^{d-1} + \ldots + A_0,\ A_i \in \mathbf{gl}_r(\mathbb{C})\,. \tag{6}$$

The stabilizer

$$\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J) = \{R \in \mathbb{P}\mathbf{GL}_r(\mathbb{C}) : RJR^{-1} = J\}$$

of $\mathbb{P}\mathbf{GL}_r(\mathbb{C})$ at $J \in \mathbf{gl}_r(\mathbb{C})$ is a commutative algebraic group isomorphic to $(\mathbb{C}^*)^{s-1} \times \mathbb{C}^{\deg(m)-s}$. It is a well known fact that M_P^J is a smooth manifold, $\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$ acts freely and properly on M_P^J by conjugation, and the quotient space $M_P^J/\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$ is a smooth manifold biholomorphic to $J(X) - \Theta$ [1,4]. Consider the holomorphic principal fibre bundle ξ with total space M_P^J, structural group $\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$, base $M_P^J/\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$, and natural projection map $\varphi : M_P^J \to M_P^J/\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$. Consider also the associate principal bundle η with base space $J(X) - \Theta$, total space $J(X') - \Theta'$, projection map ϕ, and structural group G (see (5)).

The main result of the present paper is the following

Theorem 2.1 *The holomorphic principal bundles ξ and η are isomorphic. In particular the isospectral manifold M_P^J is smooth and bi-holomorphic to the Zariski open subset $J(X') - \Theta'$ of the generalized Jacobian $J(X')$ of the singularized spectral curve X'.*

We may resume Theorem 2.1 in the following commutative diagram

$$\begin{tikzcd}
 & & M_P^J \arrow[r, "\varphi"] \arrow[d, "l'"] & M_P^J/\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J) \arrow[d, "l"] & \\
 & & J(X') - \Theta' \arrow[r, "\phi"] \arrow[d] & J(X) - \Theta \arrow[d] & \\
0 \arrow[r] & G \arrow[r] & J(X') \arrow[r] & J(X) \arrow[r] & 0
\end{tikzcd} \tag{7}$$

in which the maps l, l' are biholomorphic,

$$l' : \varphi^{-1}(b) \to \phi^{-1} \circ l(b)$$

is an isomorphism of algebraic groups for every $b \in M_P^J / \mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$, and the exact sequence in (7) is an extension of the algebraic group $J(X)$ by G.

Proof of Theorem 2.1 Recall that a matrix $B \in \mathbf{gl}_r(\mathbb{C})$ is called *regular* if one of the following equivalent conditions is satisfied

- all eigenspaces of B are of dimension one
- the minimal and the characteristic polynomials of B are equal
- the variety $\mathbf{gl}_r(\mathbb{C}; B)$ is of dimension r

We shall use the following

Proposition 2.2 *If the spectral curve X of the matrix polynomial $A(x)$ is smooth, then $A(x)$ is regular for any fixed $x \in \mathbb{P}^1$ (for $x = \infty$ this means that the leading term A_d of $A(x)$ is regular).*

Indeed, if for some $x_0 \in \mathbb{C}$ the matrix $A(x_0) = (a_{ij}(x_0))_{i,j}$ is not regular, then there exists y_0 such that $(x_0, y_0) \in X$ and $\mathrm{rank}(A(x_0) - y_0 I_r) \leq r - 2$. If we denote by $\Delta_{ij}(x, y)$ the (i, j)th minor of the matrix $A(x) - yI_r$, then we have

$$\begin{aligned} \Delta_{ij}(x_0, y_0) &= 0\,, \\ P_x'(x_0, y_0) = \sum_{i,j} (-1)^{i+j} a_{ij}'(x) \Delta_{ij}(x_0, y_0) &= 0\,, \\ P_y'(x_0, y_0) = - \sum_i \Delta_{ii}(x_0, y_0) &= 0 \end{aligned}$$

and hence the curve X is not smooth at (x_0, y_0). The regularity of A_d is proved in the same way, we only change the local coordinates on the weighted projective space $\mathbb{P}^2(d)$ as

$$x \to \frac{1}{x}\,, \quad y \to \frac{y}{x^d}\,.$$

Let $A(x)$ be a matrix with a spectral polynomial $P(x, y)$. By Proposition 2.2 the leading term J of $A(x)$ is a regular matrix. The characteristic polynomial of J obviously coincides with the highest order weight-homogeneous part of $P(x, y)$. If $m = \sum n_i p_i$ is the modulus of the spectral curve, then without loss of generality we shall suppose that $J = \mathrm{diag}(J(\lambda_1), J(\lambda_2), \ldots$ $\ldots, J(\lambda_s))$, where $J(\lambda_i)$ is a Jordan block of dimension n_i with eigenvalue λ_i, and $\lambda_i \neq \lambda_j$.

We recall first the definition of the eigenvector map l. Consider a line bundle over a curve X which is a sub-bundle of the trivial vector bundle

$X \times \mathbb{C}^r$. It is defined by a meromorphic vector $f(p) = (f_1, f_2, \dots, f_r)$. We shall always suppose that f is *normalized*, that is to say the meromorphic functions $f_1, f_2, \dots, f_r$ have not a common zero.The map

$$X \to \mathbb{P}^{r-1} : p \to [f_1(p), f_2(p), \dots, f_r(p)]$$

is holomorphic, so we obtain a holomorphic line bundle over X. We denote its dual by L. If D is the pole divisor of f, that is to say the minimal effective divisor such that $(f_i) \geq -D$ for any i, then $L = L_D$.

Definition 3 *(eigenvector line bundle on the spectral curve X) Let $\mathbf{f}(x, y) =^t (f_1, f_2, \dots, f_r)$, $p = (x, y) \in X$ be a normalized eigenvector of the matrix $A(x) \in M_P$, $A(x)\mathbf{f} = y\mathbf{f}$. It defines a line bundle over the spectral curve X which will be called eigenvector line bundle. Denote its dual by L and the corresponding sheaf of sections by $\mathcal{L}$.*

Of course if D is the pole divisor of the normalized eigenvector $\mathbf{f}$ then $L = L_D$ and $\mathcal{L} = \mathcal{L}(D)$. The following properties of L are well known (see [4,2,17])

Proposition 2.3 *If $\pi : X \to \mathbb{P}^1$ is the projection defined above, then the sheaf $\pi_*\mathcal{L}$ is a trivial $\mathcal{O}_{\mathbb{P}^1}$ module of rank r*

$$\pi_*\mathcal{L} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}$$

and the functions $f_1, f_2, \dots, f_r$ form a basis of global sections of $\pi_*\mathcal{L}$. Moreover $\deg(L) = \deg(D) = p_a = p_g + r - 1$, where $p_g = (r-1)(dr-2)/2$, $\dim H^0(X, \mathcal{L}) = r$ and $f_1, f_2, \dots, f_r$ form a basis of $H^0(X, \mathcal{L}(D))$.

The above proposition shows that if the matrices $A(x)$, $\tilde{A}(x)$ define isomorphic eigenvector bundles, then for some $R \in \mathbf{gl}_r(\mathbb{C})$, $A(x) = R\tilde{A}(x) \times R^{-1}$. Indeed, if $\mathbf{f}$, $\tilde{\mathbf{f}}$ are the corresponding normalized eigenvectors with equivalent pole divisors $D \sim \tilde{D}$, then there exists a meromorphic function φ on X such that $(\varphi) = \tilde{D} - D$. As $\varphi\tilde{f}_i$ form a basis of $H^0(X, \mathcal{L}(D))$, then there exists a matrix $R \in \mathbf{gl}_r(\mathbb{C})$ such that $\mathbf{f} = \varphi R\tilde{\mathbf{f}}$, and hence $A(x) = R\tilde{A}(x)R^{-1}$. Thus we obtain a holomorphic map

$$\begin{gathered}\{\text{ a matrix } A(x) \in M_P \text{ up to conjugation by a matrix in } \mathbb{P}\mathbf{GL}_r(\mathbb{C})\}\\ \downarrow l \\ \{\text{ an isomorphism class of a line bundle } L \in \mathrm{Pic}^{p_a}(X) - \Theta\}.\end{gathered}$$

The following beautiful argument of Beauville [4] shows that l is a bijection (see also Sect. 8.2. of the survey [17] by A.G.Reyman and M.A. Semenov-Tian-Shansky). Take a degree $p_a = p_g + r - 1$ invertible sheaf $\mathcal{L}$ on X. By Riemann-Roch theorem

$$\begin{aligned}\chi(\mathcal{L}) &= \deg(\mathcal{L}) - p_g + 1 = r\,,\\ \chi(\pi_*\mathcal{L}) &= \deg(\pi_*\mathcal{L}) + (1 - p_g(\mathbb{P}))\,\mathrm{rank}(\pi_*\mathcal{L}) = \deg(\pi_*\mathcal{L}) + r\,,\end{aligned}$$

by Grothendieck-Riemann-Roch

$$\chi(\mathcal{L}) = \chi(\pi_* \mathcal{L})\,,$$

and hence $\deg(\pi_* \mathcal{L}) = 0$. If we suppose in addition that $\mathcal{L} \in \mathrm{Pic}^{p_a}(X) - \Theta$ we conclude that $\pi_* \mathcal{L}$ is the rank r trivial vector bundle $(\mathcal{O}_{\mathbb{P}^1})^r$. The invertible sheaf $\mathcal{L}$ on X can be equivalently described as a locally trivial $\mathcal{O}_{\mathbb{P}^1}$ module $\pi_* \mathcal{L}$ equipped with an additional structure of a $\pi_* \mathcal{O}$ module, that is to say a homomorphism of algebras $a : \pi_* \mathcal{O} \to End(\pi_* \mathcal{L})$. To describe the homomorphism a amounts to give a linear map (multiplication by y)

$$\pi_* \mathcal{L} \to \pi_* \mathcal{L}(d)$$

that is to say a polynomial $r \times r$ matrix of degree d. Denote the transposed to this matrix by $A(x)$. Clearly $A(x)$ satisfies $P(x, A(x)) = 0$ and as $P(x, y)$ is irreducible over $\mathbb{C}(x)$ then by the Cayley-Hamilton theorem the spectral polynomial of $A(x)$ is $P(x, y)$. Note that the matrix $A(x)$ is determined only modulo an automorphism of $\pi_* \mathcal{L}$. Thus the matrix $A(x)$ is determined only up to conjugation $A(x) \to R^{-1} A(x) R$ by a matrix $R \in \mathbb{P}\mathbf{GL}_r(\mathbb{C})$.

The next step is to define the eigenvector line bundle on the singularized spectral curve X' and the corresponding map l'.

Definition 4 *(eigenvector sheaf on the singularized spectral curve X') Let $\mathbf{f}(x, y)$ be an eigenvector of $A(x)$ normalized by the condition*

$$\sum_{i=1}^{r} f_i \equiv 1$$

and let D be the minimal divisor, such that $(f_i) \geq -D$, $i = 1, 2, \ldots s$. Then D is an effective divisor, $D \subset X_{reg}$, and we define the invertible eigenvector sheaf on the singularized spectral curve X' to be $\mathcal{L}' = \mathcal{L}'(D)$, where

$$\mathcal{L}'_p(D) = \begin{cases} \mathcal{L}_p(D), & p \neq \infty \\ \mathcal{O}'_p, & p = \infty. \end{cases}$$

We denote by L' the line bundle over X' associated to the invertible sheaf $\mathcal{L}'$. To prove the correctness of the above definition it remains to check that $D \subset X_{reg}$, where D is the pole divisor of the normalized eigenvector $\mathbf{f}$.

Let $S = \sum_1^s p_i$ be the support of the modulus $m = \sum_1^s n_i p_i$ and we may suppose that $p_i \in X$ corresponds to the Jordan block $J(\lambda_i)$ of the matrix J. An easy computation shows that $\mathbf{f}(p_i)$ determines a line over p_i, collinear with the eigenvector $(0, 0, .., 0, 1, 0, \ldots, 0)$ of J corresponding to the Jordan block $J(\lambda_i)$. The eigenvector $\mathbf{f}$ has a pole at p_i if and only if the line determined by $\mathbf{f}(p_i)$ is contained in the plane $f_1 + f_2 + \ldots + f_r = 0$, so p_i is not a pole.

Let $\mathrm{Pic}^{p_a}(X')$ be the "shifted" Picard group $\mathrm{Pic}^0(X') = J(X')$ of degree p_a line bundles on X'. It is isomorphic to the Jacobian variety $J(X')$ and $J(X') - \Theta'$ is the subset of line bundles $L' \in \mathrm{Pic}^{p_a}(X')$ with one non-zero holomorphic section $h^0\mathcal{L}' = \dim H^0(X', L') = 1$. Definition 4 establishes a holomorphic map

$$\begin{gathered}\{ \text{ a matrix } A(x) \in M_P^J\} \\ \downarrow l' \\ \{ \text{ an isomorphism class of a line bundle } L' \in \mathrm{Pic}^{p_a}(X') - \Theta'\}.\end{gathered}$$

Clearly the map l' is such that the diagram (7) commutes: $\phi \circ l' = l \circ \varphi$. As the map l is a bijection, then to show that l' is a bijection too it suffices to check that

i) the fibres $\varphi^{-1}(b)$ and $\phi^{-1} \circ l(b)$ have the same dimension.

ii) $l' : \varphi^{-1}(b) \to \phi^{-1} \circ l(b)$ is an injective homomorphism of algebraic groups.

Step i) is obvious and the dimension of the fibres is r. To check that l' is injective we take a sheaf $\mathcal{L}' \in \mathrm{Pic}^{p_a}(X') - \Theta'$ in the image of l'. By Riemann-Roch theorem [19]

$$\begin{aligned}\chi(\mathcal{L}') &= \deg(\mathcal{L}') - p_a + 1 = 1\,, \\ \chi(\pi_*\mathcal{L}') &= \deg(\pi_*\mathcal{L}') + (1 - p_g(\mathbb{P}))\,\mathrm{rank}(\pi_*\mathcal{L}') = \deg(\pi_*\mathcal{L}') + r\,,\end{aligned}$$

by Grothendieck-Riemann-Roch

$$\chi(\mathcal{L}') = \chi(\pi_*\mathcal{L}').$$

We conclude that $\pi_*\mathcal{L}'$ is a degree $1 - r$ and rank r locally trivial $\mathcal{O}_{\mathbb{P}^1}$ module, having one holomorphic section, $h^0\pi_*\mathcal{L}' = h^0\mathcal{L}' = 1$, so

$$\pi_*\mathcal{L}' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\,.$$

The invertible sheaf $\mathcal{L}'$ on X' can be equivalently described as a locally trivial $\mathcal{O}_{\mathbb{P}^1}$ module $\pi_*\mathcal{L}'$ equipped with an additional structure of a $\pi_*\mathcal{O}'$ module, that is to say a homomorphism of algebras $a : \pi_*\mathcal{O}' \to End(\pi_*\mathcal{L}')$. It is easy to compute $\pi_*\mathcal{O}'$: a basis over the affine plane $\mathbb{C}$ is given by $\{1, y, y^2, \ldots, y^{r-1}\}$ and over $\mathbb{P} - \{0\}$ by $\{1, y/x^{d+1}, y^2/x^{2d+1}, \ldots, y^{r-1} /x^{(r-1)d+1}\}$, so

$$\pi_*\mathcal{O}' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d-1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(-d(r-1)-1)\,.$$

To describe the homomorphism a amounts to give a linear map (multiplication by y)

$$\pi_*\mathcal{L}' \to \pi_*\mathcal{L}'(d)$$

that is to say a polynomial $r \times r$ matrix of degree d. Denote the transposed to this matrix by $A(x)$. If $f_1, f_2, \ldots f_r$ is a normalized basis of $\pi_*\mathcal{L}'$ over

$\mathbb{C}$, $\sum f_i \equiv 1$, then ${}^t(f_1, f_2, \dots f_r)$ is an eigenvector of $A(x)$. Clearly $A(x)$ satisfies $P(x, A(x)) = 0$ and as $P(x,y)$ is irreducible over $\mathbb{C}(x)$ then by the Cayley-Hamilton theorem the spectral polynomial of $A(x)$ is $P(x,y)$. The homomorphism a is determined modulo an automorphism of $\pi_*\mathcal{L}'$. In the base $f_1, f_2, \dots, f_r$ the vector $1 \equiv \sum f_i \in H^0(\mathbb{P}^1, \pi_*\mathcal{L}')$ has coordinates $\mathbf{e} =^t (1, 1, \dots, 1)$, and hence the group $Aut(\pi_*\mathcal{L}')$ is identified to

$$\mathbf{GL}_r(\mathbb{C}; \mathbf{e}) = \{R \in \mathbf{GL}_r(\mathbb{C}) : \mathbf{e} \text{ is an eigenvector of } R\}.$$

As we supposed that $\pi_*\mathcal{L}' \in l'(M_P^J)$, then in a suitable basis of $\pi_*\mathcal{L}'$ we have $A(x) \in M_P^J$. If $\tilde{A}(x) \in M_P^J$ is another matrix which defines the same eigenvector sheaf $\mathcal{L}'$, then $\tilde{A}(x) = RA(x)R^{-1}$ for some $R \in \mathbb{P}\mathbf{GL}_r(\mathbb{C}; \mathbf{e})$. As at the same time $R \in \mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$, we conclude that $R = 1 \in \mathbb{P}\mathbf{GL}_r(\mathbb{C})$.

Finally we note that the vector fields $[J^k, A(x)]$, $r = 1, \dots, r-1$ are tangent to the fibre $\varphi^{-1}(b)$, $\mathbb{P}\mathbf{GL}_r(\mathbb{C}; J)$ invariant and linearly independent (this follows from the regularity of J). The images of these vector fields in $\mathrm{Pic}^{p_a}(X') \sim J(X')$ are well known to be translation invariant [17] and hence $l' : \varphi^{-1}(b) \to \phi^{-1} \circ l(b)$ is a homomorhism. This completes the proof of step ii).

It remains to prove that M_P^J is a smooth manifold, that is to say, to find at any point $A(x) \in M_P^J$ vector fields which span the tangent space, and such that their images in $J(X') - \Theta'$ span the tangent space too. These vector fields are given by

$$Y_a^{(i)}(A(x)) = \left[\frac{A^i(a)}{x-a}, A(x)\right], \quad a \in \mathbb{P}^1,\ i \in \mathbb{N}$$

but this will be explained in the next section. $\square$

3 Integrable systems

Let us fix a non-zero matrix $J \in \mathbf{gl}(r, \mathbb{C})$ and denote by M^J the affine space of all matrix polynomials $A(x)$ of the form

$$A(x) = Jx^d + A_{d-1}x^{d-1} + \dots + A_0,\ A_i \in \mathbf{gl}_r(\mathbb{C}) .$$

The space M^J is of dimension $\dim M^J = 2p_a + dr = dr^2$ and it carries several compatible Poisson structures of rank $2p_a = dr(r-1)$. Let us fix such a structure $\{.,.\}$. A function φ on M^J is called invariant if it is constant on each isospectral manifold

$$M_P^J = \{A(x) \in M^J : \det(A(x) - yI_r) = P(x,y)\}.$$

The algebra of invariant functions on M^J is thus generated by the $dr(r+1)/2$ non-trivial coefficients of $P(x,y)$ (which are in addition functionally independent).

It turns out that the invariant functions commute with respect to $\{.,.\}$. Moreover, the tangent space to M_P^J at any point $A(x)\in M_P^J$ is the span of all Hamiltonian vector fields $X_\varphi=\{.,\varphi\}$, where φ is an invariant function. It follows that any such Hamiltonian vector field X_φ is completely integrable in the sense of Liouville, and hence its solutions can be explicitly computed "by quadratures".

The purpose of this section is to describe briefly the Hamiltonian structure of the completely integrable system (1) (thus justifying the title of the article). The scheme is quite classical now and proofs together with historical comments may be found in [17].

We describe first the compatible Poisson structures. Let

$$\tilde{\mathbf{g}}=\mathbf{g}[x,x^{-1}]$$

be the loop algebra of the Lie algebra $\mathbf{g}$ formed by Laurent polynomials in one variable x with coefficients in $\mathbf{g}$, and commutator given by

$$[\sum_i A_i x^i,\sum_j B_j x^j]=\sum_k(\sum_{i+j=k}[A_i,B_j])x^k, A_i,B_j\in\mathbf{g}\,.$$

Let

$$\tilde{\mathbf{g}}^*=\mathbf{g}^*[x,x^{-1}]$$

be the "restricted" dual space to $\tilde{\mathbf{g}}$ consisting of Laurent polynomials.The space $\tilde{\mathbf{g}}^*$ carries a canonical Lie-Poisson structure, which is the extension of the Lie algebra of linear functions on $\tilde{\mathbf{g}}^*$ to the entire space of smooth functions on $\tilde{\mathbf{g}}^*$ (a linear function on $\tilde{\mathbf{g}}^*$ is identified to a point in $\tilde{\mathbf{g}}$). Any non-degenerate invariant bi-linear form $<.,.>$ on $\tilde{\mathbf{g}}$ identifies $\tilde{\mathbf{g}}^*$ to $\tilde{\mathbf{g}}$ so the latter space also carries a Poisson structure. To be explicit, put $\mathbf{g}=\tilde{\mathbf{gl}}_r(\mathbb{C})$ and

$$<A(x),B(x)>=\mathrm{Res}_{x=0}\,\mathrm{Trace}(A(x)B(x))\frac{dx}{x}.$$

Choose a basis e^a in $\mathbf{g}$ and let C_c^{ab} be the structure constants of $\mathbf{g}$, $[e^a,e^b]=\sum_c C_c^{ab}e^c$. Let $A(x)=\sum_i A_i x^i\in\tilde{\mathbf{g}}$, where $A_i=\sum_a A_i^a e^a$. Then

$$\{A_i^a,A_j^b\}=-\sum_c C_c^{ab}A_{i+j}^c\,.$$

The simplectic leaves of this Poisson structure are the co-adjoint orbits of the Lie group underlying $\tilde{\mathbf{g}}$. The corresponding ring of adjoint invariants (Casimir functions) is generated by

$$\varphi_{mn}(A(x))=\mathrm{Res}_{x=0}(x^{-n}\varphi(x^m A(x)))dx\,,\quad m,n\in\mathbb{N}$$

where φ is any invariant function on $\mathbf{g}$. It is clear that in such a way any Lie algebra structure on $\tilde{\mathbf{g}}$ defines a Poisson structure on $\tilde{\mathbf{g}}$. The most important class of Poisson brackets are the so called R-brackets. Namely, let $R \in End(\tilde{\mathbf{g}})$ be a linear operator, and suppose that the commutator

$$[X,Y]_R = \frac{1}{2}([RX,Y] + [X,RY]), \quad X,Y \in \tilde{\mathbf{g}} \tag{8}$$

satisfies the Jacobi identity (this happens for example if R satisfies the classical Yang-Baxter identity). This induces, according to the scheme described above, a Poisson bracket $\{.,.\}_R$ on $\tilde{\mathbf{g}}$. The importance of the R-bracket is related to the following result (due to Semenov-Tian-Shanski [18,17] and closely related to the so called Adler-Kostant-Symes theorem).

Theorem 3.1 *(i) The Casimir functions φ_{mn} on $\tilde{\mathbf{g}}$ are in involution with respect to the R-bracket.*
(ii) The Hamiltonian system associated to

$$H(A(x)) = \mathrm{Res}_{x=0}\,\mathrm{Trace}\left(\frac{A^{k+1}(x)}{k+1}\right)\frac{dx}{x},$$

$\{.,.\}_R$*, is given by*

$$\frac{d}{dt}A(x) = [A(x), M], A(x) \in \tilde{\mathbf{g}}, M = \frac{1}{2}R(A^k(x)). \tag{9}$$

The decomposition

$$\tilde{\mathbf{g}} = \tilde{\mathbf{g}}^+ \oplus \tilde{\mathbf{g}}^-$$

where

$$\tilde{\mathbf{g}}^+ = \oplus_{i=0}^{\infty}\tilde{\mathbf{g}}x^i,\ \tilde{\mathbf{g}}^- = \oplus_{i=-1}^{-\infty}\tilde{\mathbf{g}}x^i$$

defines a R matrix on $\mathbf{g}$. Namely, if $A(x) = A(x)^+ + A(x)^- \in \tilde{\mathbf{g}}$ where $A(x)^\pm \in \tilde{\mathbf{g}}^\pm$, then define

$$R(A(x)) = A(x)^+ - A(x)^-.$$

The commutator (8) is given by

$$[A(x), B(x)]_R = [A(x)^+, B(x)^+] - [A(x)^-, B(x)^-]$$

and it satisfies the Jacobi identity. The induced Poisson bracket $\{.,.\}_R$ on $\tilde{\mathbf{g}}$ is explicitly given (in the notations above) by

$$\{A_i^a, A_j^b\} = -\epsilon_{ij}\sum_c C_c^{ab} A_{i+j}^c$$

where $\epsilon_{ij} = 1$ for $i,j \geq 0$, $\epsilon_{ij} = -1$ for $i,j < 0$, and $\epsilon_{ij} = 0$ for $i \geq 0, j < 0$.

Let

$$q(x) = \sum_{i=-d+1}^{1} q_i x^i$$

be a fixed polynomial. Then the embedding

$$M^J \hookrightarrow \tilde{\mathbf{g}} : A(x) \mapsto q(x)A(x) \tag{10}$$

is a *Poisson mapping* with respect to the R-bracket. This means that the embedding induces a Poisson structure on M^J. We obtain thus a family of *compatible* Poisson structures on M^J which depend linearly on the coefficients q_i of the polynomial $q(x)$.

Corollary 3.2 *The Hamiltonian system on M^J associated to*

$$H(A(x)) = \mathrm{Res}_{x=0}\,\mathrm{Trace}\left(\frac{A^{k+1}(x)}{k+1}\right)\frac{dx}{x}$$

and to the Poisson structure induced by the embedding (10) is given by

$$\begin{aligned} \frac{d}{dt}A(x) &= [A(x), M_\pm]\,, \quad A(x) \in \tilde{\mathbf{g}}\,, \\ M_\pm &= \pm(q(x)A^k(x))_\pm \in \tilde{\mathbf{g}}^\pm\,. \end{aligned} \tag{11}$$

If we choose for example $q(x) = 1/x$, then (11) takes the form

$$\frac{d}{dt}A(x) = \left[\frac{A^k(0)}{x}, A(x)\right]\,. \tag{12}$$

The construction of the loop algebra $\tilde{\mathbf{g}}$ was related to the choice of Laurent polynomials with a pole at $x = 0$. It is obvious that all that holds true if we consider Laurent polynomials with a pole at $x = a \in \mathbb{C}$. In this case (12) takes the Beauville form

$$\frac{d}{dt}A(x) = \left[\frac{A^k(a)}{x-a}, A(x)\right] = Y_a^{(k)}(A(x))\,. \tag{13}$$

Recall now that when the spectral curve is smooth, then the invariant level set (the isospectral manifold of $A(x)$) of (13) is smooth and bi-holomorphic to the Zariski open subset $J(X') - \Theta'$ of the generalized Jacobian $J(X')$ (Theorem 2.1). It is shown in [17] that the vector fields $Y_a^{(k)}(A(x))$ induce translation invariant vector fields on $J(X')$ (although the results are formulated only on $J(X)$). The direction of $Y_a^{(k)}(A(x))$ is moreover explicitly computed (formula (8.5) on p.177, but see also [4, Corollary 2.7]). These formulae imply that the vector fields $Y_a^{(k)}(A(x))$ span, for generic a the tangent space to the generalized Jacobian $J(X')$.

We conclude that the Hamiltonian system (11) is completely integrable.

Definition 5 *A Hamiltonian system is called algebraically completely integrable, provided that it is completely integrable, and in addition each generic complex invariant level set is a Zariski open subset of a commutative algebraic group, on which the Hamiltonian vector fields generated by the first integrals are translation invariant.*

Of course in order that the above definition makes a sense we must suppose that the Poisson manifold, the Hamiltonian functions and vector fields are *algebraic* (compare to [14, p.3.53]). Taking into account the results of Sect. 2 we obtain

Corollary 3.3 *The system (11) (and hence (1)) is a* $p_a = dr(r-1)/2$ *degrees of freedom algebraically completely integrable Hamiltonian system.*

4 Examples

4.1 Lagrange top

Let X be a smooth elliptic curve, $m = P_1 + P_2$, $P_1 \neq P_2$, an effective divisor on X, and let X' be the singularized curve X relative to the modulus m. The generalized Jacobian $J(X')$ is an extension of the usual Jacobian $J(X)$ by $\mathbb{C}^*$

$$0 \to \mathbb{C}^* \to J(X') \to J(X) \to 0$$

and it is easy to check that the above extension is *never* trivial. Indeed, if the generalized Jacobian $J(X')$ is isomorphic to $J(X) \times \mathbb{C}^*$ then $J(X') = \mathbb{C}^2/\Lambda$ where

$$\Lambda = \mathbb{Z}\begin{pmatrix} 2\pi \\ 0 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 0 \\ 2\pi \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

with $\tau_2 = 0$. The generalized Riemann theta function [5,8]

$$\tilde{\theta}(z_1, z_2|\tau_1, \tau_2) = e^{z_2/2}\theta(z_1 + \tau_2/2|\tau_1) + e^{-z_2/2}\theta(z_1 - \tau_2/2|\tau_1)\,, \\ (z_1, z_2) \in \mathbb{C}^2/\Lambda$$

decomposes into the product $(e^{z_2/2} - e^{z_2/2})\theta(z_1|\tau_1)$ where $\theta(z_1|\tau_1)$ is the usual elliptic Riemann theta function ($z_1 \in \mathbb{C}/\{2\pi i\mathbb{Z} + \mathbb{Z}\tau_1\}$). It follows that the generalized Riemann theta divisor $(\tilde{\theta})$ is reducible which contradicts to the fact that it is isomorphic to the affine curve $X - \{P_1 \cup P_2\}$ [8].

Consider now the affine space M^J of matrix polynomials $A(x)$ of the form

$$A(x) = Jx^2 + A_1x + A_0,\ A_0, A_1 \in \mathbf{gl}_2(\mathbb{C})$$

where J is a fixed matrix with distinct eigenvalues. As we explained in Sect. 3 the Lax pair

$$\begin{aligned} \frac{d}{dt}A(x) &= \left[A(x), \frac{A(a)}{x-a}\right] = [A(x), Jx + aJ + A_1] \\ &= [A(x), ((q(x)A(x))_+] \end{aligned} \tag{14}$$

where $q(x) = (x+a)/x^2$, defines a completely integrable Hamiltonian system on the simplectic leaves of several compatible Poisson structures on M^J. Moreover, when the spectral curve X with affine equation $\{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\}$ is smooth, the corresponding isospectral manifold

$$M_P^J = \{A(x) \in M^J : \det(A(x) - yI_2) = P(x,y)\}$$

is smooth and is described as in Theorem 2.1. In addition the above vector field is translation invariant on the generalized Jacobian $J(X')$, so our system is algebraically completely integrable. As the modulus of the spectral curve X is $m = \infty^+ + \infty^-$, where $\infty^\pm$ are the two "infinite" points on X, then the generalized Jacobian $J(X')$ is described as above.

Our purpose is to show that, for appropriate choice of the matrix J and the parameter a, equation (14) is the classical equation of heavy symmetric top. The symmetry group $\mathbb{C}^*$ is then the complexified circle action (rotations about the symmetry axe of the top). In the sequel we put

$$J = \frac{\sqrt{2}}{\epsilon}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \epsilon = \exp\sqrt{-1}\pi/4$$

Consider the isospectral manifold

$$M_f^J = \{A(x) \in M^J : \det(A(x) - yI_2) = y^2 - f(x)\}$$

where $f(x)$ is a fixed monic polynomial

$$f(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

without double roots. We may consider M_f^J as a subvariety of the affine vector space of traceless matrices

$$V = \{A(x) \in M^J : \mathrm{Trace}(A(x)) = 0\}\,.$$

By making use of the isomorphism of Lie algebras $\mathbf{sl}_2(\mathbb{C})$ and $\mathbf{so}_3(\mathbb{C})$ given by

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon x & \epsilon z + \bar{\epsilon} y \\ \epsilon z - \bar{\epsilon} y & -\epsilon x \end{pmatrix}, \quad \epsilon = \exp\sqrt{-1}\pi/4$$

we may identify V to the affine space

$$\{L(x) : L(x) = \chi x^2 + \mathbf{M}x - \mathbf{\Gamma}\,,\quad \mathbf{M}, \mathbf{\Gamma} \in \mathbf{so}_3(\mathbb{C})\,,\quad \chi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\}\,.$$

The Hamiltonian system (14) takes the form

$$\frac{d}{dt}(x^2\chi + x\mathbf{M} - \mathbf{\Gamma}) = [x^2\chi + x\mathbf{M} - \mathbf{\Gamma}, x\chi + \mathbf{M} + aJ]\,. \tag{15}$$

If we put at last $a = -m\Omega_3$ and

$$\mathbf{M} = (\Omega_1, \Omega_2, (1+m)\Omega_3), \mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3), \mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$$

then we obtain

$$\frac{d}{dt}\mathbf{M} = [\mathbf{M}, \mathbf{\Omega}] - [\mathbf{\Gamma}, \chi], \frac{d}{dt}\mathbf{\Gamma} = [\mathbf{\Gamma}, \mathbf{\Omega}].$$

which are the equations describing the Lagrange top. Indeed, after identifying the isomorphic Lie algebras $(\mathbb{R}^3, \wedge)$ and $(\mathbf{so}(3), [.,.])$, and making obvious rescalings we obtain the system

$$\frac{d}{dt}\mathbf{M} = \mathbf{M} \times \mathbf{\Omega} + \chi \times \mathbf{\Gamma}, \quad \frac{d}{dt}\mathbf{\Gamma} = \mathbf{\Gamma} \times \mathbf{\Omega} \tag{16}$$

where

$$\mathbf{M} = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3)\,, \quad \mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)\,,$$
$$\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3)\,, \quad \chi = (\chi_1, \chi_2, \chi_3)$$

and in addition $I_1 = I_2, \chi_1 = \chi_2 = 0$. Here $\mathbf{M}, \mathbf{\Omega}$ and $\mathbf{\Gamma}$ denote respectively the angular momentum, the angular velocity and the coordinates of the unit vector in the direction of gravity, all expressed in body-coordinates. The constant vector χ is the center of mass in body-coordinates multiplied by the mass and the acceleration, I_1, I_2, I_3 are the principal moments of inertia of the body.

To resume, we proved that the Lagrange top is an algebraically completely integrable system. It linearizes on a two-dimensional complex algebraic group – the generalized Jacobian $J(X')$ of an elliptic curve X with two points $\infty^{\pm}$ identified. This result is proved directly in [9]. If we reduce further the system (16) with respect to the circle action $\mathbb{C}^*$ we obtain, as it is well known, a one degree of freedom algebraically completely integrable system linearized on the elliptic curve $J(X')/\mathbb{C}^* = X$ [12,2,16,20, 3]. Other mechanical systems linearized on non-compact algebraic groups were recently studied by Fedorov [8].

4.2 The general integral of a system of hyperelliptic differential equations

Let $f(x)$ be a fixed polynomial of degree $2n$ or $2n-1$ without double roots and consider the following *hyperelliptic system of differential equations*

$$
\begin{aligned}
\frac{dx_1}{\sqrt{f(x_1)}} + \frac{dx_2}{\sqrt{f(x_2)}} + \ldots + \frac{dx_n}{\sqrt{f(x_n)}} &= 0 \\
\frac{x_1dx_1}{\sqrt{f(x_1)}} + \frac{x_2dx_2}{\sqrt{f(x_2)}} + \ldots + \frac{x_ndx_n}{\sqrt{f(x_n)}} &= 0 \\
\cdots \quad & \cdots \\
\frac{x_1^{n-2}dx_1}{\sqrt{f(x_1)}} + \frac{x_2^{n-2}dx_2}{\sqrt{f(x_2)}} + \ldots + \frac{x_n^{n-2}dx_n}{\sqrt{f(x_n)}} &= 0
\end{aligned}
\tag{17}
$$

Suppose that the polynomial $f(x)$ is written in the form

$$f(x) = -A^2(x) + B^2(x) + C^2(x)$$

where

$$A(x) = \sum_{k=0}^{n} a_k x^k,\ B(x) = \sum_{k=0}^{n} b_k x^k,\ C(x) = \sum_{k=0}^{n} c_k x^k\,. \tag{18}$$

Jacobi [10] proved in 1846 the following

Theorem 4.1 *Let $x_1(\varphi), x_2(\varphi), \ldots, x_n(\varphi)$ be the roots of the polynomial equation*

$$A(x) = B(x)\cos(\varphi) + C(x)\sin(\varphi)\,.$$

Then $\mathbf{x}(\varphi) = (x_1(\varphi), x_2(\varphi), \ldots, x_n(\varphi))$ is an integral curve of (17).

Let us note that the phase space of the system (17) is the nth symmetric product $S^n\Gamma$ of the smooth affine curve

$$\Gamma = \{(x,y) : y^2 = f(x)\}.$$

The variables $x_1, x_2, \ldots, x_n$ provide a system of local coordinates in a neighborhood of any generic point on the smooth manifold $S^n\Gamma$. We shall give an independent proof of Jacobi's theorem in the light of the present paper. For a further discussion on the Jacobi's paper see Mumford [14, p. 3.17].

Assume first that $\deg(f) = 2n$ and consider, instead of system (17), the following generalized Jacobi inversion problem [6, 11, 19]

$$
\begin{aligned}
\frac{dx_1}{\sqrt{f(x_1)}} + \frac{dx_2}{\sqrt{f(x_2)}} + \ldots + \frac{dx_n}{\sqrt{f(x_n)}} &= dz_1 \\
\frac{x_1dx_1}{\sqrt{f(x_1)}} + \frac{x_2dx_2}{\sqrt{f(x_2)}} + \ldots + \frac{x_ndx_n}{\sqrt{f(x_n)}} &= dz_2 \\
\cdots \quad & \cdots \\
\frac{x_1^{n-1}dx_1}{\sqrt{f(x_1)}} + \frac{x_2^{n-1}dx_2}{\sqrt{f(x_2)}} + \ldots + \frac{x_n^{n-1}dx_n}{\sqrt{f(x_n)}} &= dz_n
\end{aligned}
\tag{19}
$$

It involves the differential of second kind

$$\frac{x^{n-1}dx}{\sqrt{f(x)}} \tag{20}$$

on the completed and normalized genus $n-1$ hyperelliptic curve $X = \bar{\Gamma}$, where $\Gamma = \{(x,y) : y^2 = f(x)\}$. Put $m = \infty^+ + \infty^-$, where $\infty^\pm$ are the two poles of the differential (20) and let X' be the singularized curve X relative to the effective divisor m. The general symmetric function in $x_i, \sqrt{f(x_j)}$ can be expressed as a meromorphic function in ${}^t(z_1, z_2, \ldots, z_n) \in J(X') = \mathbb{C}^n/\Lambda$, where Λ is the $\mathbb{Z}$ lattice

$$\Lambda = \left\{ {}^t\left(\oint_\gamma \frac{dx}{\sqrt{f(x)}}, \oint_\gamma \frac{xdx}{\sqrt{f(x)}}, \ldots, \oint_\gamma \frac{x^{n-1}dx}{\sqrt{f(x)}} \right) \right\}_\gamma , \\ \gamma \in H_1(X - \{\infty^+, \infty^-\}, \mathbb{Z}\} .$$

The generalized Jacobian $J(X')$ is a $\mathbb{C}^*$ extension of the usual Jacobian $J(X)$

$$0 \overset{exp}{\to} \mathbb{C}^* \to J(X') \overset{\phi}{\to} J(X) \to 0 \tag{21}$$

where ϕ is the projection $\phi(z_1, z_2, \ldots, z_n) =^t (z_1, z_2, \ldots, z_{n-1})$ (see Sect. 2).

It follows that *an integral curve of the system (17) is just the fibre*

$$\phi^{-1}(z_1^0, z_2^0, \ldots, z_{n-1}^0) \tag{22}$$

over the point ${}^t(z_1^0, z_2^0, \ldots, z_{n-1}^0) \in J(X)$*. In particular each integral curve is isomorphic to the algebraic group* $\mathbb{C}^*$*, and the set of all integral curves is parameterized by the Jacobian variety* $J(X)$.

Theorem 2.1 provides an explicit parameterization of the fibre (22). Namely, let

$$L(x) = \begin{pmatrix} -\mathrm{i}A(x) & C(x) - \mathrm{i}B(x) \\ C(x) + \mathrm{i}B(x) & \mathrm{i}A(x) \end{pmatrix} , \quad \mathrm{i} = \sqrt{-1}$$

where $A(x), B(x), C(x)$ are the Jacobi polynomials (18). The spectral polynomial of $L(x)$ is $P(x,y) = y^2 - f(x)$. Put

$$R(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, R(\infty) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and consider the eigenvector ${}^t(1, f_2(x,y;\alpha))$ corresponding to the eigenvalue y of the matrix $R(\alpha)L(x)R^{-1}(\alpha)$. This defines a divisor $D(\alpha) = (f_2)_\infty$ and hence a one-parameter family of line bundles $L'_{D(\alpha)} \in \mathrm{Pic}^n(X')$, $\alpha \in \mathbb{P}^1$ on the singular curve X'. As the line bundle $L_{D(\alpha)} \in \mathrm{Pic}^n(X)$ does not depend on α then $L'_{D(\alpha)}$, $\alpha \in \mathbb{P}^1$ parameterizes the fibre (22), that is to say an integral curve of (17). A simple computation shows that

$$f_2(x,y;\alpha) = \frac{-\mathrm{i}A(x) + \alpha(C(x) + \mathrm{i}B(x)) - y}{2\mathrm{i}\alpha(-A(x) - \frac{\sqrt{-1}}{2}(\alpha - \frac{1}{\alpha})C(x) + \frac{1}{2}(\alpha + \frac{1}{\alpha})B(x))}\,.$$

If $D(\alpha) = \sum_{k=1}^{n} p_k$ where $p_k = (y_k, x_k) \in X$, then x_k is the root of the denominator

$$\begin{aligned}
&-A(x) - \frac{\sqrt{-1}}{2}\left(\alpha - \frac{1}{\alpha}\right) C(x) + \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) B(x) \\
&= -A(x) + B(x)\cos(\varphi) + C(x)\sin(\varphi)
\end{aligned}$$

where $\alpha = e^{\sqrt{-1}\varphi}$. This completes the proof of Jacobi's theorem in the case $\deg(f) = 2n$.

Note that there are exactly two values $\alpha^{\pm}$ of α such that the pole divisor of f is not contained in the affine part of the curve X and hence the line bundle $L'_{D(\alpha)}$ is not defined. Thus topologically the integral curve of (17) is $\mathbb{P} - \{\alpha^+, \alpha^-\} \sim \mathbb{C}^*$ as we explained before. At last if $\deg(f) = 2n-1$ Jacobi's theorem holds too (although Jacobi did not study this case explicitly). The differential (20) is of third kind, $m = 2\infty$, where ∞ is its double pole, and $J(X')$ is a non-trivial extension of $J(X)$ by $\mathbb{C}$

$$0 \to \mathbb{C} \to J(X') \xrightarrow{\phi} J(X) \to 0\,.$$

Indeed, in this case $b_0^2 + c_0^2 - a_0^2 = 0$, so $\alpha^+ = \alpha^-$ and the fiber is isomorphic to $\mathbb{P} - \{\alpha^{\pm}\} \sim \mathbb{C}$.

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