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Isochronicity of plane polynomial Hamiltonian systems

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Abstract. We study isochronous centres of plane polynomial Hamiltonian systems, and more generally, isochronous Morse critical points of complex polynomial Hamiltonian functions. Our first result is that if the Hamiltonian function H is a non-degenerate semi-weighted homogeneous polynomial, then it cannot have an isochronous Morse critical point, unless the associate Hamiltonian system is linear, that is to say H is of degree two. Our second result gives a topological obstruction for isochronicity. Namely, let $\gamma(h)$ be a continuous family of one-cycles contained in the complex level set $H^{-1}(h)$, and vanishing at an isochronous Morse critical point of H, as $h \to 0$. We prove that if H is a good polynomial with only simple isolated critical points and the level set $H^{-1}(0)$ contains a single critical point, then $\gamma(h)$ represents a zero homology cycle on the Riemann surface of the algebraic curve $H^{-1}(h)$. We give several examples of 'non-trivial' complex Hamiltonians with isochronous Morse critical points and explain how their study is related to the famous Jacobian conjecture.

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1. Introduction

Let $H \in \mathbb{R}[x, y]$ be a real polynomial of the form

$$H = (x^2 + y^2)/2 + \text{'higher order terms'}.$$

The plane Hamiltonian system

$$\begin{aligned} \dot{x} &= \partial H/\partial y \\ \dot{y} &= -\partial H/\partial x \end{aligned} \tag{1}$$

has an equilibrium point (a centre) at the origin surrounded by a family of periodic solutions parameterized by the energy H. Each periodic orbit $\gamma(h)$ is contained in an unique level set

$$\{(x, y) \in \mathbb{R}^2 : H(x, y) = h\}$$

for $h > 0$ sufficiently small and its period equal to

$$T(h) = \oint \mathrm{d}t = \int_{\gamma(h)} \frac{\mathrm{d}x}{\partial H/\partial y}.$$

The centre is called *isochronous* if the period $T(h)$ of these solutions does not depend on h. Another interpretation of the period function $T(h)$ is the following. Let $S(h) = \iint_{H\leqslant h} \mathrm{d}x \wedge \mathrm{d}y$ be the area of the set bounded by the periodic orbit $\gamma(h)$ on the plane $\mathbb{R}^2$. Then the derivative of the area function $S(h)$ is the period function $T(h)$.

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The study of isochronous systems goes back at least to Galileo who discovered in 1632 the isochronicity of small oscillations of simple pendulum and the formula for its period was given by Huygens in 1673 [23, p 72]. Huygens also described the first nonlinear isochronous pendulum: a particle constrained to move on a cycloid under the action of gravity [13] (see [23, p 111, example 1] for exact formulation). Isochronous systems were later studied by Euler, Bernoulli and Lagrange (see for example [15]). The plane quadratic isochronous systems are completely classified by Loud [16]. For more recent results on isochronicity of plane systems of differential equations we refer the reader to [7] (for a local study) and to [18].

As the period function $T(h)$ is given by an Abelian integral, it is more natural to study it in a complex domain, and even for complex Hamiltonian functions H. This will be the point of view adopted in the present paper. Namely, to any complex polynomial $f \in \mathbb{C}[x, y]$ having a Morse critical point at the origin, $f(0, 0) = 0$, we associate a one-cycle $\gamma(t)$ in the fibre $f^{-1}(t)$ vanishing at the origin in $\mathbb{C}^2$ as t tends to 0, and a period function

$$T(t) = \int_{\gamma(t)} \omega$$

where

$$\omega = \frac{\mathrm{d}x \wedge \mathrm{d}y}{\mathrm{d}f} = -\frac{\mathrm{d}x}{\partial f/\partial y}$$

is the Gel'fand–Leray form of the 'volume form' $\mathrm{d}x \wedge \mathrm{d}y$. If $f = H = (x^2 + y^2)/2 + \cdots$ is a real polynomial, and the orientation of $\gamma(t)$ is appropriately chosen, then this period function coincides with the period function associated to the centre of (1). We shall say that a Morse critical point of the complex polynomial function f is *isochronous* provided that the associated period function $T(t)$ is constant in t.

Further, we shall not use the real structure of the system. Thus, we shall make no difference between (non-degenerate) saddle-points and centres. Both will be for us simply Morse critical points. Our main results are theorems 4.1 and 3.1 where we find necessary conditions for isochronicity of a large class of complex plane polynomial Hamiltonian systems. In particular theorem 4.1 suggests that the monodromy of the cycle $\gamma(t)$ is an obstruction for a Morse critical point to be isochronous. This also leads to the following question

Is it true that if a Morse critical point is isochronous, then the associated vanishing cycle $\gamma(t)$ represents a zero homology cycle on the Riemann surface of the fibre $f^{-1}(t)$?

As we show in section 6, a positive answer to the above question would imply the famous Jacobian conjecture.

The paper is organized as follows. In section 2 we summarize some basic facts on the topology of the polynomial fibration $f^{-1}(t) \to t$, $f \in \mathbb{C}[x, y]$, which are used through the paper. For the convenience of the reader we also sketch the proofs. In section 3 we prove that a non-degenerate semi-weighted homogeneous polynomial cannot have an isochronous Morse critical point, unless its degree is two (theorem 3.1 provides a natural complex generalization of results obtained earlier in [7, 18]). We show that such a polynomial defines a Milnor fibration 'at infinity' so the asymptotic behaviour of the period function can be easily studied. From that we deduce that the period function is not a constant. We note that in the same way one may compute the asymptotic behaviour of any Abelian integral along a cycle contained in the level sets of the polynomial under consideration.

Suppose that $f \in \mathbb{C}[x, y]$ has only simple (in the sense of singularity theory) isolated critical points, and that it definies a Milnor fibration 'at infinity'. We prove in section 4 that if the critical level set $f^{-1}(0)$ contains a single critical point which is Morse and isochronous, then the associated vanishing cycle $\gamma(t)$ represents a zero homology cycle on the Riemann surface of the fibre $f^{-1}(t)$ (theorem 4.1). We use a monodromy argument which in fact can be applied to a larger class of polynomials. Thus, the proof seems to be more important than the result itself. We believe that a further progress can be achieved by a more careful study of the monodromy of polynomials.

In section 5 we give examples of polynomials with isochronous Morse critical points. It is seen in particular that the genus of the Riemann surface of the generic fibre f^{-1} is not an obstruction for isochronicity. Finally, in section 6, we explain how the sudy of isochronous Morse critical points is related to the well known Jacobian conjecture.

After this paper was submitted for publication we learned that the relation between isochronous systems and the Jacobian conjecture was also noted by M Sabatini (Connection between isochronous Hamiltonian centres and the Jacobian conjecture, preprint, Università degli Studi di Trento, 1995).

2. The topology of the fibration $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$

In this section we summarize some basic facts on the topology of the polynomial fibration $f^{-1}(t) \to t$, $f \in \mathbb{C}[x, y]$, which will be used through the paper. Let f be a polynomial in two complex variables with only isolated critical points and which is written in the form

$$f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) \tag{2}$$

where $a_i(x)$ are polynomials in x of degree at most i.

To each isolated critical point $p \in \mathbb{C}^2$ of f we associate its Milnor number

$$\mu_p(f) = \dim_{\mathbb{C}} \mathcal{O}_p(x, y)/\langle f_x, f_y\rangle$$

where $\mathcal{O}_p(x, y)$ is the local ring of $\mathbb{C}^2$ at p (it may be any of the rings of rational functions defined at p, formal or convergent power series in a neighbourhood of p) and $\langle f_x, f_y\rangle$ is the Jacobian ideal in $\mathcal{O}_p(x, y)$ generated by the gradient of f. We define also the global Milnor number $\mu(f)$ of f

$$\mu(f) = \sum_p \mu_p(f) = \dim \mathbb{C}^2[x, y]/\langle f_x, f_y\rangle.$$

The polynomial f has only isolated critical points if and only if $\mu(f)$ is finite.

Note that $\mu(f)$ is a topological invariant of f. It means that if f is topologically conjugate to the polynomial g then $\mu(f) = \mu(g)$. We shall now define another topological invariant of f.

Denote by $\Delta(t, x)$ the discriminant of $f(x, y) - t$ with respect to y. Let $d(t)$ be the degree of $\Delta(t, x)$ in x and let d be the degree of $\Delta(t, x)$ for generic t. Obviously $d - d(t) \geqslant 0$ and there is only a finite number of values for t such that $d - d(t) > 0$.

Definition 1. We denote

$$\lambda^t(f) = d - d(t) \qquad \lambda(f) = \sum_{t\in\mathbb{C}} \lambda^t(f).$$

The fact that $\lambda(f)$ is a topological invariant will follow from theorem 2.2. The number $\lambda^{t_0}(f)$ counts the number of ramification points of the curve $\Gamma_t = \{f(x, y) = t\}$ which tend to infinity as t tends to t_0 and then the number $\lambda(f)$ is the total number of ramification points which tend to infinity as t varies.

Following [5, p 236], we shall give another (equivalent) interpretation of the number $\lambda^t(f)$. Let $\overline{\Gamma_t}$ be the projective closure in $\mathbb{CP}^2$ of the affine curve $\Gamma_t = \{f(x, y) = t\} \subset \mathbb{C}^2$.

For any $p \in \overline{\Gamma_t} - \Gamma_t$ let $\mu(\overline{\Gamma_t}, p)$ be the Milnor number of the germ of the analytic curve $\overline{\Gamma_t}$ at p. As $\mu(\overline{\Gamma_t}, p)$ is upper semicontinuous in t [5], then for t sufficiently close to t_0, but $t \neq t_0$ the number

$$\lambda_p^{t_0}(f) = \mu(\overline{\Gamma_t}, p) - \mu(\overline{\Gamma_{t_0}}, p)$$

is well defined.

Definition 2. We denote

$$\lambda^t(f) = \sum_{p \in \overline{\Gamma_t} - \Gamma_t} \lambda_p^t(f) \qquad \lambda(f) = \sum_{t \in \mathbb{C}} \lambda^t(f).$$

Proposition 2.1. *For any t the numbers $\lambda^t(f)$ from definitions 1 and 2 coincide.*

The proof is given for example in [9].

Theorem 2.2. *If f is a polynomial with isolated critical points then the Euler characteristic of the fibre $f^{-1}(t)$ is given by*

$$\chi(f^{-1}(t)) = 1 - \mu(f) - \lambda(f) + \mu^t(f) + \lambda^t(f). \tag{3}$$

In the case where $f^{-1}(t)$ is a generic fibre ($\mu^t(f) = \lambda^t(f) = 0$) the above theorem is yet contained in [22] (see also [5], theorem 5.2). To prove the formula in general we use that for any t (see for example lemma 8 in [12])

$$\chi(\overline{\Gamma_t}) = 2 - (d-1)(d-2) + \sum_{p \in \overline{\Gamma_t}} \mu(\overline{\Gamma_t}, p).$$

Let $A_f \subset \mathbb{C}$ be the smallest set such that $f : \mathbb{C}^2 - f^{-1}(A_f) \to \mathbb{C} - A_f$ is a locally trivial fibration. Then by definition A_f is the set of *non-generic values* of t and it is often called a set of *atypical values*. Let A_c be the set of critical values of f,

$$A_c = \{t \in \mathbb{C} : \mu^t(f) > 0\}$$

and put

$$A_\infty = \{t \in \mathbb{C} : \lambda^t(f) > 0\}.$$

It is well known that A_f is a finite set (see [12, 8] for a discussion). The following theorem is due to Ha Huy Vui and Nguyen Le Anh who described completely the set of atypical values

Theorem 2.3 ([10, 11]). $A_f = A_c \cup A_\infty$.

Definition 3 ([5]). A polynomial $f : \mathbb{C}^n \to \mathbb{C}$ is called a 'tame' polynomial if there is a compact neighbourhood U of the critical points of f such that $\|grad(f)\|$ is bounded away from the origin on the set $\mathbb{C}^n - U$.

It is known that if f is a tame polynomial then it defines a 'Milnor fibration at infinity' and in particular $\lambda(f) = 0$ [5]. On the other hand, the class of polynomials defining a fibration at infinity is larger than the class of tame polynomials.

Definition 4 ([20]). A fibre $f^{-1}(t)$ is regular at infinity if there exists a neighbourhood D of t and a compact subset K of $\mathbb{C}^2$ such that

$$f^{-1}(D) - K \xrightarrow{f} D$$

is a locally trivial fibration. If all fibres of f are regular at infinity then we say that f is good.

Here is an equivalent definition of a good polynomial

Definition 5. f is good if and only if $\lambda(f) = 0$.

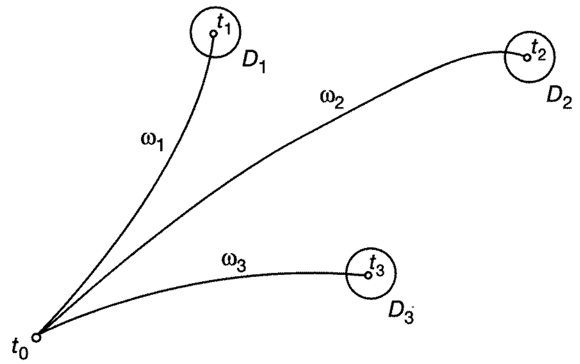

Figure 1.

Examples. All quadratic polynomials are tame. Up to linear changes of the independent and the dependent variables, the only non-good (and hence non-tame) cubic polynomial is $f(x, y) = y(xy + 1)$ (see [5, 6]). As f is not in the form (2) then we consider $\tilde{f}(x, y) = f(x + y, y)$. The discriminant of $\tilde{f}(x, y) - t$ with respect to y is

$$\Delta(x, t) = 4tx^3 + x^2 - 18tx - 27t^2 - 4$$

and hence $\lambda(f) = \lambda^0(f) = 1$, $\chi(f^{-1}(0)) = 1$, and $\chi(f^{-1}(t)) = 0$ for $t \neq 0$. As $\mu(f) = 0$ then for $t \neq 0$ the fibre $f^{-1}(t)$ has the homotopy type of a circle and hence $f^{-1}(t)$ is a Riemann sphere with two removed points.

Let $f = c(y^2 + x)^k + y$, $c \neq 0$, $k > 1$. As $\lambda(f)$, $\mu(f)$ are topological invariants and the bi-polynomial change of variables $x \to x + y^2$, $y \to y$ puts f into the form $cx^k + y$, then $\mu(f) = \lambda(f) = 0$. Thus f is a good polynomial but nevertheless it is not tame.

2.1. Vanishing cycles

According to theorem 2.2 the generic fibre $f^{-1}(t)$ of a polynomial with isolated critical points has the homotopy type of a bouquet of $\mu(f) + \lambda(f) = \dim H_1(f^{-1}(t), \boldsymbol{Z})$ circles. By analogy to the local case we may define $\mu(f) + \lambda(f)$ 'vanishing cycles' which form a base of $H_1(f^{-1}(t), \boldsymbol{Z})$. Namely, following [6], let $D_i \subset \mathbb{C}$ be small closed disks centred at the atypical points $t_i \in A_f$ of f, ω_i be continuous paths, nonintersecting except at t_0, connecting some fixed typical value $t_0 \notin A_f$ to t_i (figure 1). Denote $X_i = f^{-1}(D_i \cup \omega_i)$, $X_t = f^{-1}(t)$, $X = \cup_i X_i$. Define $V^i \subset H_1(X_{t_0}, \boldsymbol{Z})$ to be the kernel of the homomorphism

$$H_1(X_{t_0}, \boldsymbol{Z}) \to H_1(X_i, \boldsymbol{Z})$$

induced by the inclusion $X_{t_0} \to X_i$.

Theorem 2.4.

$$H_1(X_{t_0}, \boldsymbol{Z}) = \oplus_i V^i$$

where rang $V^i = \mu^{t_i}(f) + \lambda^{t_i}(f)$. *If* $\lambda^{t_i}(f) = 0$, *then* V^i *has a basis of*

$$\mu^{t_i}(f) = \sum_{p \in f^{-1}(t_i)} \mu_p(f)$$

1-cycles in the fibre $f^{-1}(t_0)$ *that vanish as* t *tends to* t_i *along the path* ω_i.

Proof. As X is a deformation retract of $\mathbb{C}^2$ then

$$H_1(X_{t_0}, \boldsymbol{Z}) \cong H_2(X, X_{t_0}) \tag{4}$$

and by the direct sum theorem we have the decomposition

$$H_2(X, X_{t_0}) = \oplus_i H_2(X_i, X_{t_0}). \tag{5}$$

Consider the long exact sequence associated with the pair (X_i, X_{t_0})

$$\cdots \to H_2(X_i, X_{t_0}) \xrightarrow{\partial^*} H_1(X_{t_0}, \boldsymbol{Z}) \to H_1(X_i, \boldsymbol{Z}) \to \cdots.$$

By (4) and (5) the map ∂^* is an injection and hence $H_1(X_{t_0}, \boldsymbol{Z}) = \oplus_i V^i$. Further, if $\lambda^{t_i}(f) = 0$ then the singular fibre $f^{-1}(t_i)$ is a deformation retract of X_i (the proof is the same as in the 'local' case, see [1]). It follows that there are exactly

$$\mu^{t_i}(f) = \sum_{p \in f^{-1}(t_i)} \mu_p(f)$$

one-cycles in the fibre $f^{-1}(t)$ that vanish as $t \to t_i$. Finally, if $\lambda^{t_i}(f) \neq 0$, then the formula for the rank of V^i is a by-product from the proof of theorem 3.1 in [21].

Remark. Note that if $\lambda^{t_i}(f) \neq 0$ then the singular fibre $f^{-1}(t_i)$ may not be a deformation retract of X_i. Nevertheless the notion 'vanishing cycle' still has a sense but the fibre $f^{-1}(t)$ should be replaced by its projective closure $\overline{f^{-1}(t)} \subset \mathbb{CP}^2$ (see [21] for details).

2.2. (λ, μ) constant deformations

Let $f \in \mathbb{C}_d[x, y]$ be a polynomial of degree d with isolated critical points, and consider a polynomial deformation $f_\theta \in \mathbb{C}_d[x, y]$ of f depending continuously on the parameter θ.

Definition 6. We shall say that f_θ is a (λ, μ) constant deformation provided that $\lambda = \lambda(f_\theta)$, $\mu = \mu(f_\theta)$ do not depend on θ.

Theorem 2.5. *Consider a (λ, μ) constant polynomial deformation f_θ, $0 \leqslant \theta \leqslant 1$, of the polynomial $f(x, y) = f_0(x, y)$, and suppose in addition that $0 \in \mathbb{C}$ is a typical value of f_θ for all θ. Then the fibration*

$$[0, 1] \times \mathbb{C}^2 \to [0, 1] : (\theta, f_\theta^{-1}(0)) \mapsto \theta$$

is trivial.

The above theorem claims that if two polynomials f_0, f_1 are connected by a (λ, μ) constant deformation, then their generic fibres are equivalent up to an isotopy. Note, however, that f_0 and f_1 may have different atypical points and values with different fibre numbers λ^t and μ^t and only the global numbers λ, μ are the same. Thus (in contrast to the local case [17]) f_0 and f_1 may not be topologically conjugate.

Proof of theorem 2.5. As $0 \in \mathbb{C}$ is a typical value of f_θ then the fibre numbers $\lambda^0(f_\theta)$, $\mu^0(f_\theta)$ are equal to zero and hence for t sufficiently small and all $\theta \in [0, 1]$ holds

$$\lambda^t(f_\theta) = \mu^t(f_\theta) = 0.$$

Without loss of generality we may suppose that the polynomial f_θ is written in the form

$$f_\theta(x, y) = \sum_{i=0}^{d} a_{d-i,\theta}(x) y^i$$

where $a_{d-i,\theta}(x)$ are polynomials in x of degree at most $d - i$ which depend continuously on the parameter θ. After an appropriate linear change of the variables x, y we may suppose that the leading coefficient $a_{0,0} = a_{0,0}(x)$ is a non-zero constant. To simplify the notations we shall also suppose that $a_{0,\theta} = a_{0,\theta}(x) \neq 0$ for all $\theta \in [0, 1]$. This is not a restriction as, to prove theorem 2.5, it is enough to establish the triviality of the fibration for θ sufficiently small. For such θ clearly holds $a_{0,\theta} \neq 0$.

Let

$$\Delta_\theta(t, x) = \sum_{i=0}^{N} \delta_{i,\theta}(t) x^i$$

be the discriminant of the polynomial $f_\theta(x, y) - t$ with respect to y. We have $\delta_{N,\theta}(0) \neq 0$ (definition 1) and hence there exists $c_0 > 0$, $\epsilon > 0$, such that if $\theta \in [0, 1]$, $|t| < \epsilon$, then $\Delta_\theta(t, x) \neq 0$ on the set $\{x \in \mathbb{C} : |x| \geqslant c_0\}$. Then an elementary calculation shows that for t, θ, c such that $\theta \in [0, 1]$, $|t| < \epsilon$, $c \geqslant c_0$ the cylinder

$$C_c = \{(x, y) \in \mathbb{C}^2 : |x| = c\}$$

is transverse to the smooth affine curve $\{(x, y) \in \mathbb{C}^2 : f_\theta(x, y) = t\}$. Thus $f_\theta^{-1}(t) \cap C_c$, $c \geqslant c_0$, is smooth and as $a_{0,\theta} \neq 0$ then $f_\theta^{-1}(t) \cap C_c$ is a finite unramified covering over the circle $\{x : |x| = c\}$. We conclude that for any $\theta \in [0, 1]$, $|t| < \epsilon$, $c \geqslant c_0$, the set $f_\theta^{-1}(t) \cap C_c$ is a finite disjoint union of circles and hence we obtain the following two proper submersions

$$f_\theta^{-1}(t) \cap \{(x, y) \in \mathbb{C}^2 : |x| \leqslant c_0\} \to (t, \theta) \qquad |t| < \epsilon,\ \theta \in [0, 1] \tag{6}$$

and

$$f_\theta^{-1}(t) \cap C_c \to (t, c) \qquad |t| < \epsilon,\ \theta \in [0, 1],\ c \geqslant c_0. \tag{7}$$

The Ehresmann fibration theorem implies that (6) and (7) are locally trivial fibrations. Tying them up together we obtain a locally trivial fibration

$$(\theta, f_\theta^{-1}(t)) \to (\theta, t) \qquad |t| < \epsilon,\ \theta \in [0, 1].$$

which implies the local triviality (and hence triviality) of the fibration

$$[0, 1] \times \mathbb{C}^2 \to [0, 1] : (\theta, f_\theta^{-1}(0)) \mapsto \theta.$$

3. Semiweighted homogeneous systems

A function $f : \mathbb{C}^n \to \mathbb{C}$ is called weighted homogeneous (wh) of weighted degree d and type $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$, $w_i = \text{weight}(x_i)$ if

$$f(t^{w_1} x_1, t^{w_2} x_2, \ldots, t^{w_n} x_n) = t^d f(x_1, x_2, \ldots, x_n) \qquad \forall t \in \mathbb{C}^*.$$

We shall also suppose that $d \geqslant 2w_i > 0$, $i = 1, 2, \ldots, n$. A polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is called semiweighted homogeneous (swh) of weighted degree d and type $\boldsymbol{w}$ if it can be written as $f = \sum_{i=0}^{d} f_i$, where f_i are wh polynomials of weighted degree i and type $\boldsymbol{w}$.

Definition 7. A swh polynomial $f = \sum_{i=0}^{d} f_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ of degree d and type $\boldsymbol{w}$ is called non-degenerated if its highest weighted homogeneous part f_d is a polynomial with isolated critical points.

Theorem 3.1. *A Morse critical point of a non-degenerate swh polynomial in two complex variables cannot be isochronous unless the polynomial is of degree two.*

The above theorem is a generalization of theorem 7.2 [18].

Example ([7, p 466]). Let $f = y^2 + V(x)$, $V(x) = x^2 + a_3x^3 + \cdots + a_nx^n$, $a_n \neq 0$. f is a non-degenerate swh polynomial and hence the Hamiltonian system

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x = -\frac{\mathrm{d}}{\mathrm{d}x}V(x)$$

is not isochronous.

To prove theorem 3.1 we need the following

Proposition 3.2. *Let* $f = \sum_{i=0}^{d} f_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ *be a non-degenerate swh polynomial of degree* d *and type* $\boldsymbol{w}$. *Then* f *is tame and its global Milnor number is given by the formula*

$$\mu(f) = \prod_{i=1}^{n}\left(\frac{\mathrm{d}}{w_i} - 1\right).$$

Proof. Consider the family of topological spheres

$$S_t = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : |x_1|^{1/w_1} + |x_2|^{1/w_2} + \cdots + |x_n|^{1/w_n} = t\}.$$

As f_d is a wh polynomial with isolated critical points then the spheres S_t are transversal to its fibres $f_d^{-1}(t)$ for $t > 0$. In particular there exists $c > 0$ such that on the compact set S_1 holds

$$\max_i \left|\frac{\partial}{\partial x_i} f_d\right| \geqslant c$$

and hence on the sphere S_t we have

$$\max_i |\frac{\partial}{\partial x_i} f_d| \geqslant ct^{d-w_0} \qquad w_0 = \max\{w_1, w_2, \ldots, w_n\}. \tag{8}$$

Consider now the polynomial deformation

$$f_\theta = f_d + \theta f' \qquad f' = \sum_{i=0}^{d-1} f_i \qquad 0 \leqslant \theta \leqslant 1.$$

If

$$C = \max_{i, x \in S_1} \left|\partial f'/\partial x_i\right|$$

then on S_t holds

$$\left|\frac{\partial}{\partial x_i} f'\right| \leqslant Ct^{d-1-w_i}. \tag{9}$$

Comparing (8) with (9) we conclude that there exists $t_0 > 0$ such that for any $\theta \in [0, 1]$ and $t \geqslant t_0$ the function f_θ has no critical points on S_t. This shows that f_θ (and hence f) is tame. On the other hand, the global Milnor number $\mu(f_\theta)$ is the degree of the map [19]

$$(x_1, x_2, \ldots, x_n) \to \frac{\nabla f_\theta}{\|\nabla f_\theta\|} \qquad \nabla f_\theta = (\partial f_\theta/\partial x_1, \ldots, \partial f_\theta/\partial x_n) \qquad x \in S_t$$

for t sufficiently large and hence

$$\mu(f) = \mu(f_\theta) = \mu(f_d).$$

Finally the global Milnor number $\mu(f)$ of a wh polynomial with isolated critical points is easily computed by the Poincaré series of the corresponding gradient map ∇f_d (see for instance [4, p 104]). $\square$

Remark. The above result can be considered as a special case of Kushnirenko's theorem [14]. Suffice it to note that a non-degenerate swh polynomial is also non-degenerate with respect to its Newton boundary (at infinity). Moreover, as f is tame, then without loss of generality it may be also supposed convenient (in the sense of [14]). Thus, the global Milnor number of a non-degenerate swh polynomial coincides with its Newton number.

Proof of theorem 3.1. Let $f \in \mathbb{C}[x_1, x_2]$ be a non-degenerate swh polynomial of type $\boldsymbol{w} = (w_1, w_2)$ and degree d, $f = \sum_{i=0}^{d} f_i$, f_i—weighted homogeneous of weighted degree i. Consider the polynomial deformation

$$g_t(x_1, x_2) = (f(x_1 t^{w_1/d}, x_2 t^{w_2/d}) - t)/t = f_d(x_1, x_2) + t^{-1/d} f_{d-1}(x_1, x_2) + \cdots + t^{-1} f_0 - 1.$$

defined for $t \in [1, \infty]$, $g_1(x_1, x_2) = f(x_1, x_2) - 1$, $g_\infty(x_1, x_2) = f_d(x_1, x_2) - 1$. According to proposition 3.2 the polynomial g_t is tame (so $\lambda(g_t) = 0$) and $\mu(g_t) = \mu(f_d)$ is a constant in t. Thus, theorem 2.5 applies and the fibration

$$[t_0, \infty] \times \mathbb{C}^2 \to [t_0, \infty] : (t, g_t^{-1}(0)) \to t \tag{10}$$

is trivial, provided that $0 \in \mathbb{C}$ is a typical value of g_t for all $t \geqslant t_0$. Clearly the last condition is satisfied for t_0 sufficiently large.

Let $\tilde{\gamma}(t_0) \in H_1(g_{t_0}^{-1}(0), \boldsymbol{Z})$ be any cycle. Trivializing the fibration (10) we obtain a continuous family of cycles $\tilde{\gamma}(t) \in H_1(g_t^{-1}(0), \boldsymbol{Z})$ defined for all $t \geqslant t_0$. Denote by $\gamma(t)$ the image of the cycle $\tilde{\gamma}(t)$ in $H_1(f^{-1}(t), \boldsymbol{Z})$ under the map

$$(x_1, x_2) \to (x_1 t^{-w_1/d}, x_2 t^{-w_2/d}).$$

To compute the asymptotic behaviour of any Abelian integral along the cycle $\gamma(t)$ we have just to change the variables. In particular consider the 'area' function $S(t)$. We have

$$S(t) = \int_{\gamma(t)} x_2 \, \mathrm{d}x_1 = \int_{\tilde{\gamma}(t)} t^{w_2/d} x_2 \, \mathrm{d}t^{w_1/d} x_1 = t^{(w_1+w_2)/d} \int_{\tilde{\gamma}(t)} x_2 \, \mathrm{d}x_1.$$

As $\int_{\tilde{\gamma}(\infty)} x_2 \, \mathrm{d}x_1$ is well defined and finite then for t sufficiently large and some non-zero constant c holds

$$|S(t)| \leqslant c t^{(w_1+w_2)/d}.$$

If the period function $T(t) = \int_{\gamma(t)} \mathrm{d}x_1/f_{x_2} = S'(t)$ is identically a constant, say 2π, then $S(t) = 2\pi t$ and hence $w_1 + w_2 \geqslant d$. On the other hand, we suppose that $w_1, w_2 \leqslant d/2$. We conclude that $w_1 = w_2 = d/2$ and the polynomial $f = \sum_{i=0}^{d} f_i$ is of (non-weighted) degree at most two. $\square$

4. Systems with simple critical points

We recall that an isolated singularity of a germ of a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is called simple (or du Val singularity, or rational double point) if its modality is 0 [1]. Such singularities are classified according to the Coxeter groups A_k, D_k, E_6, E_7, E_8 (i.e. according to regular polyhedra in $\mathbb{R}^3$).

Theorem 4.1. *Let $f \in \mathbb{C}[x, y]$ be a good polynomial having only simple singularities. If a critical level set of f contains a single critical point which is Morse and isochronous, then the corresponding vanishing cycle represents a zero homology cycle on the Riemann surface of the algebraic curve $f^{-1}(t)$.*

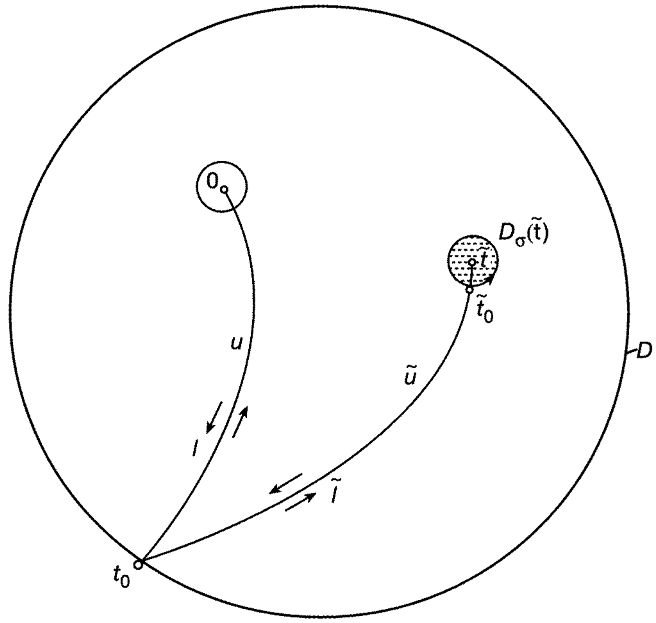

Figure 2.

Proof. Suppose that $(0, 0)$ is an isolated isochronous Morse critical point of the good polynomial $f = (x^2 + y^2)/2 + \cdots$. Let us suppose that the associated cycle $\gamma(t) \in H_1(f^{-1}(t), \boldsymbol{Z})$ vanishing at $(0, 0)$ as $t \to 0$ is not homologous to zero on the Riemann surface of $f^{-1}(t)$. This implies that the genus of this surface is at least one and hence one may always find another non-zero cycle $\tilde{\gamma}(t) \in H_1(f^{-1}(t), \boldsymbol{Z})$ having a non-zero intersection number with $\gamma(t)$. Further, according to theorem 2.4, we may suppose that $\tilde{\gamma}(t)$ is a vanishing cycle. Namely, let $\tilde{t}$ be the critical value corresponding to $\tilde{\gamma}(t)$. Choose a disk $D \subset \mathbb{C}$ centred at the origin and containing the set of critical points A_c. Let $t_0 \in \partial D$ be a non-critical point, and u and $\tilde{u}$ be non-intersecting paths in D connecting t_0 and the critical values $t = 0$ and $t = \tilde{t}$, and $l, \tilde{l} \in \pi_1(D - A_c, t_0)$ be loops correspond to u and $\tilde{u}$ as on figure 2. Further we shall suppose that $\gamma(t)$ $(\tilde{\gamma}(t))$ is a cycle vanishing along the path u $(\tilde{u})$ as $t \to 0$ $(t \to \tilde{t})$, and that the intersection number $(\gamma(t_0) \circ \tilde{\gamma}(t_0))$ is non-zero. Note that we do not suppose that $f^{-1}(\tilde{t})$ contains a single critical point.

Trivializing the fibration $f^{-1}(t) \to t$ along l and $\tilde{l}$ we obtain homeomorphisms

$$h_l, h_{\tilde{l}} : f^{-1}(t_0) \to f^{-1}(t_0)$$

which induce automorphisms

$$h_{l*}, h_{\tilde{l}*} : H_1(f^{-1}(t_0), \boldsymbol{Z}) \to H_1(f^{-1}(t_0), \boldsymbol{Z}).$$

We claim that the intersection number $(h_{\tilde{l}*}\gamma(t_0) \circ \gamma(t_0))$ is not zero. Assuming that it is not difficult to prove theorem 4.1. Indeed, let $\tilde{\delta}(t)$, $\delta(t)$ be a continuous family of cycles in the fibre $f^{-1}(t)$ defined for $t \sim t_0$ and such that

$$\tilde{\delta}(t_0) = h_{\tilde{l}*}\gamma(t_0), \delta(t_0) = h_{l*} \circ h_{\tilde{l}*}\gamma(t_0).$$

Then the Picard–Lefschetz formula reads

$$\delta(t_0) = h_{l*}\tilde{\delta}(t_0) = \tilde{\delta}(t_0) - (\tilde{\delta}(t_0) \circ \gamma(t_0))\gamma(t_0)$$

and hence for $t \sim t_0$

$$\delta(t) = \tilde{\delta}(t) - (\tilde{\delta}(t_0) \circ \gamma(t_0))\gamma(t)$$

or equivalently

$$\int_{\delta(t)} \omega = \int_{\tilde{\delta}(t)} \omega - (\tilde{\delta}(t_0) \circ \gamma(t_0)) \int_{\gamma(t)} . \tag{11}$$

On the other hand, the analytic continuation of the period function $T(t)$ along the loops $\tilde{l}$ and l gives (for $t \sim t_0$)

$$T(t) = \int_{\gamma(t)} \omega = \int_{\tilde{\delta}(t)} \omega = \int_{\delta(t)} \omega \equiv 2\pi$$

which, combined with (11) implies that the intersection number

$$(h_{\tilde{l}*}\gamma(t_0) \circ \gamma(t_0)) = (\tilde{\delta}(t_0) \circ \gamma(t_0))$$

is zero.

Finally to show that the above intersection number is non-zero we use the fact that the quadratic form of simple singularity is negative definite [1]. More precisely, let $\tilde{\gamma}(t)$ be a cycle vanishing at the critical point $(\tilde{x}, \tilde{y})$ as $t \to \tilde{t}$. For simplicity, suppose first that $(\tilde{x}, \tilde{y})$ is the only critical point contained in the fibre $f^{-1}(\tilde{t})$. Denote by f_t the local Milnor fibre of the singularity

$$f : (\mathbb{C}^2, (\tilde{x}, \tilde{y})) \to (\mathbb{C}, \tilde{t}).$$

This means that $f_t = f^{-1}(t) \cap B_\epsilon(\tilde{x}, \tilde{y})$, $t \in D_\delta(\tilde{t})$, where

$$B_\epsilon(\tilde{x}, \tilde{y}) = \{(x, y) \in \mathbb{C}^2 : |x - \tilde{x}|^2 + |x - \tilde{y}|^2 \leqslant \epsilon\} \qquad D_\delta(\tilde{t}) = \{t \in \mathbb{C} : |t - \tilde{t}| \leqslant \delta\}$$

and $0 < \delta \ll \epsilon \ll 1$. The fibration

$$(f_t, \partial f_t) \to t \qquad t \in D_\delta(\tilde{t}) - \tilde{t} \tag{12}$$

is locally trivial. We may suppose that the path $\tilde{u}$ is transversal to $\partial D_\delta(\tilde{t})$. Then the path $\tilde{u} \cap D_\delta(\tilde{t})$ connecting $\tilde{t}_0 = \tilde{u} \cap \partial D_\delta(\tilde{t})$ and $\tilde{t}$ defines a loop $\tilde{l}' \in \pi_1(D_\delta(\tilde{t}) - \tilde{t}, \tilde{t}_0)$. Trivializing the fibration (12) along $\tilde{l}'$ we obtain a smooth map (monodromy)

$$h_{\tilde{l}'} : f_{\tilde{t}_0} \to f_{\tilde{t}_0}$$

which on its turn induces an automorphism (monodromy operator)

$$h_{\tilde{l}'*} : H_1(f_{\tilde{t}_0}, \boldsymbol{Z}) \to H_1(f_{\tilde{t}_0}, \boldsymbol{Z})$$

and a homomorphism (variation operator)

$$\mathrm{var}_{\tilde{l}'} = \mathrm{Var}_f : H_1(f_{\tilde{t}_0}, \partial f_{\tilde{t}_0}) \to H_1(f_{\tilde{t}_0}, \boldsymbol{Z}).$$

Consider also the natural 'restriction' homomorphism

$$H_1(f^{-1}(\tilde{t}_0), \boldsymbol{Z}) \to H_1(f_{\tilde{t}_0}, \partial f_{\tilde{t}_0}) : \gamma(\tilde{t}_0) \to \gamma_r(\tilde{t}_0)$$

which maps a cycle in the global fibre $f^{-1}(\tilde{t}_0)$ to its 'part' lying in the local Milnor fibre $f_{\tilde{t}_0}$. We have now

$$(h_{\tilde{l}*}\gamma(t_0) \circ \gamma(t_0)) = (h_{\tilde{l}'*}\gamma(\tilde{t}_0) \circ \gamma(\tilde{t}_0)) = (\mathrm{Var}_f\, \gamma_r(\tilde{t}_0), \gamma_r(\tilde{t}_0)) = S(\mathrm{Var}_f\, \gamma_r(\tilde{t}_0), \mathrm{Var}_f\, \gamma_r(\tilde{t}_0))$$

where

$$S : (H_1(f_t, \boldsymbol{Z}), H_1(f_t, \boldsymbol{Z})) \to \boldsymbol{Z}$$

is the Seifert bilinear form [1] of the singularity

$$f : (\mathbb{C}^2, (\tilde{x}, \tilde{y})) \to (\mathbb{C}, \tilde{t}).$$

As the quadratic form $Q(a, b) = S(a, b)+S(b, a)$ is negative definite then $(h_{\tilde{l}*}\gamma(t_0)\circ\gamma(t_0))$ is not zero unless $\mathrm{Var}_f\, \gamma_r(\tilde{t}_0) \in H_1(f_{\tilde{t}}, \boldsymbol{Z})$ is homologous to zero. But the variation operator is in fact an isomorphism and the relative homology group $H_1(f_{\tilde{t}_0}, \partial f_{\tilde{t}_0})$ is identified canonically with the dual group $(H_1(f_{\tilde{t}}, \boldsymbol{Z}))^*$ via the intersection form. Thus, it remains to prove that $\gamma_r(\tilde{t}_0)$ is not in the kernel of the intersection form on $H_1(f_{\tilde{t}_0}, \boldsymbol{Z})$. As $(\gamma(t_0) \circ \tilde{\gamma}(t_0))$ is not zero then $(\gamma_r(\tilde{t}_0) \circ \tilde{\gamma}(\tilde{t}_0)) \neq 0$ and hence $(h_{\tilde{l}*}\gamma(t_0) \circ \gamma(t_0)) \neq 0$.

Suppose at last that the critical level set $f^{-1}(\tilde{t})$ contains several critical points $(\tilde{x}^i, \tilde{y}^i)$. We associate to each critical point $(\tilde{x}^i, \tilde{y}^i)$ a local Milnor fibre f_t^i and a variation operator

$$\mathrm{Var}^i_f : H_1(f^i_{\tilde{t}_0}, \partial f^i_{\tilde{t}_0}) \to H_1(f^i_{\tilde{t}_0}, \boldsymbol{Z}).$$

If $\gamma(\tilde{t}_0) \in H_1(f_{\tilde{t}_0}, \boldsymbol{Z})$ then we denote, as before, by $\gamma^i_r(\tilde{t}_0)$ its 'part' lying in the local Milnor fibre f_t^i. We have

$$\begin{aligned}(h_{\tilde{l}*}\gamma(t_0) \circ \gamma(t_0)) &= \left(\sum_i \mathrm{Var}^i_f\, \gamma^i_r(\tilde{t}_0), \sum_i \gamma^i_r(\tilde{t}_0)\right) = \sum_i (\mathrm{Var}^i_f\, \gamma^i_r(\tilde{t}_0), \gamma^i_r(\tilde{t}_0)) \\ &= \sum_i S^i(\mathrm{Var}^i_f\, \gamma^i_r(\tilde{t}_0), \mathrm{Var}^i_f\, \gamma^i_r(\tilde{t}_0))\end{aligned}$$

where S^i is the Seifert bilinear form of the singularity

$$f : (\mathbb{C}^2, (\tilde{x}^i, \tilde{y}^i)) \to (\mathbb{C}, \tilde{t}).$$

The same argument as before shows that $(h_{\tilde{l}*}\gamma(t_0) \circ \gamma(t_0)) < 0$ which completes the proof of theorem 4.1. □

5. Examples

In this section $f \in \mathbb{C}^2[x, y]$ will be a polynomial with a Morse critical point at the origin and $\gamma(t)$ will be a one-cycle in the fibre $f^{-1}(t)$ that vanish at the origin as t tends to 0. We denote by $S(t)$ the area function $\int_{\gamma(t)} y\,\mathrm{d}x$ and by $T(t)$ the period function $\int_{\gamma(t)} \omega$, $\omega = \mathrm{d}x/f_y$. The generic level set $f^{-1}(t)$ and the vanishing cycle $\gamma(t)$ in the examples that follow are shown in figure 3.

An algebraic automorphism of $\mathbb{C}^2$ is a bi-polynomial map $(x, u) \to (u, v)$. Clearly $\mathrm{d}x \wedge \mathrm{d}y = c\,\mathrm{d}u \wedge \mathrm{d}v$ for some non-zero constant c that may be supposed equal to 1. Now if we put $f(x, y) = ((u(x, y) - u(0, 0))^2 + (v(x, y) - v(0, 0))^2)/2$, then the canonical change of variables

$$(x, y) \to (u(x, y) - u(0, 0), v(x, y) - v(0, 0))$$

transforms the Hamiltonian system

$$\begin{aligned}\dot{x} &= \partial f/\partial y \\ \dot{y} &= -\partial f/\partial x\end{aligned}$$

into a linear one $\dot{u} = v, \dot{v} = -u$. As a time-independent inversible (complex) change of variables preserves the isochronicity of a centre (Morse critical point), we conclude that any centre of the initial Hamiltonian system is isochronous too. The first natural guess is that in this way we obtain all polynomial Hamiltonian isochronous systems. The following example shows that it is not so.

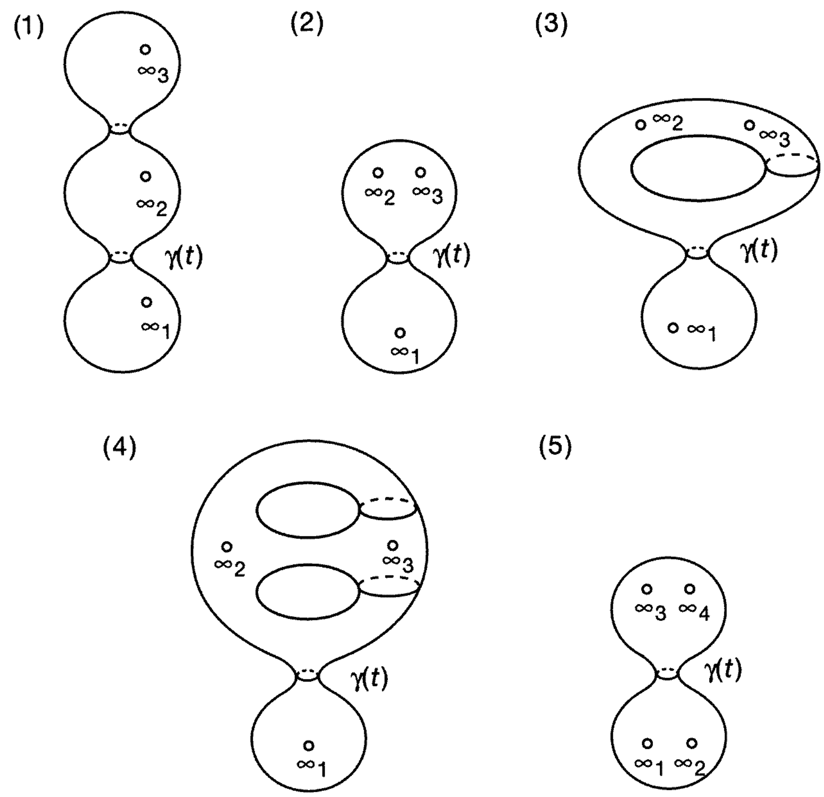

Figure 3.

Example 1. The generic fibre of the polynomial $f = yx(x-1)$ is $\mathbb{C}^{**}$ (Riemann sphere with three removed points) and hence there is no algebraic automorphism that puts f into the form x^2+y^2. On the other hand, f has an isochronous Morse critical point at the origin. Indeed, its period function is given by the residue at the origin of the form $\mathrm{d}x/x(x-1)$.

Example 2. The cubic polynomial $f = x(xy+x+y)$ has an isochronous Morse critical point at the origin. To check the above assertions we compute that f is a good polynomial ($\lambda(f) = 0$) with a global Milnor number $\mu(f) = 2$. The fibre $f^{-1}(0)$ has two components: $\mathbb{C}$ (Riemann sphere with a removed point) and $\mathbb{C}^*$ (Riemann sphere with two removed points) with one common point which is a normal crossing. We have

$$S(t) = \int_{\gamma(t)} y\,\mathrm{d}x = 2\pi\sqrt{-1}\,\mathrm{Re}\,s_{\infty_1}(y\,\mathrm{d}x) = 2\pi\sqrt{-1}\,\mathrm{Re}\,s_0\frac{t-x^2}{x(1+x)} = 2\pi\sqrt{-1}t$$

and hence $T(t) = S'(t) = 2\pi\sqrt{-1} = \text{constant}$.

The next guess may be that if a polynomial has an isochronous Morse critical point then its generic fibre is a Riemann sphere with several removed points. The next examples, suggested by E Artal and I Luengo, show that it is not so.

Example 3. The polynomial $f = y(x^2y^2+x+y)$ has an isochronous Morse critical point at the origin. Indeed, $\lambda(f) = 0$, $\mu(f) = 4$, and the generic fibre of f is a genus one

Riemann surface with three removed points. As above

$$S(t) = \int_{\gamma(t)} x\,\mathrm{d}y = 2\pi\sqrt{-1}\,\mathrm{Re}\,s_{\infty_1}(x\,\mathrm{d}y) = 2\pi\sqrt{-1}t.$$

Example 4. The polynomial $f = y((y+x^2)y^2 + x - y)$ has an isochronous Morse critical point at the origin. We check that $\lambda(f) = 0$, $\mu(f) = 6$, and the generic fibre of f is a genus two Riemann surface with three removed points. As before the area function $S(t)$ is the residue of $x\,\mathrm{d}y$ at ∞_1 and hence it equals to $2\pi\sqrt{-1}t$.

In the above examples the vanishing cycle is homologous to zero on the Riemann surface of the fibre $f^{-1}(t)$, as it was conjectured in the introduction. The next example shows, however, that it does not guarantee the isochronicity.

Example 5. The generic fibre of the polynomial $f = (x^2 + y^2(x+1)^2)/2$ is a Riemann sphere with four removed points. Nevertheless the Morse critical point $(0,0)$ of f is not isochronous. Indeed

$$T(t) = \int_{\gamma(t)} \frac{\mathrm{d}x}{f_y} = \int_{\gamma(t)} \frac{\mathrm{d}x}{y(x+1)^2} = 2\pi\sqrt{-1}\,\mathrm{Re}\,s_{\infty_1}\frac{\mathrm{d}x}{y(x+1)^2} = \frac{2\pi}{\sqrt{1-t}}.$$

It is seen that the period function is not single-valued. Thus, although the intersection form on $H_1(f^{-1}(t), \boldsymbol{Z})$ is identically zero, the cycle $\gamma(t)$ has a monodromy. This is explained by the fact that f is not good. We have $\mu(f) = 1$, $\lambda^1(f) = \lambda(f) = 2$.

6. Isochronous systems and the Jacobian conjecture

Jacobian conjecture *any polynomial canonical map*

$$\mathbb{C}^2 \to \mathbb{C}^2 : (x, y) \mapsto (u, v) \qquad \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}u \wedge \mathrm{d}v$$

is globally inversible.

The Jacobian conjecture was first formulated by O H Keller in 1939 (see [3] for a survey). It is, in fact, equivalent to prove that the map $(x, y) \mapsto (u, v)$ is injective. We shall show, however, that this is not compatible to the conclusion of theorem 4.1.

Let $(x, y) \mapsto (u, v)$ be a polynomial map, such that $\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}u \wedge \mathrm{d}v$, and put

$$f(x, y) = (u^2(x, y) + v^2(x, y))/2.$$

We may also suppose that $u(0,0) = 0$, $v(0,0) = 0$ so the origin in $\mathbb{C}^2$ is an isochronous Morse critical point of f (section 5). Denote, as usual, by $\gamma(t)$ a continuous family of one-cycles contained in the fibre $f^{-1}(t)$ and vanishing at the origin as $t \to 0$.

Proposition 6.1. *If the map $(x, y) \mapsto (u, v)$ is not injective, then $\gamma(t)$ represents a non-zero homology cycle on the Riemann surface of the algebraic curve $f^{-1}(t)$.*

The above proposition raises the natural question (asked in the introduction), whether there are isochronous Morse critical points with non-zero homology cycle. A negative answer would imply the injectivity of the map $(x, y) \mapsto (u, v)$, and hence the Jacobian conjecture.

Proof of proposition 6.1. Suppose that the map $(x, y) \mapsto (u, v)$ is not injective, that is to say there are two distinct points (x_0, y_0) and (x_1, y_1) such that $u(x_0, y_0) = u(x_1, y_1) = 0$, $v(x_0, y_0) = v(x_1, y_1) = 0$. Without loss of generality we shall put $(x_0, y_0) = (0, 0)$. Suppose that the vanishing cycle $\gamma(t)$ associated to $(0,0)$ is homologous to zero on the compactified algebraic curve $f^{-1}(t)$, or equivalently, the intersection number $(\gamma(t) \circ \alpha(t))$ is zero for any cycle $\alpha(t) \in H_1(f^{-1}(t), \boldsymbol{Z})$. We shall prove that this leads to a contradiction.

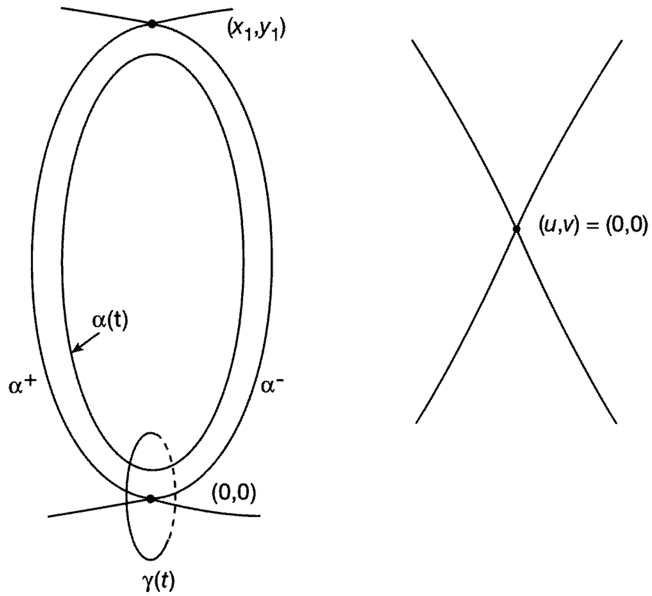

Figure 4.

As u, v are polynomials with isolated critical points (in fact they have no critical points at all) then their generic level sets are smooth and irreducible. Further without loss of generality we shall suppose that the affine curves $u(x, y) + \sqrt{-1}v(x, y) = 0$ and $u(x, y) - \sqrt{-1}v(x, y) = 0$ are smooth and irreducible (and hence connected). Let α^+ (α^-) be a continuous path on the curve $u(x, y) + \sqrt{-1}v(x, y) = 0$ ($u(x, y) - \sqrt{-1}v(x, y) = 0$) connecting the two points (x_0, y_0) and (x_1, y_1). We may suppose that the only intersection points of α^+ and α^- are their ends and denote $\alpha = \alpha^+ \cup \alpha^-$. We claim that by continuity the closed loop α defines, for all sufficiently small t, a closed loop $\alpha(t) \subset f^{-1}(t)$, $\alpha(0) = \alpha$. Indeed $f = (u + \sqrt{-1}v)(u - \sqrt{-1}v)/2$ and it suffice to define $\alpha(t)$ in a neighbourhood of the Morse critical points (x_0, y_0) ((x_1, y_1)). We may suppose that $(x_0, y_0) = (0, 0)$, $u + \sqrt{-1}v = x$, $u - \sqrt{-1}v = y$, $t \in \mathbb{R}$, $t > 0$, and define in a neighbourhood of $(0, 0)$ the loop $\alpha(t)$ to be the real curve $xy = 2t$ where $x, y \geqslant 0$. As $t \to 0$ the loop $\alpha(t)$ tends to $\alpha^+ \cup \alpha^-$, where α^+ is defined by $x = 0$, $y \geqslant 0$, and α^- by $y = 0$, $x \geqslant 0$. The loop $\alpha(t)$ is shown on figure 4.

We note finally that if

$$\gamma(t) = \{x = \sqrt{2t}\,\exp^{i\varphi}, y = \sqrt{2t}\,\exp^{-i\varphi}, \varphi \in [0, 2\pi]\}$$

is a cycle vanishing at $(0, 0)$ as $t \to 0$, then the intersection number $(\alpha(t) \circ \gamma(t))$ equals to ± 1 and we arrived at a contradiction. $\square$

Acknowledgments

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