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ON THE TOPOLOGY OF POLYNOMIALS IN TWO COMPLEX VARIABLES

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On the topology of polynomials in two complex variables

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Abstract

Let $f:\mathbb{C}^2 - \mathbb{C}$ be a polynomial map with isolated critical points. We describe the Euler characteristic of its fiber $f^{-1}(t)$ in terms of $\mu(f)$ and $\rho(f)$, where $\mu(f)$ is the global Milnor number of f and $\rho(f)$ is another topological invariant which counts the non-bounded ramification points of the curve $f^{-1}(t)$ as t varies.

f defines a trivial fibration at infinity if and only if $\rho(f)=0$ and we show that in this case the first homology group of the general fiber $f^{-1}(t)$ has a distinguished basis of vanishing cycles. As a simplest example we compute the Dynkin diagrams of cubic polynomials.

Contents

1 Introduction

Let $f:\mathbb{C}^n \to \mathbb{C}$ be a holomorphic map in a neighborhood of $0 \in \mathbb{C}^n$, $f(0)=0$. It is well known that the local fiber $f^{-1}(t)$ has the homotopy type of a bouquet of μ spheres of dimension $n-1$ [11]. The Milnor number $\mu_0(f)$ of f at $0 \in \mathbb{C}^n$ may be defined as the number of cycles of dimension $n-1$ in the fiber $f^{-1}(t)$ that vanish at 0 as $t \to 0$. In the case when f is a polynomial, by analogy with the "local" case, the middle homology of the general global fiber $f^{-1}(t)$ is a direct sum of vanishing homologies corresponding to the atypical points of f (see [4], Proposition 3). On the other hand the vanishing homology is also important as the monodromy of the fibers $f^{-1}(t)$ may be determined in terms of vanishing cycles and their intersection numbers (Picard-Lefschetz formula [1]). For arbitrary n and for a large class of polynomials (the so called tame polynomials) the vanishing homology is studied by Broughton in [3,4]. It turns out that $H_{n-1}(f^{-1}(t),\mathbf{Z}) = \mathbf{Z}^{\mu(f)-\mu^t(f)}$, $H_i(f^{-1}(t),\mathbf{Z}) = 0$, $i \neq n-1$, where

$$\mu(f) = \sum_p \mu_p(f)\,,\ \mu^t(f) = \sum_{p\in f^{-1}(t)} \mu_p(f)$$

are the global and the fiber Milnor numbers of f.

The study of the vanishing homology in general turns out to be a difficult (still unsolved) question. It is well known that the number of values for t (called atypical) such that $f : \mathbb{C}^n \to \mathbb{C}$ is not locally trivial over t is finite (see [10,5] for a discussion). It is not clear, however, how to determine these points except in the simplest case n=2. According to a result of Ha and Nguyen [8,9] if the fibration $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$, $f \in \mathbb{C}[x,y]$, is not locally trivial over $t_0 \in \mathbb{C}$, then either t_0 is a critical value, or there is a ramification point of the projection $\{(x,y) \in \mathbb{C}^2 : f(x,y) = t\} \to x$ which tends to infinity as $t \to t_0$. Using this and Broughton's results [3,4], we give a description of the Euler characteristic of the fiber $f^{-1}(t), n = 2$, and any $t \in \mathbb{C}$, in the case of a polynomial with isolated critical points ($\mu(f) < \infty$). This is our main result and it is formulated in Theorem 3.3 of section 3.

The paper is organized as follows. In section 2 we give some formulae which will be used in the proof of Theorem 3.3. In particular we explain how to compute $H_1(\Gamma,\mathbf{Z})$ for any smooth affine plane curve (Corollary 2.3) . In section 3 we define a new topological invariant $\rho(f)$ of a polynomial f, similar to the Milnor number $\mu(f)$. The fibre $f^{-1}(t)$ is then described in terms of these two numbers. If $\rho(f) = 0$ then the polynomial f is good in the sense that it defines a trivial fibration at infinity. In this case the description of $f^{-1}(t)$ coincides with the one of Broughton [3,4] but nevertheless the class of good polynomials is larger than the class of tame polynomials.

In section 4 we apply our main result Theorem 3.3 to classification of polynomials. We prove that, if two polynomials belong to the same connected component of the set $\mathcal{A}_{\mu,\rho}$ of degree n polynomials with fixed $\rho(f) = \rho$, $\mu(f) = \mu$, then their general fibres are equivalent up to a proper isotopy. In the case of a good polynomial we define a distinguished basis of vanishing cycles and prove that it generates the first homology group of the general fibres. As a simplest example we study the space of cubic polynomials and describe the corresponding Dynkin diagrams (table 1).

2 The homology of a smooth affine plane curve

Suppose that $\overline{\Gamma}$ and $\overline{\Gamma}'$ are compact Riemann surfaces and let

$$\pi : \overline{\Gamma} \to \overline{\Gamma}'$$

be a non-constant holomorphic mapping with mapping degree $deg(\pi) = n$. Consider a closed set $S \subset \overline{\Gamma}'$ such that its boundary ∂S is homeomorphic to a finite disjoint union of circles and points, and define

$$\Gamma' = \overline{\Gamma}' - S \, , \; \Gamma = \overline{\Gamma} - \pi^{-1}(S) \, .$$

For any $p \in \Gamma$ let $v(p)$ be the multiplicity of π at p. The Euler characteristics $\chi(\Gamma)$, $\chi(\Gamma')$ of Γ and Γ' are related by the following

Theorem 2.1 *(Riemann-Hurwitz formula)*

$$\chi(\Gamma) = n\chi(\Gamma') - \sum_{p \in \Gamma} (v(p) - 1) \, .$$

Proof. If $S = \emptyset$ then Γ and Γ' are compact and this is the usual Riemann-Hurwitz formula. If $S \neq \emptyset$ we may always find an open neighborhood S_ϵ of S in $\overline{\Gamma}'$ and such that

- ∂S_ϵ is a disjoint union of circles
- there are no ramification points of π in the open set $S_\epsilon - S$
- $\Gamma'_\epsilon = \overline{\Gamma}' - S_\epsilon$ is a deformation retract of Γ'.

It follows that $\Gamma_\epsilon = \overline{\Gamma} - \pi^{-1}(S_\epsilon)$ will be a deformation retract of Γ and we have a well defined holomorphic map

$$\pi : \Gamma_\epsilon \rightarrow \Gamma'_\epsilon$$

between bordered closed surfaces. The same proof as in the case $S = \emptyset$ (see for example [7]) shows that

$$\chi(\Gamma_\epsilon) = n\chi(\Gamma'_\epsilon) - \sum_{p \in \Gamma_\epsilon} (v(p) - 1) \, . \square$$

Consider a smooth irreducible affine curve $\Gamma = \{(x, y) \in \mathbb{C} : f(x, y) = 0\}$. We shall use the Riemann-Hurwitz formula to compute the first homology group $H_1(\Gamma) = H_1(\Gamma, Z)$ in terms of ramification points.

Definition 1 *A linear function l is general with respect to f provided that for any $c \in \mathbf{C}$ the intersection index of the straight line $\{l = c\} \subset \mathbf{C}$ and the affine curve Γ is exactly $n = deg(H)$.*

To each non-constant linear function l corresponds a direction $\{l(x, y) = l(0, 0)\}$ in $\mathbb{C}^2$ and there are exactly $n = deg(f)$ such directions corresponding to non-general linear functions.

Consider the projection

$$\pi : \Gamma \rightarrow \mathbb{C} : (x, y) \rightarrow l(x, y).$$

where l is a general linear function and for any $z \in \Gamma$ let $v(z)$ be the ramification index of π.

Proposition 2.2 $dim(H_1(\Gamma)) = 1 - n + \sum_{z \in \Gamma} (v(z) - 1)$.

The above proposition implies that the number of ramification points $\sum_{z \in \Gamma}(v(z) - 1)$ does not depend on the general projection π. Let $f(x, y) = a_0 y^n + a_1 y^{n-1} + \ldots + a_n$ where $a_i = a_i(x) \in \mathbb{C}[x]$ and $deg(a_i(x)) \leq i$. The linear function x is general if and only if $a_0 \neq 0$. Without loss of generality we may suppose (after a suitable linear change of variables) that $l = x$. Then, as it is easily seen, the number of ramification points equals to the degree in x of the discriminant $\Delta_f(x)$ of f with respect to y. On the other hand $deg_x \Delta_f(x) \leq n(n - 1)$ and we get

Corollary 2.3

$$dim\, H_1(\Gamma) = 1 - deg(f) + deg_x\Delta_f(x) \leq (n-1)^2.$$

Examples

- Let $f = \frac{(y^2-x)^2}{2} - x$. As x is general with respect to f, and the discriminant $\Delta_f(x)$ of f with respect to y is $-16x^3$ then $dimH_1(\Gamma) = 1 - 3 + 3 = 1$. The curve $\Gamma = \{f = 0\}$ is a Riemann sphere with two removed points.

- Let $f = x^n + y^n + 1$. One easily computes $\Delta_f(x) = c.(x^n + 1)^{n-1}, c = const. \neq 0$ and hence $dimH_1(\Gamma) = 1 - n + n(n-1) = (n-1)^2$. On the other hand Γ is a genus g Riemann surface with n removed points and $dimH_1(\Gamma) = 2g - n + 1$. Thus we obtain $g = (n-1)(n-2)/2$.

Proof of Proposition 2.2. Let $\overline{\Gamma}$ be the smooth model of the compactified curve Γ, $\overline{\Gamma} = \Gamma \cup D_\infty$, $\mathbf{CP}^1 = \mathbb{C} \cup \infty$. The projection map π can be continued to a holomorphic map $\pi : \overline{\Gamma} \to \mathbf{CP}^1$. As l is general then $\pi^{-1}(\infty) = D_\infty$. The Riemann-Hurwitz formula applied to $\overline{\Gamma}$, $\overline{\Gamma}' = \mathbf{CP}^1$, $S = \infty \in \mathbf{CP}^1$ gives

$$\chi(\Gamma) = n\chi(\mathbb{C}) - \sum_{p\in\Gamma}(v(p) - 1) .$$

But Γ has the homotopy type of a bouquet of $dim\, H_1(\Gamma)$ circles and hence $\chi(\Gamma) = 1 - dim\, H_1(\Gamma)$. $\triangle$

Further we shall study rather the fibration

$$f : \mathbb{C}^2 \to \mathbb{C}$$

and its general fibers $f^{-1}(t)$, than a single affine curve. To each isolated critical point $p \in \mathbb{C}^2$ of f we associate its Milnor number

$$\mu_p(f) = \dim_{\mathbb{C}} \mathcal{O}_p(x,y)/< f_x, f_y >$$

where $\mathcal{O}_p(x,y)$ is the local ring of $\mathbb{C}^2$ at p (it may be any of the rings of rational functions defined at p, formal or convergent power series in a neighborhood of p) and $< f_x, f_y >$ is the Jacobian ideal in $\mathcal{O}_p(x,y)$ generated by the gradient of f. We define also the global Milnor number $\mu(f)$ of f

$$\mu(f) = \sum_p \mu_p(f) = dim\mathbb{C}^2[x,y]/< f_x, f_y > .$$

f has only isolated critical points if and only if $\mu(f)$ is finite.

Proposition 2.4 *If $\mu(f) < \infty$ then the general fiber of the polynomial map $f : \mathbb{C}^2 \to \mathbb{C}$ is smooth and irreducible. If $\mu(f) = \infty$ then $f = \lambda g^2 + c$ for some constant c and $\lambda \in \mathbb{C}[x,y]$, $g \in \{\mathbb{C}[x,y] - \mathbb{C}\}$.*

Thus, if f has only isolated critical points, Proposition 2.2 applies to the general fiber of $f : \mathbb{C}^2 \to \mathbb{C}$.

Proof of Proposition 2.4. Let us suppose first that $\mu(f) < \infty$. By Bertini's theorem the general fiber $f^{-1}(t)$ is smooth and for t in a Zariski closed set the fiber is reducible. We have to prove that this set is not $\mathbb{C}$.

Let us suppose that for all $t \in \mathbb{C}$ the fiber $f^{-1}(t)$ is reducible. Then Bertini's theorem ([14, chapter 2.6]) implies that the curve $\Gamma_t = \{(x,y) \in \mathbb{C}^2 : f(x,y) = t\}$ is reducible over $\mathbb{C}(t)$. In other words $f(x,y) - t = P_t(x,y)\, Q_t(x,y)$ where P and Q are polynomials in x, y with coefficients algebraic functions in t and $deg(P), deg(Q) \geq 1$.

We may also choose $P_t(x,y)$, $Q_t(x,y)$ in such a way that their coefficients are well defined for any t. Indeed, as the highest order homogeneous component of f is a product of the highest order homogeneous components of $P_t(x,y)$ and $Q_t(x,y)$ then it is reducible over $\mathbb{C}$. Thus we may choose the highest order homogeneous components of $P_t(x,y)$ and $Q_t(x,y)$ to be polynomials which do not depend on t. If for some t_0 some coefficient of $P_t(x,y)$ tends to ∞ as t tends to t_0 then for (x,y) in a Zariski open set holds $\lim_{t\to t_0} P_t(x,y) = \infty$ and $\lim_{t\to t_0} Q_t(x,y) = 0$. The last contradicts however to the choice of the highest order homogeneous component of $Q_t(x,y)$.

For almost all t the curves

$$\{(x,y) \in \mathbb{C}^2 : P_t(x,y) = 0\}, \{(x,y) \in \mathbb{C}^2 : Q_t(x,y) = 0\}$$

do not intersect each other (otherwise Γ_t would be singular for almost all t) and hence the resultant $R(t)$ of $P_t(x,y)$ and $Q_t(x,y)$ with respect to y is a polynomial in x of degree 0. If for some t the algebraic function $R(t)$ vanishes then the curves $\{P_t(x,y) = 0\}$ and $\{Q_t(x,y) = 0\}$ have a common component. This component is a (one-dimensional) critical set of $f(x,y)$ and hence $\mu(f) = \infty$.

Finally we shall consider the case $R(t) = const. \neq 0$, that is to say for all t $\{P_t(x,y) = 0\} \cap \{Q_t(x,y) = 0\} = \emptyset$. Let for some $t = t_0$ $(x_1, y_1) \in \{P_t(x,y) = 0\}$ and $(x_2, y_2) \in \{Q_t(x,y) = 0\}$ and let l be the line passing through these two points. We may suppose that $l : \{y = c\}$ and then x_1, x_2 will be two distinct roots of the polynomial $g(x) = f(x,c) - t_0$. Now using the fact that the Dynkin diagram of a polynomial is connected we conclude by a standard argument (see for example [1], vol.2 example 2.9.) that there is a path on the $\mathbb{C}$ - plane connecting t_0 to some $\tilde{t}_0$ such that along this path the roots x_1 and x_2 are continuously deformed to some double root $\tilde{x}_1 = \tilde{x}_2$ of the polynomial $f(x,c) - \tilde{t}_0$. As the point $(\tilde{x_1}, c) = (\tilde{x_2}, c)$ for $t = \tilde{t}_0$ belongs both to $\{P_t(x,y) = 0\}$ and $\{Q_t(x,y) = 0\}$ we arrive to the desirable contradiction.

At last if $f = \lambda g^2 + c$ then f_x and f_y have a common factor g in $\mathbb{C}[x,y]$ and hence $\mu(f) = \infty$. If $\mu(f) = \infty$ then f_x and f_y have a common factor g and let C be the curve $\{(x,y) \in \mathbb{C}^2 : g(x,y) = 0\}$. We may suppose that C is irreducible and let z be an uniformizing parameter in a neighborhood of a simple point of C. We have

$$\frac{d}{dz} f(x(z), y(z)) = f_x \frac{d}{dz} x + f_y \frac{d}{dz} y \equiv 0$$

and hence f is a constant on C. Thus there exists a constant c such that $f - c = \lambda\, g^m$ for some integer m and $\lambda \in \mathbb{C}[x,y]$. Finally $m \geq 2$ as if $m = 1$ then f_x, f_y will not vanish identically on C. $\triangle$

3 The general fiber of the fibration $f : \mathbb{C}^2 \to \mathbb{C}$

Definition 2 *A polynomial $f : \mathbb{C}^n \to \mathbb{C}$ is called a "tame" polynomial if there is a compact neighborhood U of the critical points of f such that $||grad(f)||$ is bounded away from the origin on $\mathbb{C}^n - U$.*

Let

$$\mu^t(f) = \sum_{p \in f^{-1}(t)} \mu_p(f)$$

be the fiber Milnor number of f. The following theorem is proved by Broughton [3,4]

Theorem 3.1 *If f is a tame polynomial then for any $t \in \mathbb{C}$ the fiber $f^{-1}(t)$ has the homotopy type of a bouquet of $\mu(f) - \mu^t(f)$ spheres of dimension $n-1$.*

The tame polynomials can be characterized by the fact that their critical points stay in a finite plane $\mathbb{C}^n$ after a small perturbation with an arbitrary linear function.

Proposition 3.2 [3] *A polynomial f is tame if and only if $\mu(f) = \mu(f + \epsilon l)$ for each linear function l and all sufficiently small ϵ*

If f is not tame, but $\mu(f) < \infty$ and under some additional condition which always holds for $n = 2$, it is proved by Broughton [3] that for the general fibers $f^{-1}(t)$ holds $H_{n-1}(f^{-1}(t), \mathbb{Z}) \approx \mathbb{Z}^{\mu(f)+\rho(f)}$. Here $\mu(f)$ is the global Milnor number of f and $\rho(f)$ is the "jump of Milnor numbers at infinity". The number $\rho(f)$ as it is defined in [3] is however not very computable. Further we shall specialize to the case $n = 2$ and we shall characterize $\rho(f)$ in a different way.

The total Milnor number $\mu(f)$ of a polynomial is easily computed for any given polynomial $f \in \mathbb{C}^2[x, y]$. Suppose that x is general with respect to f. If $R_{\nabla f}(x)$ is the resultant of f_x and f_y with respect to y then

$$\mu(f) = deg R_{\nabla f}(x)$$

and the condition that x is general is indeed necessary. Note that $\mu(f)$ is a topological invariant of f. It means that if f is topologically conjugate to the polynomial g then $\mu(f) = \mu(g)$. We shall define now another topological invariant of f.

Let us suppose that $l = x$ is general with respect to f and for any $t \in \mathbb{C}$ denote by $\Delta(t, x)$ the discriminant of $f(x, y) - t$ with respect to y. Let $d(t)$ be the degree of $\Delta(t, x)$ in x and let d be the degree of $\Delta(t, x)$ for general t. Obviously $d - d(t) \geq 0$ and there is only a finite number of values for t such that $d - d(t) > 0$.

Definition 3 *We denote*

$$\rho^t(f) = d - d(t), \quad \rho(f) = \sum_{t \in \mathbb{C}} \rho^t(f) .$$

The fact that $\rho(f)$ is a topological invariant will follow from Theorem 3.3. The number $\rho^{t_0}(f)$ counts the number of ramification points of the curve $\Gamma_t = \{f(x,y) = t\}$ which tend to ∞ as t tends to t_0 and then the number $\rho(f)$ is the total number of ramification points which tend to infinity as t varies. For any given polynomial $f(t)$ the number $\rho(f)$ is also easily computed.

The main result of this section is the following

Theorem 3.3 *If f is a polynomial with isolated critical points then the Euler characteristic of the fiber $f^{-1}(t)$ is given by*

$$\chi(f^{-1}(t)) = 1 - \mu(f) - \rho(f) + \mu^t(f) + \rho^t(f) . \tag{1}$$

Theorem 3.3 implies immediately

Corollary 3.4 *If the fibre $f^{-1}(t)$ is connected then it has the homotopy type of a bouquet of $\mu(f) + \rho(f) - \mu^t(f) - \rho^t(f)$ circles.*

If t is general then $\mu^t(f) = \rho^t(f) = 0$, the fibre $f^{-1}(t)$ is smooth and irreducible (Proposition 2.4) and hence it is homeomorphic to a (connected) Riemann surface with several, but at least one, removed points. We shall see below (in the proof of Theorem 3.8 for example) that if $\rho(f) = 0$ then f defines a trivial fibration "at infinity". On the other hand the local Milnor fibre of an analytic function is always connected which implies that if $\rho(f) = 0$ then $f^{-1}(t)$ is connected. Now Corollary 3.4 gives the following

Corollary 3.5 *The general fibre $f^{-1}(t)$ of a polynomial with isolated critical points has the homotopy type of a bouquet of $\mu(f) + \rho(f)$ circles and hence*

$$dim\ H_1(f^{-1}(t)) = \mu(f) + \rho(f) . \tag{2}$$

Corollary 3.6 *If $\rho(f) = 0$ then $f^{-1}(t)$ has the homotopy type of a bouquet of $\mu(f) - \mu^t(f)$ circles.*

Corollary 3.6 indicates that a class of well-behaved polynomials are those with $\rho(f) = 0$. For such polynomials the conclusion of Theorem 3.1 holds but the class is larger than the one of the tame polynomials. This justifies the following

Definition 4 *The polynomial f is good provided that $\rho(f) = 0$.*

The properties of good polynomials will be studied in more details in section 4.

Proposition 3.7 *Each tame polynomial is good but there exist good polynomials which are not tame.*

Theorem 3.3 and Proposition 3.7 will be proved later in this section.

Let $A_f \subset \mathbb{C}$ be the smallest set such that $f : \mathbb{C}^2 - f^{-1}(A_f) \to \mathbb{C} - A_f$ is a locally trivial fibration. Then by definition A_f is the set of *non-general values* of t and it is often called a set of *atypical values*. Let A_c be the set of critical values of f,

$$A_c = \{t \in \mathbb{C} : \mu^t(f) > 0\} ,$$

and put

$$A_\infty = \{t \in \mathbb{C} : \rho^t(f) > 0\} \,.$$

It is well known that A_f is a finite set (see [10,5] for a discussion).The following two theorems du to Ha Huy Vui, Lê Dung Trang and Nguyen Le Anh are closely related to our Theorem 3.3

Theorem 3.8 ([8,9]) $A_f = A_c \cup A_\infty$.

For completeness we give below a proof of Theorem 3.8.

Theorem 3.9 ([10], but see also [5], Proposition 4.6 on p.22) *The fibration $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$ is locally trivial over $t_0 \in \mathbb{C}$ provided that the Euler characteristic of $f^{-1}(t_0)$ equals the Euler characteristic of the general fibre $f^{-1}(t)$.*

Note that that in Theorem 3.8 and Theorem 3.9 f is an arbitrary polynomial, possibly with non-isolated critical points.

Proof of Theorem 3.8. It is well known that the fibration $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$ is not locally trivial over A_c ([10], remark 4). Let $t_0 \in \mathcal{A}_\infty - (\mathcal{A}_c \cap \mathcal{A}_\infty)$ and suppose that $l = x$ is a general function with respect to $f(x, y)$. Then it is general with respect to $f(x, y) - t$ for any constant $t \in \mathbb{C}$ and let

$$\Delta(t, x) = \sum_{i=0}^{d} a_i(t) x^i \tag{3}$$

be the discriminant of the polynomial $f(x, y) - t$ with respect to y. By Definition 3 $t_0 \in \mathcal{A}_\infty$ means that $a_0(t_0) = 0$. Thus the smooth affine curves $f^{-1}(t)$ and $f^{-1}(t_0)$, $t \sim t_0$, have different number of ramification points under the projection $(x, y) \to x$. Now Corollary 2.3 implies that $dim\, H_1(f^{-1}(t)) \neq dim\, H_1(f^{-1}(t_0))$ and the fibration $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$ is not locally trivial over t_0 .

Suppose now that $t_0 \notin A_f$. We have to prove that $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$ is locally trivial over t_0. We may use a vector field argument as in [9], or use directly the Ehresmann fibration theorem (see for example [5]) in the following way.

As $t_0 \notin A_f$ then $a_0(t_0) \neq 0$ and there exist $c_0 > 0, \epsilon > 0$, such that if $\mid t - t_0 \mid < \epsilon$, then $\Delta(t, x) \neq 0$ on the set $\{x : \mid x \mid \geq c_0\}$. It follows that for any $t \in \mathbb{C}, c \in \mathbb{R}$, such that $\mid t - t_0 \mid < \epsilon$, $c \geq c_0$, the cylinder

$$C_c = \{(x, y) \in \mathbb{C}^2 : \mid x \mid = c\}$$

is transverse to the smooth affine curve $\{(x, y) \in \mathbb{C}^2 : f(x, y) = t\}$. Thus $f^{-1}(t) \cap C_c$, $c \geq c_0$, is smooth and as x is general with respect to f then $f^{-1}(t) \cap C_c$ is a finite unramified covering over the circle $\{x : \mid x \mid = c\}$. We conclude that $f^{-1}(t) \cap C_c$ is a finite disjoint union of circles for any $c \geq c_0$ and hence we obtain the following two proper submersions

$$f^{-1}(t) \cap \{(x, y) \in \mathbb{C}^2 : \mid x \mid \leq c_0\} \xrightarrow{f} t \,,\ \mid t - t_0 \mid < \epsilon \tag{4}$$

and

$$f^{-1}(t) \cap C_c \to (t,c)\,,\ | t - t_0 |< \epsilon,\ c \geq c_0\,. \tag{5}$$

The Ehresmann fibration theorem implies that f fibres the pair

$$(f^{-1}(t) \cap \{(x,y) \in \mathbb{C}^2 :| x |\leq c_0\}, f^{-1}(t) \cap C_{c_0})$$

locally trivially over the disc $\{t \in \mathbb{C} :| t - t_0 |< \epsilon\}$, and that (5) is a locally trivial fibration over the set

$$\{t \in \mathbb{C} :| t - t_0 |< \epsilon\} \times \{c \in \mathbb{R} : c \geq c_0\}\,.$$

The latter claim however implies that

$$f^{-1}(t) \cap \{(x,y) \in \mathbb{C}^2 :| x |\geq c_0\} \xrightarrow{f} t \tag{6}$$

is locally trivial over the disc $\{t \in \mathbb{C} :| t - t_0 |< \epsilon\}$. This together with the local triviality of (4) gives the local triviality of $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$ over t_0. □

Definition 5 *A polynomial f defines a trivial fibration at infinity, provided that for any $t_0 \in \mathbb{C}$ there exist $\epsilon > 0$, $c_0 > 0$, such that (6) defines a trivial fibration over the disc $\{t \in \mathbb{C} :| t - t_0 |< \epsilon\}$.*

From the proof of Theorem 3.8 we obtain

Corollary 6 *f defines a trivial fibration at infinity if and only if $\rho(f) = 0$, that is to say f is a good polynomial (Definition 4).*

f is defines a trivial fibration at infinity if for any fiber $f^{-1}(t_0)$ the nearby fibres "look like it" at infinity. Thus our definition of a good polynomial is the same as in [12].

Proof of Theorem 3.3. If f is tame then the result follows from Theorem 3.1. If f is not tame then almost all linear perturbations make it tame. It remains to compare the fibres of f with the fibres of the perturbed polynomial. Let us give the details.

We shall study first the general fibres of f, i.e. the fibres $f^{-1}(t)$ with $t \notin A_f$. In this case the affine curve

$$\Gamma_t = \{(x,y) \in \mathbb{C} : f(x,y) = t\}$$

is smooth and irreducible (Proposition 2.4) and we may apply Proposition 2.2. Without loss of generality we shall suppose that the linear function $l = x$ is general with respect to the polynomial f (Definition 1) and hence with respect to $f - t$ for any $t \in \mathbb{C}$. As in Proposition 2.2 we consider the projection

$$\pi : \Gamma_t \to \mathbb{C} : (x,y) \to x \tag{7}$$

and let $v(p)$ be the ramification index of π at $p \in \Gamma_t$. Then we have

$$dim(H_1(\Gamma_t)) = 1 - deg(f) + \sum_{p \in \Gamma_t} (v(p) - 1) \tag{8}$$

and we wish to prove (2) which in this case is equivalent to (1) . Consider also the affine curve (possibly singular and reducible)

$$C = \{(x, y) \in \mathbb{C}^2 : \frac{\partial f}{\partial y} = 0\}.$$

The ramification index $\sum_{p\in\Gamma_t}(v(p) - 1)$ can be interpreted as the number of zeros of the (holomorphic) function f on C. More precisely, for a fixed $p \in C$ let $\hat{f}$ be the image of f in the local ring $\mathcal{O}_p(C)$. Then the multiplicity of the zero of $f|_C$ at p is defined as $ord_p f = ord_p \hat{f} = dim_{\mathbb{C}} \mathcal{O}_p(C)/(\hat{f})$ where $(\hat{f})$ is the ideal generated by $\hat{f}$ in $\mathcal{O}_p(C)$. On the other hand $ord_p f$ is the intersection index $I(C \cap \Gamma, p)$ of C and Γ at q ([6, page 81]) which also equals to $v(p) - 1$. Keeping the same notation for the map

$$\pi : \Gamma_t^\epsilon \rightarrow \mathbb{C} : (x, y) \rightarrow x$$

where $\Gamma_t^\epsilon = \{(x, y) \in \mathbb{C} : f(x, y) + \epsilon x = t\}$, ϵ - sufficiently small, we obtain by Proposition 2.2

$$dim(H_1(\Gamma_t^\epsilon)) = 1 - deg(f) + \sum_{p\in\Gamma_t^\epsilon}(v(p) - 1). \tag{9}$$

We may also suppose that for all sufficiently small non-zero ϵ the polynomial $f + \epsilon x$ is tame. Indeed, consider the affine space $V = \{f + l : l \in \mathbb{C}_1[x, y]\} \subset \mathcal{A}$ where $f \subset \mathcal{A}_{\mu,\rho}$ is fixed and l is an arbitrary linear function. As $\mu(f)$ is a lower semi-continuous function on $\mathcal{A} - \mathcal{A}^\infty$ ([3], Proposition 2.3) then there is a Zariski open dense subset $\tilde{V}$ in $V - V \cap \mathcal{A}^\infty$ such that $\mu(f + l)$ is a constant on it. According to Proposition 3.2 $\tilde{V}$ consists of tame polynomials and we conclude that for a fixed general linear function and any sufficiently small non-zero ϵ the polynomial $f + \epsilon l$ is tame. Without loss of generality we may suppose that $l = x$.

Now Theorem 3.1 implies $\dim H_1(\Gamma_t^\epsilon) = \mu(f + \epsilon x)$ for $t \notin A_{f+\epsilon x}$. Thus according to (8), (9), the identity (2) is equivalent to

$$\mu(f + \epsilon x) - \mu(f) - \rho(f) = \sum_{p\in\Gamma_t^\epsilon}(v(p) - 1) - \sum_{p\in\Gamma_t}(v(p) - 1). \tag{10}$$

Let $\overline{C} \rightarrow C$ be the normalization of C. $\overline{C}$ will be in general a disjoint union of several Riemann surfaces and let $D_\infty = \sum_i p_i$ be the infinity divisor. Thus $\overline{C} - D_\infty$ is the pre-image of C and for any polynomial g let $\hat{g}$ be the image of g in the function field $\mathbb{C}(\overline{C})$. Let $ord_p\hat{g}$ be the order of the meromorphic function $\hat{g}$ at p, $\hat{g} = z^{ord_p\hat{g}}\, u(z)$ where z is an uniformizing parameter in a neighborhood of p, $u(0) \neq 0$. As g is a polynomial then $\hat{g}$ will have no poles on the affine part $\overline{C} - D_\infty$ of $\overline{C}$ and hence

$$\text{"the number of zeros of } g \text{ on } C\text{"} = - \sum_{p\in D_\infty} ord_p\hat{g}. \tag{11}$$

Note now that $\mu(f + \epsilon x)$ (respectively $\mu(f)$) is exactly the number of zeros of $\frac{\hat{\partial f}}{\partial x} + \epsilon$ (respectively $\frac{\hat{\partial f}}{\partial x}$) on $C \sim \overline{C} - D_\infty$ and $\sum_{p\in\Gamma_t^\epsilon}(v(p) - 1)$ (respectively $\sum_{p\in\Gamma_t}(v(p) - 1)$) is the number of zeros of $\hat{f}+t+\epsilon\hat{x}$ (respectively $\hat{f}+t$) on $C\sim\overline{C}-D_\infty$. This combined with (11) shows that for general t (10) is equivalent to the identity

$$\sum_i ord_{p_i}\frac{\hat{f}+t+\epsilon\,\hat{x}}{\hat{f}+t}-\rho(f)=\sum_i ord_{p_i}\frac{\frac{\hat{\partial f}}{\partial x}+\epsilon}{\frac{\hat{\partial f}}{\partial x}}.$$

If $p(t)$ is a ramification point on the curve Γ_t that tends to infinity as t tends to some finite value, then also $p(t)\in C$ and as a point on $\overline{C}$ it tends to some $p_i\in D_\infty$. If $\rho_{p_i}(f)$ is the number of such points then

$$\rho(f)=\sum_i\rho_{p_i}(f)$$

and hence it will be enough to prove that for any i and for general values of t holds

$$ord_{p_i}\frac{\hat{f}+t+\epsilon\,\hat{x}}{\hat{f}+t}-\rho_{p_i}(f)=ord_{p_i}\frac{\frac{\hat{\partial f}}{\partial x}+\epsilon}{\frac{\hat{\partial f}}{\partial x}}. \tag{12}$$

Let z be an uniformizing parameter on $\overline{C}$ in a neighborhood of $p=p_i\in D_\infty$, $p(0)=p_i$. If $\rho_{p_i}>0$ then

$$\hat{f}(z)=\hat{f}(0)+z^{\rho_{p_i}(f)}.u(z),u(0)\neq 0. \tag{13}$$

On the other hand if $\rho_{p_i}=0$ then

$$ord_{p_i}\hat{f}<0. \tag{14}$$

We have

$$\frac{d}{dz}\hat{f}(z)=\widehat{\frac{\partial f}{\partial x}}\frac{d}{dz}\hat{x}+\widehat{\frac{\partial f}{\partial y}}\frac{d}{dz}\hat{y}=\widehat{\frac{\partial f}{\partial x}}\frac{d}{dz}\hat{x}$$

$$\frac{d}{dz}(\hat{f}(z)+\epsilon\hat{x})=(\widehat{\frac{\partial f}{\partial x}}+\epsilon)\frac{d}{dz}\hat{x}$$

and hence

$$\frac{\frac{d}{dz}(\hat{f}(z)+\epsilon\hat{x})}{\frac{d}{dz}\hat{f}(z)}=\frac{\widehat{\frac{\partial f}{\partial x}}+\epsilon}{\widehat{\frac{\partial f}{\partial x}}}. \tag{15}$$

If $\rho_{p_i}(f)=0$, then (14) and (15) imply (12). On the other hand if $\rho_{p_i}(f)>0$ and $ord_{p_i}\hat{x}<0$ then (13) combined with (15) implies again (12). Thus it remains to prove that $ord_{p_i}\hat{x}<0$.

As we noted before the condition that x is general with respect to f means that $f(x,y)=\sum_{i=0}^n a_i(x)y^{n-i}$, where $deg(a_i(x))\leq i$ and $a_0\neq 0$.It follows that x is also general with respect to the function f_y. Thus no zeros of $\hat{x}-c$ tend to $D_\infty=\sum p_i$ as c varies and hence $ord_{p_i}\hat{x}<0$. Formula (2) is proved.

Next, we shall find the homotopy type of the non-general fibre $f^{-1}(t_0),t_0\in A_f$. If $\rho^{t_0}(f)+\mu^{t_0}(f)\neq 0$ but $\rho^{t_0}(f)=0$ this can be done as in [3]. If $\mu^{t_0}(f)=0$ but $\rho^{t_0}(f)\neq 0$ we note that for $t\sim t_0$ the fibre $f^{-1}(t)$ is smooth and according to Riemann-Hurwitz formula for $t\neq t_0$

$$\chi(f^{-1}(t_0))-\chi(f^{-1}(t))=\rho^{t_0}(f)\ .$$

In general we may reason in the following way. Let $p_i' = (x_i, y_i)$, $i = 1,2,...,k$, be the critical points of f on the fibre $f^{-1}(t_0)$, $S^i_\epsilon = \{x \in \mathbb{C} :| x - x_i |\leq \epsilon\}$, $S = \sum_i S^i_\epsilon$, and with abuse of notation we denote by π also the map (compare with (7)) $\pi : \mathbb{C}^2 \to \mathbb{C} : (x,y) \to x$. To compute $\chi(f^{-1}(t_0)) - \chi(f^{-1}(t))$ we shall use that

$$\begin{aligned}\chi(f^{-1}(t)) &= \chi(f^{-1}(t) \cap \pi^{-1}(S_\epsilon)) \\ &\quad +\chi(f^{-1}(t) - int\, \pi^{-1}(S_\epsilon)) \\ &\quad -\chi(f^{-1}(t) \cap \pi^{-1}(\partial S_\epsilon)).\end{aligned} \tag{16}$$

As the ramification points of $f^{-1}(t_0)$ under the projection π are isolated then there exists $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$ the cylinders $\{(x,y)\, \mathbb{C}^2 :| x - x_i |= \epsilon\}$ are transversal to $f^{-1}(t_0)$ and in addition for all t sufficiently close to t_0 the cylinders $\{(x,y)\, \mathbb{C}^2 :| x - x_i |= \epsilon_0\}$ are transversal to $f^{-1}(t)$. Thus $\pi^{-1}(\partial S_\epsilon) \cap f^{-1}(t)$ for $t \sim t_0$ is a disjoint union of circles and hence its Euler characteristic is zero. We shall prove that

$$\mu^{t_0}(f) = \chi(f^{-1}(t_0) - int\, \pi^{-1}(S_\epsilon)) - \chi(f^{-1}(t) - int\, \pi^{-1}(S_\epsilon)) \tag{17}$$

and

$$\rho^{t_0}(f) = \chi(f^{-1}(t_0) \cap \pi^{-1}(S_\epsilon)) - \chi(f^{-1}(t) \cap \pi^{-1}(S_\epsilon)). \tag{18}$$

Indeed, the Riemann-Hurwitz formula applied to the projection

$$f^{-1}(t_0) - int\, \pi^{-1}(S_\epsilon) \xrightarrow{\pi} \mathbb{C} - int\, S_\epsilon$$

implies (17). To prove (18) we note that for $t = t_0$ the fibre $f^{-1}(t)$ is transverse to $\pi^{-1}(\partial S_\epsilon)$ for any $\epsilon \leq \epsilon_0$. Thus $f^{-1}(t_0) \cap \pi^{-1}(S_\epsilon)$ is a disjoint union of cones over the N pre-images $q_1, q_2, ..., q_N$ of $x_1, x_2, ..., x_k$ in $f^{-1}(t_0)$ under π and hence $\chi(f^{-1}(t_0) \cap \pi^{-1}(S_\epsilon)) = N$. On the other hand for t sufficiently close to t_0 but $t \neq t_0$ the set $f^{-1}(t) \cap \pi^{-1}(S_\epsilon)$ is a smooth bordered surface and it is not difficult to see that it is homeomorphic to the disjoint union of local Milnor fibres obtained by taking the intersection of $f^{-1}(t)$ with sufficiently small balls centered at the N pre-images $q_1, q_2, ..., q_N$. Thus $f^{-1}(t) \cap \pi^{-1}(S_\epsilon)$ has homotopy type of a disjoint union on N bouquets of $\mu_{q_i}(f)$ circles. This implies that

$$\chi(f^{-1}(t) \cap \pi^{-1}(S_\epsilon)) = \sum_{i=1}^{N}(1 - \mu_{q_i}(f)) = N - \mu^{t_0}(f)$$

and (17) is proved. Now (16), (17), and (18) imply

$$\chi(f^{-1}(t_0)) - \chi(f^{-1}(t)) = \mu^{t_0}(f) + \rho^{t_0}(f)$$

This combined with (2) implies (1). Theorem 3.3 is proved. □

Proof of Proposition 3.7. As in the proof of Theorem 3.3 let $\overline{C}$ be the normalization of $C = \{(x,y) \in \mathbb{C}^2 : f_y(x,y) = 0\}$, $D_\infty = \sum_i p_i$ its infinity divisor, and $\hat{f}_x$ the image of f_x in the function field $\mathbb{C}(\overline{C})$.

Let us suppose that f is tame.If $\hat{f}_x$ has a zero at p_i then there exists a sequence $\{q_i\}_i$, $q_i \in C$, and such that $|q_i| \to \infty$, $f_x(q_i) \to 0$ which shows that f is not tame. It follows that

$ord_{p_i}\hat{f}_x \leq 0$. On the other hand $ord_{p_i}x < 0$ (see the proof of Theorem 3.3) and if z is a local parameter in a neighborhood of p_i then

$$\frac{d}{dz}\hat{f} = \hat{f}_x \frac{d}{dz}\hat{x}$$

and hence $ord_{p_i}\hat{f} < 0$. The last inequality implies that for any i $\rho_{p_i}(f) = 0$ and hence $\rho(f) = 0$ and the polynomial f is good.

To prove the second part of Proposition 3.7 it suffices to give a counterexample. We claim that the polynomial $f(x,y) = \frac{(y^2-x)^2}{2} - y$ is good but not tame. Of course this may be elementary checked by Definition 2. We prefer to give another proof which, together with Proposition 3.2, describes any good but non-tame polynomial.

As x is a general linear function with respect to f then according to Definition 4 we compute the discriminant $\Delta(t,x)$ of $f(x,y) - t$ with respect to y

$$\Delta(t,x) = -16\,x^3 + 16\,t^2x^2 + 36\,tx - \frac{27}{4} - 32\,t^3$$

and hence $d(t) = d = 3$ and the function f is good. Moreover the discriminant of the function $f + \epsilon x$ equals to $\Delta(t - \epsilon x, x)$ which shows that $f + \epsilon x$ is also good (but $d(t) = d = 4$ in this case). If f is tame then by Proposition 3.2 for sufficiently small ϵ holds $\mu(f) = \mu(f + \epsilon x)$. Thus the general fibre of the two polynomial functions has the same homology which contradicts to Corollary 2.3, as $deg_x\Delta(t,x) \neq deg_x\Delta(t - \epsilon x, x)$.Δ

4 On the classification of polynomials

Let $\mathcal{A}$ be the set of all complex polynomials in two variables and degree exactly n. Then we have

$$\mathcal{A} = \mathcal{A}^\infty + \sum_{\mu,\rho} \mathcal{A}_{\mu,\rho}$$

where $\mathcal{A}^\infty = \{f \in \mathcal{A} : \mu(f) = \infty\}$ and $\mathcal{A}_{\mu,\rho} = \{f \in \mathcal{A} : \mu(f) = \mu, \rho(f) = \rho\}$. Thus we have a kind of "stratification" of the affine variety $\mathcal{A}$ and conjecturally each stratum $\mathcal{A}_{\mu,\rho}$ is a smooth algebraic variety. The set $\mathcal{A}^\infty$ is not smooth so it should be further decomposed in a similar way, but we shall not study this here. The main fact about the set $\mathcal{A}_{\mu,\rho}$ is the following

Proposition 4.1 *If two polynomials f_0, f_1 belong to one and the same connected component of the set $\mathcal{A}_{\mu,\rho}$, then any two general fibers $f_0^{-1}(t_0), f_1^{-1}(t_1)$ are equivalent up to proper isotopy.*

Proof. Let $a \in \mathbb{C}^N$, $N = (n+1)(n+2)/2$ be the vector of coefficients of an arbitrary polynomial in $\mathcal{A}$ which we denote by f_a. Suppose that $f_{a_0} = f_0$ and let $f_{a_0}^{-1}(t_0)$ be a general fiber of the polynomial f_{a_0}. It suffice to prove that the map F

$$F : \mathbb{C}^N \times \mathbb{C}^2 \to \mathbb{C}^N \times \mathbb{C} : (a,x,y) \to (a,t)\ ,\ t = f_a(x,y) \tag{19}$$

defines a fibration with a base $\mathcal{A}_{\mu,\rho} \times \mathbb{C}$, which is locally trivial over (a_0, t_0).

Note first that the critical points of f_a depend continuously on a and their number is fixed to $\mu = \mu(f_a)$. Then $\mu^{t_0}(f_{a_0}) = 0$ implies that $\mu^t(f_a) = 0$ for (a,t) sufficiently close to (a_0, t_0) and hence the fibres $f^{-1}(t)$ are smooth. As before we suppose that x is general with respect to f_{a_0} (and hence to f_a for $a \sim a_0$) and consider the projection (7) $\pi : f_a^{-1}(t) \to x$. The number of ramification points of π is equal to (Proposition 2.2, and Theorem 3.3)

$$dim\, H_1(f_a^{-1}(t)) + n - 1 = \mu + \rho + n - 1 - \rho^t(f_a)$$

and as $\rho^{t_0}(f_{a_0}) = 0$ then

$$dim\, H_1(f_a^{-1}(t) \leq dim\, H_1(f_{a_0}^{-1}(t_0)\,. \tag{20}$$

On the other hand the ramification points of $f^{-1}(t)$ under the map π depend continuously on a and t and using once again Proposition 2.2 we obtain

$$dim\, H_1(f_a^{-1}(t) \geq dim\, H_1(f_{a_0}^{-1}(t_0)\,. \tag{21}$$

The inequalities (20) and (21) imply that for all (a,t) sufficiently close to (a_0, t_0) we have also $\rho^t(f_a) = 0$.

Further we prove the local triviality of (19) along the same lines as Theorem 3.8. Namely, if $t_0 \notin A_f$ then there exist $c_0 > 0$, such that for any complex t sufficiently close to t_0, $c \geq c_0$, the cylinder

$$C_c = \{(x,y) \in \mathbb{C}^2 : \mid x \mid = c\}$$

is transverse to the smooth affine curve $\{(x,y) \in \mathbb{C}^2 : f_{a_0}(x,y) = t\}$. Thus, for $a = a_0$ and t sufficiently close to t_0, we obtain the following two proper submersions

$$f_a^{-1}(t) \cap \{(x,y) \in \mathbb{C}^2 : \mid x \mid \leq c_0\} \xrightarrow{f_a} (a,t)\,, \tag{22}$$

and

$$f_a^{-1}(t) \cap C_c \to (a,t,c)\,,\; c \geq c_0\,. \tag{23}$$

Now the point is that (22), (23), remain proper submersions for all (a,t) sufficiently close to (a_0, t_0) and such that $f_a \in \mathcal{A}_{\mu,\rho}$. Indeed, (22) is a submersion because, for $(a,t) \sim (a_0, t_0)$ the cylinder C_{c_0} is still transverse to $f_a^{-1}(t)$, and f_a has no critical points on the fibre $f_a^{-1}(t)$ (that is to say $\mu^t(f_a) = 0$).

On the other hand (23) is a submersion if and only if the cylinder C_c is transverse to $f_a^{-1}(t)$ for $c \geq c_0$. The last is equivalent to $\partial f_a / \partial y \neq 0$ on the set

$$f_a^{-1}(t) \cap \{(x,y) \in \mathbb{C}^2 : \mid x \mid \geq c_0\}$$

and hence we have to prove that the projection

$$\pi : f_a^{-1}(t) \cap \{(x,y) \in \mathbb{C}^2 : \mid x \mid \geq c_0\} \xrightarrow{f} x \tag{24}$$

has no ramification points. This is true for $(a,t) = (a_0, t_0)$ and the local triviality of (22) implies that it is also true for any (a,t) sufficiently close to (a_0, t_0), $f_a \in \mathcal{A}_{\mu,\rho}$ - otherwise $f_a^{-1}(t)$ will have more ramification points than $f_{a_0}^{-1}(t_0)$ which contradicts to $\rho^t(f_a) = 0$. We conclude that (23) is a proper submersion.

The same arguments as in the proof of Theorem 3.8 show that the map (19) defines a fibration which is locally trivial over $(a_0, t_0) \in \mathcal{A}_{\mu,\rho} \times \mathbb{C}$.

At last, if two polynomials $f_0 = f_{a_0}$, $f_1 = f_{a_1}$ belong to the same connected component of $\mathcal{A}_{\mu,\rho}$ then we may connect them by a continuous compact arc $a = a(s), 0 \leq s \leq 1$, $a(0) = a_0$, $a(1) = a_1$, $f_{a(s)} \in \mathcal{A}_{\mu,\rho}$. We proved, however, that any two sufficiently close polynomials $f_{a(s)}$ have their general fibres equivalent up to a proper isotopy. Proposition 4.1 is proved. □

To the end of this section we shall study in more details the set of good polynomials $\mathcal{A}_{\mu,0}$, $\rho(f) = 0$ (Definition 4).

Let $t_1, t_2, ..., t_s$ be the critical points of a good polynomial $f \in \mathcal{A}_{\mu,\prime}$ and let $D \subset \mathbb{C}$ be a closed disc centered at the origin and such that $t_i \in D$, $i = 1, 2, .., s$. We consider a system of paths $u_1, u_2, ..., u_s$ connecting $t_1, t_2, ..., t_s$ and some fixed non-critical value $t_0 \in \partial D$ of f and such that (see fig.1)

i) each path has no self-intersection points

ii) two distinct paths u_i and u_j meet only at their common origin $u_i(0) = u_j(0) = t_0$.

iii) the points t_i and the paths u_i are numbered in the order they start from the point t_0, counting clockwise.

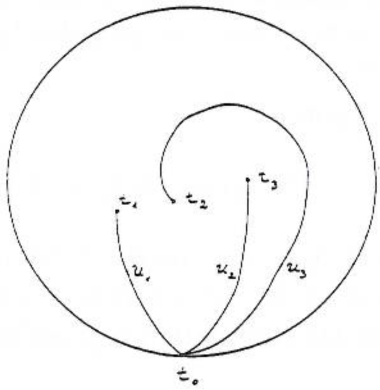


fig.1

We may also suppose that f is a Morse function. Indeed if it is not so then let $e : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone C^∞ real function such that for some $c > 0$ holds $e(r) \equiv 0$ on $[0, c]$ and $e(r) \equiv 1$ on $[2c, \infty]$. Then let $g(x, y)$ be a polynomial and consider

$$\tilde{f}(x, y) = f(x, y) + \epsilon e(|\, x \,| + |\, y \,|)g(x, y). \tag{25}$$

If ϵ is sufficiently small and c sufficiently big, then $\tilde{f}^{-1}(t) \to t$ is locally trivial on the complement of the set of critical values of $\tilde{f}$, $\mu(\tilde{f}) = \mu(f)$, and the general fiber $\tilde{f}^{-1}(t)$ is homeomorphic to the general fiber $f^{-1}(t)$. Although the function $\tilde{f}$ is not a polynomial, it coincides with f in a complement of a compact subset of $\mathbb{C}^2$, and equals to $f(x, y) + \epsilon g(x, y)$ in a disc containing the critical points of f. If $g(x, y)$ is for example a general linear function then Sard theorem implies that $\tilde{f}$ as a complex function on $\mathbb{R}^4 \sim \mathbb{C}^2$ has only simple critical points and we may also suppose that the corresponding critical values are all different (see [1],vol.2 chapter 1).

Now we define in a standard way cycles $\gamma_i(t_0) \in H_1(f^{-1}(t_0))$ vanishing at the critical points corresponding to the critical values of f. Namely, let $P \in \mathbf{C}^2$ be a critical point of f and $f(P) = t_i$. The fibration $f^{-1}(t) \to t$ is locally trivial along the path $u_i - t_i$ and its "limit" fiber $f^{-1}(t_i)$ has a simple singular point P which appears by contracting a cycle $\gamma_i(t)$ in the fiber $f^{-1}(t)$ to the point P. $\gamma_i(t)$ is called a *vanishing cycle* at t_i along the path u_i. Thus we have a family of cycles $\gamma_i(t_0), i = 1, 2, ..., \mu(f)$ in the fiber $f^{-1}(t_0)$.

Definition 7 *The set $\gamma_i(t_0), i = 1, 2, ..., \mu(f)$ of cycles with the numbering as described above, is called a distinguished basis of vanishing cycles for $H_1(f^{-1}(t_0))$.*

The above definition is justified by the following

Proposition 4.2 *The vanishing cycles form a basis of the first homology group of the general fiber $f^{-1}(t)$, $t \notin A_c$, of any good polynomial f.*

Note that according to Corollary 3.6 for a good polynomial we have $H_1(f^{-1}(t)) = \mu(f) - \mu^t(f)$. As in each singular fibre exactly $\mu^t(f)$ cycles vanish then by Proposition 4.2 the vanishing cycles form a basis in any fibre $f^{-1}(t)$. Before proving Proposition 4.2 let us note that a natural consequence is that we may completely describe, as in the local case, the monodromy of the fibres of f.

Consider the fundamental group

$$\pi = \pi_1(D \backslash \{t_1, t_2, ..., t_s\}, t_0)$$

and its monodromy representation $\pi \to Aut H_1(f^{-1}(t))$. If $\alpha_i \in \pi$ corresponds to the path u_i and goes once around t_i anticlockwise, where t_i is a simple critical value, then the corresponding classical monodromy transformation is given by the usual Picard-Lefschetz formula

$$T_i : \gamma \to \gamma - < \gamma, \gamma_i > \gamma_i, \forall \gamma \in H_1(f^{-1}(t)) \tag{26}$$

where γ_i is the cycle vanishing at t_i and $< \gamma, \gamma_i >$ is the intersection number. If t_i is a non-simple critical value then the classical monodromy transformation is obtained by composing the monodromy transformations $T_{i_1}, T_{i_2}, ..., T_{i_k}$ associated to the cycles $\gamma_{t_{i_1}}, \gamma_{t_{i_2}}, ..., \gamma_{t_{i_k}}$ vanishing at t_i and ordered as in the distinguished basis of $H_1(f^{-1}(t_0))$ defined before. Thus the classical monodromy of the general fiber $f^{-1}(t)$ is completely determined by the intersection form $< .,. >$ and the distinguished basis.

Definition 8 *The matrix*

$$I_\gamma = (< \gamma_i, \gamma_j >)_{i,j=1,1}^{\mu,\mu}, \mu = \mu(f)$$

is called the intersection matrix of the good polynomial f with respect to the distinguished basis $\gamma_i, i = 1, 2, ..., \mu(f)$.

We recall that a *Dynkin diagram* corresponding to the intersection matrix I_γ is a graph such that to each cycle γ_i corresponds a vertex and two distinct vertices $\gamma_i, \gamma_j, i < j$ are joined by k edges (respectively k dotted edges) if their intersection number is k (respectively $-k$). If two polynomials belong to one and the same connected component of the set $\mathcal{A}_{\mu,\rho}$ then their general fibres are equivalent up to a proper isotopy. It follows that they have the same Dynkin diagram.

As a simplest example we shall classify the set of all cubic polynomials that we denote by $\mathcal{A}$. For doing that we need some normal forms. We shall say that the polynomials f and g are linearly conjugate if there exists a linear bijective change of the independent variables $H : \mathbb{C}^2 \to \mathbb{C}$ and a linear bijective change of the dependent variable $h : \mathbb{C} \to \mathbb{C}$ such that $f \circ H = h \circ g$. If f_3 and g_3 are the highest homogeneous parts of f and g we have $f_3 \circ H = h \circ g_3$. On the other hand a cubic homogeneous polynomial f_3 is linearly conjugate either to x^3 (f_3 is a third power of a linear form) or to $x(y^2 - ax^2)$ (f_3 is not a third power). With further linear changes of the variables we obtain the following four families of cubic polynomials.

I. $xy^2 + ey - ax^3 - bx^2 - cx$
II. $xy - ax^3 - bx^2 - cx$
III. $y^2 - ax^3 - bx^2 - cx$
IV. $y - ax^3 - bx^2 - cx$

Note that if f is a real polynomial then the linear changes of variables may be chosen real. The above list was first obtained by Newton in his investigation of real cubics (see [2], p.92 for details). Further we compute $\mu(f), \rho(f)$ and the corresponding Dynkin diagram for any value of a, b, c, e. It turns out that

$$\mathcal{A} = \sum_{\mu=0}^{4} \mathcal{A}_{\mu,0} + \mathcal{A}_{0,1} + \mathcal{A}^{\infty}$$

where the set $\mathcal{A}_{2,0}$ has two connected components and the other are connected. The corresponding normal form with respect to the left-right action of the group of real linear bijections is shown in the second column of table 1 (the parameter c is equal to 0 or ± 1). If two polynomials belong to the same connected component $\mathcal{A}_{\mu,\rho}$ then, as we noted, they have the same Dynkin diagram which is shown in the third column. We denote the first 6 families of polynomials by $D_4^{\pm}, A_3, A_2, A_1 + A_1, A_1, A_0$ according to the type of their Dynkin diagram. In each family there is a polynomial with a "most singular" fiber. This polynomial is given in the last column of table 1 and the family is a deformation of it in the class of cubic polynomials with fixed $\mu(f)$ and $\rho(f)$. We note that the polynomial $xy^2 + ey \pm x$, although it is good, has a disconnected Dynkin diagram. The polynomial $xy^2 + y$ is the only non-good, and hence non-tame (Proposition 3.7) cubic polynomial. This was observed also by Broughton [3,4].

set	normal form	Dynkin diagram	notation	representative
$\mathcal{A}_{4,0}$	$xy^2 + ey \pm x^3 - bx^2 - cx$		$D_4^{\pm}$	$x(y^2 \pm x^2)$
$\mathcal{A}_{3,0}$	$xy^2 + ey - x^2 - cx$		A_3	$x(y^2 - x)$
$\mathcal{A}_{2,0}$	$y^2 - x^3 - bx^2 - cx$		A_2	$y^2 - x^3$
$\mathcal{A}_{2,0}$	$xy^2 + ey \pm x,\ e = 0, 1$		$A_1 + A_1$	$xy^2 \pm x$
$\mathcal{A}_{1,0}$	$xy - x^3 - bx^2 - cx$		A_1	$xy - x^3$
$\mathcal{A}_{0,0}$	$y - x^3 - bx^2 - cx$		A_0	$y - x^3$
$\mathcal{A}_{0,1}$	$xy^2 + y$			$xy^2 + y$
$\mathcal{A}^{\infty}$	xy^2			xy^2

Real normal forms for cubic polynomials and their Dynkin diagrams ($c = 0, \pm 1$)
Table 1

Proof of Proposition 4.2. As the proof will be similar to the one in the case of an isolated singularity then we shall omit some of the details referring the reader to [1]. Consider the real valued function

$$F(x, y) = \mid f(x, y) \mid : \mathbb{C}^2 \to \mathbb{R}^+.$$

If R is the radius of the disc D then it is easily seen that $F^{-1}(r) \to r$ is a locally trivial (and hence trivial) fibration on the interval (R, ∞). Note that each fiber $F^{-1}(r)$ on its hand is the total space of the locally trivial fibration $f^{-1}(re^{\sqrt{-1}\varphi}) \to \varphi \in S^1$ with a base the circle S^1 and which is not trivial in general.

It is concluded that the space

$$V = \bigcup_{|t| \leq R} f^{-1}(t) = \bigcup_{r \leq R} F^{-1}(r)$$

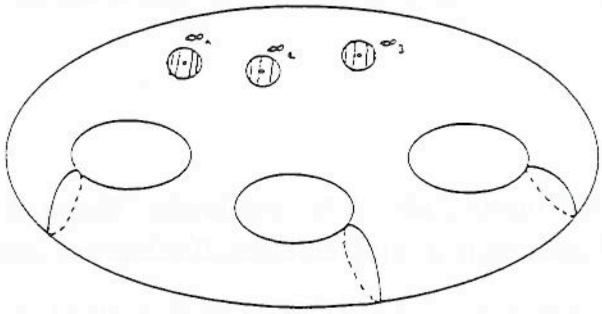
is a deformation retract of $\mathbb{C}^2 = F^{-1}(\mathbb{R}^+)$.Thus we shall restrict our attention to the fibration $f^{-1}(t) \to t$ with $t \in D = \{\mid t \mid \leq R\}$. Further we shall replace the non-compact fibers $f^{-1}(t)$ with some compact subset f_t of it and such that f_t is a deformation retract of $f^{-1}(t)$.

Consider the cylinder $\{\mid x \mid = c\} \subset \mathbb{C}^2, c > 0$. As usual we suppose that x is general with respect to f. The set $f^{-1}(t) \cap \{\mid x \mid = c\}$ is compact and as a real analytic subset of $\mathbb{R}^4 \approx \mathbb{C}^2$ it is of dimension one. Further the set $\{\mid x \mid = c\}$ intersects transversally the fiber $f^{-1}(t)$ at some point P if and only if $\mid f_y(P) \mid \neq 0$. On the other hand f is a good polynomial and no ramification points $P \in f^{-1}(t) \cap f_y^{-1}(0)$ tend to infinity as t varies in the compact set $\overline{D}$. Thus there exists $c_0 \in \mathbb{R}^+$ and such that for any fixed $c \geq c_0$ and $t \in D$ the set $f^{-1}(t) \cap \{\mid x \mid = c_0\}$ is smooth and compact and hence it is a disjoint union of circles. It is clear now that each connected component of the set $V \cap \{\mid x \mid = c\}$ is homeomorphic to $D \times S^1$ provided that $c \geq c_0$. The fibration $V \cap \{\mid x \mid = c\} \to c$ is locally trivial on (c_0, ∞) and hence $V \cap \{\mid x \mid = c_0\}$ is a deformation retract of $V \cap \{\mid x \mid \geq c_0\}$. We shall denote

$$f_t = f^{-1}(t) \cap \{(x, y) \in \mathbb{C}^2 : \mid x \mid \leq c_0\}$$

and we shall study from now on the fibration $f_t \to t$ for $t \in D$. It is clear that its total space $\bigcup_{t \in D} f_t$ is a deformation retract of the space V and hence it has homotopy type of a point.

The fiber f_t is obtained from $f^{-1}(t)$ by removing a small disk around each "infinite" point on the compactified algebraic curve $\overline{f^{-1}(t)}$ (fig.2).



f_t is the Riemann surface $\overline{f^{-1}(t)}$ with removed small disks around each "infinite" point

fig.2

Consider further the union $U = \cup_i u_i$ of the paths connecting t_0 to t_i. It is a deformation retract of the disk D and the covering homotopy theorem implies that $Y = \bigcup_{t \in U} f_t$ is a deformation retract of $\bigcup_{t \in D} f_t$. It follows that Y also has homotopy type of a point. If we remove from Y the singular fibers $f_{t_i}, i = 1, 2, .., s$ then we obtain a space fibered over the set $U - \bigcup_{i=1}^{s} \{t_i\}$ the last being contractible to a point. It follows that $Y - \cup_i f_{t_i}$ has the same homotopy type as the fiber f_{t_0}.

Finally we may use a standard argument (see for example [1])to show that, up to homotopy , the space Y can be built up from the fiber f_{t_0} by adjoining to each vanishing cycle $\gamma_{t_0}^{j}$ a two-dimensional disk D_j^2. On the other hand the (reduced) homology of Y is trivial which shows that f_{t_0} (and hence $f^{-1}(t_0)$) has a homotopy type of $\mu(f)$ circles and $H_1(f^{-1}(t_0))$ is generated by the vanishing cycles of f.

Indeed, as

$$H_1(Y) = H_2(Y) = 0, H_1(Y - \cup_{i=1}^{s} f_{t_i}) = H_1(f_{t_0}) = H_1(f^{-1}(t_0)) \tag{27}$$

then the long exact sequence associated to the pair $(Y, Y - \cup_{i=1}^{s} f_{t_i})$

$$\dots \to H_2(Y) \to H_2(Y, Y - \cup_{i=1}^{s} f_{t_i}) \to H_1(Y - \cup_{i=1}^{s} f_{t_i}) \to H_1(Y) \to \dots \tag{28}$$

gives

$$H_1(f^{-1}(t_0) \sim H_2(Y, Y - \cup_{i=1}^{s} f_{t_i}).$$

On the other hand the fibers f_t are compact, $\partial f_t \to t$ is trivial on $U = \cup_i u_i$ and exactly as in [1] we obtain

$$H_2(Y, Y - \cup_{i=1}^{s} f_{t_i}) = \bigoplus_{i=1}^{\mu(f)} H_2(D_i^2, \partial D_i^2) = \bigoplus_{i=1}^{\mu(f)} H_2(S_i^2) = \mathbf{Z}^{\mu(f)}. \tag{29}$$

The identity (29) clearly holds if f has only simple critical points and different critical values ($s = \mu(f)$ in this case). If it is not so, then we replace f by $\tilde{f}$, where the function $\tilde{f}$ is defined as in (25). All the preceding reasonings hold also for $\tilde{f}$, and the general fiber $\tilde{f}_t$ is homeomorphic to the general fiber f_t. We conclude that $H_1(f^{-1}(t_0)) = \mathbf{Z}^{\mu(f)}$ and moreover the image of the generator of $H_2(D_j^2, \partial D_i^2)$ under (28) is the vanishing cycle $\gamma_j(t_0)$. $\triangle$

References

[1] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko, *Singularities of Differentiable Maps*, vols. 1 and 2, Monographs in mathematics, Birkhauser, 1985 and 1988.

[2] E. Brieskorn, H. Knörrer, *Plane Algebraic Curves*, Birkhaäuser, Boston, 1986.

[3] S.A. Broughton, Milnor numbers and the topology of polynomial hypersurfaces, Invent.math. vol. 92, 217-241 (1988).

[4] S.A. Broughton, On the topology of polynomial hypersurfaces, Proc.Symp. Pure Math. 40, Part 1 (Arcata Singularities Conference), American Mathematical Society, 1983, p.167-178.

[5] A. Dimca, *Singularities and Topology of Hypersurfaces*, Springer, 1992.

[6] W. Fulton,*Algebraic curves*, Addison-Wesley, 1989.

[7] L.Fourrier, Topologie d'un polynôme de deux variables complexes au voisinage de l'infini, prébublication n39, Laboratoire de Topologie et Géométrie, Université Toulouse III, 1994.

[8] P.A. Griffiths, J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons, 1978.

[9] Ha Huy Vui, Sur la fibration globale des polynômes de deux variables complexes, C.R.Acad.Sci.Paris, t.309, série I, p.231-234, 1989

[10] Ha Huy Vui, Nguyen Le Anh,Le comportement géométrique à l'infinie des polynômes de deux variables complexes, C.R.Acad.Sci.Paris, t. 309, série I, p.183-186, 1989.

[11] Ha Huy Vui, Lê Dung Trang, Sur la topolgie des polynômes complexes, Acta. Math. Vietnamica, vol.9, No1,1984, p.21-32.

[12] S. Kaliman, Two remarks on polynomials in two variables, Pacific J. of Math., vol. 154 (1992) 285-295.

[13] J. Milnor, *Singular Points of Complex Hypersurfaces*, Ann.Math.Studies vol.61, Princeton University Press, 1968.

[14] W.D. Neumann, L.Rudolph, Unfoldings in knot theory, Math.Ann. vol.278 (1987) 409-439 and Corrigendum: ibid. vol.282 (1988) 349-351.

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