Separability and Lax pairs for Hénon–Heiles system

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The Hamiltonian system corresponding to the (generalized) Hénon–Heiles Hamiltonian $H=\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}Ax^2+\frac{1}{2}By^2+x^2y+\epsilon y^3$ is known to be integrable in the following three cases: ($A=B$, $\epsilon=\frac{1}{3}$); ($\epsilon=2$); ($B=16A$, $\epsilon=\frac{16}{3}$). In the first two the system has been integrated by making use of genus one and genus two theta functions. We show that in the third case the system can also be integrated by making use of elliptic functions. Finally, using the Fairbanks theorem, we find Lax pairs for each of the three integrable systems under consideration.

I. INTRODUCTION

The (generalized) Hénon–Heiles Hamiltonian

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+\tfrac{1}{2}Ax^2+\tfrac{1}{2}By^2+x^2y+\epsilon y^3 \tag{1}$$

has been extensively studied in nonintegrable and integrable regimes. Only three integrable cases are known:[1–4]

$$\begin{aligned}
&\text{(i)} && A=B, && \epsilon=\tfrac{1}{3},\\
&\text{(ii)} && && \epsilon=2,\\
&\text{(iii)} && B=16A, && \epsilon=\tfrac{16}{3},
\end{aligned}$$

and the nonintegrability in some of the remaining cases (including the "historical" Hénon–Heiles Hamiltonian[5] corresponding to $A=B$ and $\epsilon=-\frac{1}{3}$) has been established by Ziglin,[6] Ito,[7] and Fordy.[8] By integrability here we mean existence of a second (global) integral of motion, and in this case the Liouville theorem implies that the problem can be solved by quadratures. This, however, can be done only after the finding of special new variables which separate the associated Hamilton–Jacobi equation. Such separating variables are known for the case (i) and (ii) above: the case (i) trivially separates in Cartesian coordinates

$$x=\lambda-\mu,\ y=\lambda+\mu,$$

though the case (ii) separates in translated parabolic coordinates

$$x^2=-4\lambda\mu,\ y=\lambda+\mu+(B-4A)/4 \quad \text{(see Ref. 9)}. \tag{2}$$

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In the present paper we shall find separating variables for the case (iii) (Sec. II). For doing that we shall make use of the algebraic structure of the problem. Namely, it is well known that the majority of integrable polynomial systems are algebraically completely integrable, i.e., the complexified system linearizes on an appropriate Abelian variety.[10] If it is irreducible and of dimension two, then there exists a general procedure, due to Pol Vanhaecke[11,12] for finding the separating variables. It turns out, however, that the Abelian variety in our case is a reducible one (i.e., it is a direct product of two elliptic curves). Nevertheless the separating variables can be found by expecting the Painlevé expansions of the solutions near some special divisors on the compactified invariant manifolds of the problem. Finally, in Sec. III we apply Fairbanks theorem[13] to derive Lax pairs for the three integrable cases above.

II. SEPARATION OF THE VARIABLES

The Hamiltonian system

$$\frac{d^2}{dt^2}x=-x(A+2y),\quad \frac{d^2}{dt^2}y=-16Ay-x^2-16y^2, \tag{3}$$

corresponding to the Hénon–Heiles Hamiltonian

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+\tfrac{1}{2}Ax^2+8Ay^2+x^2y+\tfrac{16}{3}y^3, \tag{4}$$

possesses a second integral of motion[3,4]

$$F=9p_x^4+18(A+2y)p_x^2x^2-12p_xp_yx^3+x^4(9A^2-12Ay-2x^2-12y^2). \tag{5}$$

Let us conjecture that the complex invariant set

$$\mathbf{V}_C=\{(x,y,p_x,p_y)\in\mathbf{C}^4:H=h,F=f\}$$

is an affine part of an Abelian variety. Then this variety contains elliptic curves as

$$\mathbf{V}_C\cap\{x=0\}=\{9p_x^4=f,\tfrac{1}{2}p_x^2+\tfrac{1}{2}p_y^2+8Ay^2+\tfrac{16}{3}y^3=h\}$$

and hence it should be reducible,[14] that is to say a product of elliptic curves. Denote by Γ_1 and Γ_2 the elliptic curves defined by

$$\Gamma_1:=\left\{(w_1,z_1)\in\mathbf{C}^2:\frac{w_1^2}{2}+8Az_1^2+\frac{16}{3}z_1^3-h+\frac{\sqrt{f}}{6}=0\right\},$$

$$\Gamma_2:=\left\{(w_2,z_2)\in\mathbf{C}^2:\frac{w_2^2}{2}+8Az_2^2+\frac{16}{3}z_2^3-h-\frac{\sqrt{f}}{6}=0\right\}.$$

The functions z_1 and z_2 on $\Gamma_1\times\Gamma_2$ are uniquely determined (modulo multiplication and addition of a constant) by their infinity divisors

$$(z_1)_\infty=2\Gamma_2,\quad (z_2)_\infty=2\Gamma_1\,. \tag{6}$$

As we shall see later z_1 and z_2 are separating variables and the curves Γ_1 and Γ_2 will serve as "coordinates axes" on $\Gamma_1\times\Gamma_2$ in the same way as the coordinates axes in $\mathbf{R}^2$ (see Fig. 1).

To verify our conjecture we shall find the functions z_1 and z_2 having the property (6). This can be done by expecting the Painlevé expansions of the solutions, and also their Taylor expansions around the zero divisor of the function x restricted to $\mathbf{V}_C$.

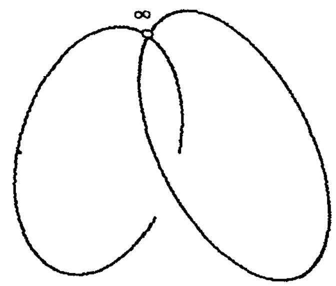

FIG. 1. Coordinates axes on $\Gamma_1 \times \Gamma_2$.

Let us compute first the Painlevé expansions. This is a part of the so-called Painlevé property test (see Refs. 15 and 16 for instance). If we assign to x and y weighed degrees 2, and substitute the Painlevé expansions

$$x=\frac{1}{t^2}\left(\sum_{i=0}^{\infty} a_i t^{i/n}\right), \quad n\in\mathbf{N},$$
$$y=\frac{1}{t^2}\left(\sum_{i=0}^{\infty} b_i t^{i/n}\right), \tag{7}$$

in (3) we find

$$6a_0=-2a_0b_0, \quad 6b_0=-a_0^2-16b_0^2.$$

It follows that there are two possible leading behaviors

$$x=a_0/t^2, \quad y=b_0/t^2, \tag{8}$$

where

$$b_0=-3, \quad a_0=3\sqrt{-14}$$

or

$$a_0=0, \quad b_0=-3/8.$$

The Kovalevskaya matrix K corresponding to the system (3) and to leading behavior (8) is

$$K=\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2b_0 & -2a_0 & 3 & 0 \\ -2a_0 & -32b_0 & 0 & 3 \end{pmatrix}.$$

In the first case we find

$$\det(K-\lambda I)=(\lambda+1)(\lambda+7)(\lambda-6)(\lambda-12)$$

and hence the series (7) can not depend upon three parameters. In the second case

$$\det(K-\lambda I)=(\lambda+1)(\lambda-6)(\lambda-7)(2\lambda-3)/4.$$

Thus we may take $n=2$ in (7) and after calculations we find that indeed the system (3) posseses a three-parameter family of solutions of the form

$$\begin{aligned} x&=\frac{\alpha}{\sqrt{t}}+\beta t\sqrt{t}-\frac{\alpha}{18}t^2\sqrt{t}+\frac{\alpha A^2}{10}t^3\sqrt{t}-\frac{\alpha^2\beta}{18}t^4\sqrt{t}+\cdots, \\ y&=-\frac{3}{8t^2}-\frac{A}{2}+\frac{\alpha^2}{12}t-\frac{2A^2}{5}t^2+\frac{\alpha\beta}{3}t^3-\gamma t^4+\cdots, \end{aligned} \tag{9}$$

where α, β, γ are the free parameters. It is seen that x and $p_x=(d/dt)x$ are not meromorphic functions on our Abelian variety, but rather their squares. This suggests to us to consider the involution

$$\iota:(x,p_x)\mapsto(-x,-p_x),$$

acting on $\mathbf{V}_C$ and to consider the variety $\mathbf{A}_C=\mathbf{V}_C/\iota$. We note the following obvious identity

$$\mathbf{A}_C\cap\{x=0\}\equiv\Gamma_1\cup\Gamma_2.$$

It is easy to compute the Taylor expansions of the solutions around the zero divisor of x restricted to $\mathbf{V}_C$:

$$\begin{aligned} x&=c_1t-c_1[(2d_0+A)/6]t^3+\cdots, \\ y&=d_0+d_1t-8d_0(d_0+A)t^2-\tfrac{8}{3}d_1(2d_0+A)t^3+\cdots. \end{aligned} \tag{10}$$

After substituting (10) in the identities $H=h$ and $F=f$ we determine the following relations between the parameters c_1, d_0, d_1:

$$9c_1^4=f,\ \frac{c_1^2}{2}+\frac{d_1^2}{2}+8Ad_0^2+\frac{16}{3}d_0^3=h. \tag{11}$$

Let us find now the functions z_1 and z_2 having the property (6). Using (10) and (11) we have

$$\left(\frac{p_x}{x}\right)^2=\frac{1}{t^2}-\frac{2(A+2d_0)}{3}+\cdots,\quad \frac{\sqrt{f}}{x^2}=3\frac{1}{t^2}+\cdots,$$

and hence for the divisors of the functions $\pm\sqrt{f}/x^2+(p_x/x)^2$ restricted to $\mathbf{A}_C$ holds

$$\left[\frac{\sqrt{f}}{x^2}+\left(\frac{p_x}{x}\right)^2\right]\geqslant-2\Gamma_1,\quad \left[-\frac{\sqrt{f}}{x^2}+\left(\frac{p_x}{x}\right)^2\right]\geqslant-2\Gamma_2.$$

We are looking now for functions which do not blow up at infinity. Using the expansions (9) we compute

$$\left(\frac{p_x}{x}\right)^2=\frac{1}{4t^2}+\cdots,\quad \frac{\sqrt{f}}{x^2}=\frac{\sqrt{f}t}{\alpha^2}+\cdots,\quad 2y=-\frac{3}{4t^2}-A+\cdots.$$

Then

$$\pm\frac{\sqrt{f}}{x^2}+3\left(\frac{p_x}{x}\right)^2+2y=\frac{6\beta}{\alpha}+o(t)$$

and we conclude that the functions

$$z_1=\frac{\sqrt{f}}{x^2}-\left[3\left(\frac{p_x}{x}\right)^2+2y\right],\quad z_2=-\frac{\sqrt{f}}{x^2}-\left[3\left(\frac{p_x}{x}\right)^2+2y\right],$$

restricted to $\mathbf{A}_C$ blow up only on $\Gamma_1+\Gamma_2$ and $(z_1)_\infty=2\Gamma_2$, $(z_2)_\infty=2\Gamma_1$.

Let us define now on each invariant level set $\mathbf{V}_C\equiv\{H=h,F=f\}$ new variables u and v by

$$u=z_1|_{\mathbf{V}_C},\quad v=z_2|_{\mathbf{V}_C}.$$

A straightforward computation shows that on $\mathbf{V}_C$ the following identities hold:

$$x^2=-2\sqrt{f}/(u-v), \tag{12}$$

$$p_x^2=\sqrt{f}(u+v+4y)/3(u-v), \tag{13}$$

$$p_y=\frac{1}{2\sqrt{-6(u+v+4y)}}\left[\frac{-2\sqrt{f}}{u-v}-(3A-u)(3A-v)+24Ay+4(u+v)y+32y^2\right], \tag{14}$$

$$y=-\frac{u+v}{4}-\left[\frac{\sqrt{P(u)}+\sqrt{Q(v)}}{2(u-v)}\right]^2, \tag{15}$$

where

$$\begin{aligned}P(u)&=u^3-3Au^2-9A^2u+27A^3-2\sqrt{f}-12h,\\ Q(v)&=v^3-3Av^2-9A^2v+27A^3+2\sqrt{f}-12h.\end{aligned} \tag{16}$$

Differentiating (13) and (12) with respect to the time t and using (3) we get

$$\frac{du}{dt}-\frac{dv}{dt}=2\,\frac{\sqrt{P(u)}-\sqrt{Q(v)}}{\sqrt{6}},\quad \frac{du}{dt}+\frac{dv}{dt}=2\,\frac{\sqrt{P(u)}+\sqrt{Q(v)}}{\sqrt{6}},$$

and hence

$$\frac{du}{dt}=\sqrt{\frac{2}{3}P(u)},\quad \frac{dv}{dt}=\sqrt{\frac{2}{3}Q(v)}. \tag{17}$$

Thus u,v (and hence x,y,p_x,p_y) can be expressed in terms of Weierstrass elliptic functions. Finally we note that, according to formulas (12), (15), and (17), the functions $x^2,y,x.p_x=\frac{1}{2}(d/dt)x^2$, $p_y=(d/dt)y$, are single-valued on $\Gamma_1\times\Gamma_2$. On the other hand these functions live on the variety $\mathbf{A}_C=\mathbf{V}_C/\iota$ and following Ref. 17 we may show that $\mathbf{A}_C$ can be identified with an affine part of $\Gamma_1\times\Gamma_2$.

III. LAX PAIRS

In this chapter we shall find Lax pairs for the three integrable cases of Hénon–Heiles Hamiltonian (1).

Let U, V, and W be functions in t with the property $UW+V^2=c$, $c=\text{const}$. Then the following obvious identity holds

$$\frac{d}{dt}L=[L,M], \quad [L,M]=LM-ML, \tag{18}$$

where

$$L=\begin{pmatrix} V & U \\ W & -V \end{pmatrix},$$
$$M=\frac{1}{2V}\begin{pmatrix} 0 & \left(\frac{d}{dt}\right)U \\ -\left(\frac{d}{dt}\right)W & 0 \end{pmatrix}, \quad M=\frac{1}{2V}\begin{pmatrix} 0 & \frac{d}{dt}U \\ -\frac{d}{dt}W & 0 \end{pmatrix}. \tag{19}$$

Suppose now that a completely integrable Hamiltonian system is given which linearizes (for almost all values of the constants of motion) on a Jacobian variety $\text{Jac}(\Gamma)$ of a hyperelliptic curve Γ: $w^2=f(z)$, where $f(z)$ is a polynomial with coefficients depending upon the constants of motion. Then, as it has been noted first by Fairbanks[13] (see also Pol Vanhaecke[11]), we may take $c=f(z)$, and define U, V, W to be the Jacobi polynomials (see Refs. 13 and 18) on $S^g\Gamma\sim\text{Jac}(\Gamma)$ (g is the genus of Γ). Thus we obtain a Lax pair (18) depending on a spectral parameter z, and the the coefficients of $f(z)$ (and hence the first integrals) are reconstructed from the identities

$$\det(L-wI)=w^2-V^2-UW=w^2-f(z)=\text{const}.$$

Remark: In Fairbanks[13] the matrix M is given in the form

$$M=\frac{1}{U}\begin{pmatrix} 0 & 0 \\ \frac{dV}{dt} & -\frac{dU}{dt} \end{pmatrix},$$

which differs from ours in a linear combination of L and the identity matrix. Our choice of M, however, gives simpler expressions in the initial phase variables.

Suppose first that Γ is a genus 2 curve. Let p_1 and p_2 be points on Γ and denote $\lambda=z(p_1)$, $\mu=z(p_2)$. Then the Jacobi polynomials associated with Γ read

$$U(z)=(z-\lambda)(z-\mu), \quad V(z)=\frac{\sqrt{f(\lambda)}(z-\mu)-\sqrt{f(\mu)}(z-\lambda)}{\lambda-\mu},$$

$$W(z)=\frac{f(z)-V(z)^2}{U(z)}.$$

Note that $W(z)$ is in fact a polynomial in z.

Let us find now a Lax pair for the integrable case (ii), $\epsilon=2$. The corresponding Hamiltonian system has a second integral of motion:

$$F=x_1^4+4x^2y^2-4p_x(p_xy-p_yx)+4Ax^2y+(4A-B)(p_x+Ax^2).$$

and the Hamilton–Jacobi equation separates in λ, μ coordinates[9] given by (2). On each Liouville torus $\{H=h,F=f\}$ the Hamiltonian system takes the form

$$\frac{d\lambda}{\sqrt{f(\lambda)}}+\frac{d\mu}{\sqrt{f(\mu)}}=0,$$
$$\frac{\lambda\,d\lambda}{\sqrt{f(\lambda)}}+\frac{\mu\,d\mu}{\sqrt{f(\mu)}}=dt \Leftrightarrow \dot{\lambda}=\frac{\sqrt{f(\lambda)}}{\lambda-\mu},\quad \dot{\mu}=\frac{\sqrt{f(\mu)}}{\mu-\lambda}, \tag{20}$$

where

$$f(z)=-(z/4)[16z^4+8(6A-B)z^3+(12A-B)(4A-B)z^2-(8h-A(4A-B)^2)z-f].$$

Thus our system linearizes on the Jacobi variety $\mathrm{Jac}(\Gamma)$ of the genus 2 hyperelliptic curve Γ: $w^2=f(z)$, and we are in a position to apply the Fairbanks theorem. By making use of the first integrals, and (2) we compute the Jacobi polynomials

$$U(z)=z^2-(\lambda+\mu)z+\lambda\mu=\frac{4z^2-(4y+B-4A)z-x^2}{4},$$

$$V(z)=\frac{\sqrt{f(\lambda)}(z-\mu)-\sqrt{f(\mu)}(z-\lambda)}{\lambda-\mu},$$

$$\dot{U}(z)=\frac{2p_yz+xp_x}{2},\quad W(z)=\frac{f(z)-v^2(z)}{u(z)},$$

and after some calculation

$$W(z)=-4z^3-z^2(4y+8A-B)+p_x^2-z(x^2+4y^2+4Ay+A(4A-B)).$$

At last

$$\frac{1}{2V}\frac{dW}{dt}=\frac{1}{2V}\frac{dU}{dt}=\frac{az+2a(\lambda+\mu)+b}{2},$$

where

$$\lambda+\mu=y+(4A-B)/4,\quad a=1,\quad b=-(6A-B)/2.$$

and hence $(1/2V)(dW/dt)=2z+4y+2A$.

Thus we found the Lax pair:[19] $\dot{L}=[L,M]$ where

$$L(z)=\begin{pmatrix} 2p_yz+xp_x & 4z^2-4zy-(4A-B)z-x^2 \\ \begin{array}{c}4z^3+z^2(4y+B-8A)+p_x\\ +z(x^2+4y^2-4Ay+(4A-B))\end{array} & -2p_yz-xp_x \end{pmatrix},$$

$$M=\begin{pmatrix} 0 & -1 \\ -z-2y+A & 0 \end{pmatrix}.$$

In the case $A=B=0$ this Lax pair is found first by Newell *et al.*[20] by making use of Painlevé expansions. On the other hand, the case of arbitrary A and B is also interesting, as the constants A,B,f,h provide moduli of the (three-dimensional) space of all Abelian varieties of polarization (1,2) (see Ref. 17). Suppose now that Γ: $w^2=f(z)$ is a genus 1 curve. The Jacobi polynomials associated with Γ read

$$U(z)=z-\lambda, \quad V(z)=\sqrt{f(\lambda)}, \quad W(z)=\frac{f(z)-V^2(z)}{U(z)}.$$

As in the case (i) and (iii) the Hamiltonian system linearizes on $\Gamma\times\Gamma$ and for the corresponding variables λ,μ holds $\dot{\lambda} = \sqrt{f(\lambda)}$, $\dot{\mu} = \sqrt{f(\mu)}$, then we find a "pair" of Lax pairs $(d/dt)L_i=[L_i,M_i]$, $i=1,2$ where L_1,M_1 depend only on λ, and L_2,M_2 depend only on μ (λ is just replaced by μ in L_1,M_1). Thus we may define 4×4 matrices

$$L=\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad M=\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

and we get a Lax pair $\dot{L}=[L,M]$ from which the differential equation can be reconstructed.

Finally after some calculations we find for the case (i)

$$L_1=\begin{pmatrix} p_y-p_x & y-x-k \\ \dfrac{2k^2+k[3A+2(y-x)]+3A(y-x)+2(y-x)^2}{12} & p_x-p_y \end{pmatrix}, \tag{21}$$

$$M_1=\begin{pmatrix} 0 & 1 \\ -\dfrac{y-x}{3}-\dfrac{3A+2k}{12} & 0 \end{pmatrix}, \tag{22}$$

$$L_2=\begin{pmatrix} p_y+p_x & y+x-k \\ \dfrac{2k^2+k[3A+2(y+x)]+3A(y+x)+2(y+x)^2}{12} & -p_x-p_y \end{pmatrix}, \tag{23}$$

$$M_2=\begin{pmatrix} 0 & 1 \\ -\dfrac{y+x}{3}-\dfrac{3A+2k}{12} & 0 \end{pmatrix}. \tag{24}$$

In a similar way for the case (iii) we get

$$U=\pm\sqrt{F}/x^2-3(p_x^2/x^2)-2y-k, \tag{25}$$

$$V=\pm\frac{\sqrt{F}p_x}{x^3}-3\frac{p_x^3}{x^3}-3\frac{p_x}{x}(A+2y)+p_y, \tag{26}$$

$$W=\pm\frac{\sqrt{F}}{x^2}\left(\frac{A}{2}-\frac{k}{6}+\frac{p_x^2}{x^2}+\frac{2y}{3}\right)+3\frac{p_x^4}{x^4}+\frac{p_x^2}{x^2}\left(8y+\frac{9A-k}{2}\right)$$

$$+\tfrac{1}{6}[k^2-k(3A+2y)-2(3Ay+x^2+4y^2)]-2\frac{p_x}{x}p_y, \tag{27}$$

where the signs correspond to L_1 and L_2, the last being defined as in (19), and in a similar way we find for M_1 and M_2 the matrices

$$M_{1,2}=\begin{pmatrix} 0 & -1 \\ \pm\frac{\sqrt{F}}{3x^2}-\frac{p_x^2}{x^2}-\frac{2}{3}y+\frac{k}{6}-\frac{a}{2} & 0 \end{pmatrix}.$$

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