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THE PERIOD FUNCTION OF A HAMILTONIAN QUADRATIC SYSTEM*

W.A. Coppel
Department of Theoretical Physics, Institute of Advanced Studies
Australian National University,
GPO Box 4, Canberra, A.C.T. 2601, Australia

L. Gavrilov
U.R.A. 1408, Université Paul Sabatier, Laboratoire de Topologie et Géométrie
118, route de Narbonne, 31062 Toulouse Cedex, France

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Abstract. It is shown that, for a planar Hamiltonian quadratic system with a center, the period of the associated periodic orbits is a strictly increasing function of the energy.

1. Introduction. Let

$$dx/dt = H_y, \quad dy/dt = -H_x \tag{1}$$

be a planar Hamiltonian system with a center, which for definiteness we assume located at the origin. The origin is surrounded by a continuous family of periodic orbits. Each periodic orbit in this continuous family lies on an energy level set $H(x,y) = h$ and may be denoted by $\gamma(h)$, since it is uniquely determined by h. The *period function* $T(h)$ is the (least) period of $\gamma(h)$.

The dependence of the period on the energy has been extensively studied. On the one hand there is interest in *isochronous* systems, for which $T(h)$ is a constant. On the other hand, in studying the perturbation of periodic orbits by Melnikov's method (see, e.g., Guckenheimer and Holmes [8]) it is assumed that the derivative $T'(h)$ is nonzero, so that $T(h)$ is a strictly monotonic function.

Conditions which ensure that $T'(h) \neq 0$ have been given for specific types of Hamiltonian system by several authors. Schaaf [10] has considered systems of the form

$$dx/dt = g(y), \qquad dy/dt = -f(x),$$

and the special case

$$d^2x/dt^2 + f(x) = 0 \tag{2}$$

has received particular attention. It is known that the period function for a center of (2) is monotonic if $\int_0^x f(\xi)\, d\xi / f^2(x)$ is convex (Chicone [1]), or if f is a polynomial

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with only simple real zeros (Schaaf [10]), or if f is a quadratic polynomial (Chow and Sanders [4], Schaaf [10]). On the other hand, if f is a cubic polynomial then $T(h)$ may have one, but not more than one, critical point (Chow and Sanders [4], Gavrilov [7]).

In the present work we show that $T'(h) > 0$ whenever the Hamiltonian system (1) is quadratic, i.e., when $H(x, y)$ is a cubic polynomial in x and y. It is interesting that the proof depends on reducing the Hamiltonian system to a differential equation of the form (2), although with a different independent variable t (which changes the period function).

2. The differential equation (2). We are going to study first the period function of the second-order differential equation (2) under the following general hypotheses on the function f :

(*) f is a three times continuously differentiable function on an open interval (α, β), where $\alpha < 0 < \beta$, such that $f(0) = 0$, $f'(0) > 0$ and $f(x) \neq 0$ if $x \neq 0$.

The hypotheses $(*)$ imply that $f(x) < 0$ for $x \in (\alpha, 0)$ and $f(x) > 0$ for $x \in (0, \beta)$. Hence, if we put

$$F(x) := \int_0^x f(\xi)\, d\xi, \tag{3}$$

then $F(x) > 0$ for all $x \in (\alpha, \beta)$ with $x \neq 0$. If we further put

$$\psi(x) := F(x)/f^2(x), \tag{4}$$

then also $\psi(x) > 0$ for all $x \in (\alpha, \beta)$ with $x \neq 0$. In fact $\psi(x)$ is well-defined and positive even for $x = 0$. For, when $x \to 0$,

$$f(x) = f'(0)x + \tfrac{1}{2} f''(0)x^2 + \tfrac{1}{6}[f'''(0) + o(1)]x^3$$

and hence

$$\psi(x) = \frac{1}{2f'(0)} - \frac{f''(0)}{3f'(0)^2}x + \frac{1}{2}\left[\frac{(5/3)f''(0)^2 - f'(0)f'''(0)}{4f'(0)^3} + o(1)\right]x^2.$$

Thus, $\psi(0) = 1/2f'(0)$, $\psi'(0) = -f''(0)/3f'(0)^2$ and

$$\psi''(0) = [(5/3)f''(0)^2 - f'(0)f'''(0)]/4f'(0)^3. \tag{5}$$

It follows that ψ is twice continuously differentiable on the interval (α, β).

It is convenient to replace the differential equation (2) by the equivalent system

$$dx/dt = y, \qquad dy/dt = -f(x). \tag{6}$$

The system (6) is Hamiltonian with Hamiltonian function

$$H(x, y) = y^2/2 + F(x). \tag{7}$$

It has a center at the origin surrounded by a continuous family of periodic orbits, each lying on an energy level set. We denote the periodic orbit lying on the energy level set $H(x,y) = h$ by $\gamma(h)$ and its period by $T(h)$. Since the origin is the only critical point of (6) under the hypotheses $(*)$, we may call $T(h)$ the period function of the differential equation (2).

The orbit $\gamma(h)$ is symmetric with respect to the x-axis. The portion above the x-axis is given explicitly by

$$y = y(x,h) := 2^{1/2}[h - F(x)]^{1/2} \tag{8}$$

and is defined for $x_1(h) \leq x \leq x_2(h)$, where $x_1(h)$ and $x_2(h)$ are the (unique) negative and positive roots of the equation $F(x) = h$.

Since $dx/dt = y$, the period function $T(h)$ is given by the line integral

$$T(h) = \oint_{\gamma(h)} dx/y.$$

Moreover, since $\gamma(h)$ is described clockwise, we can write this in the form

$$T(h) = 2\int_{x_1(h)}^{x_2(h)} dx/y, \tag{9}$$

where $y = y(x,h)$.

Lemma 1. *Let the function f satisfy the hypotheses $(*)$. Then the period function $T(h)$ for the differential equation (2) has a derivative with respect to h given by*

$$T'(h) = \frac{1}{2h}\iint_{\sigma(h)} (F/f^2)''\,dx\,dy, \tag{10}$$

where $\sigma(h)$ is the compact region bounded by $\gamma(h)$.

Proof. For $y = y(x,h)$ we have $\partial y/\partial x = -f/y$. Hence, by integrating by parts repeatedly we obtain

$$\begin{aligned}
\frac{1}{3}\int_{x_1(h)}^{x_2(h)} y^3(F/f^2)''\,dx &= \int_{x_1(h)}^{x_2(h)} (F/f^2)'yf\,dx = \int_{x_1(h)}^{x_2(h)} (F/y - Ff'y/f^2)\,dx \\
&= \int_{x_1(h)}^{x_2(h)} \{F/y - y + (F/f)'y\}dx = \int_{x_1(h)}^{x_2(h)} (2F/y - y)\,dx \\
&= \int_{x_1(h)}^{x_2(h)} (2h/y - 2y)\,dx.
\end{aligned}$$

Therefore, by (9),

$$hT(h) = 2\int_{x_1(h)}^{x_2(h)} y\,dx + \frac{1}{3}\int_{x_1(h)}^{x_2(h)} y^3(F/f^2)''dx. \tag{11}$$

We now differentiate (11) with respect to h. Noting that $y\partial y/\partial h = 1$, and that $y[x_j(h),h] = 0$ and $f[x_j(h)]x_j'(h) = 1$ for $j = 1,2$, we obtain

$$hT'(h) + T(h) = 2\int_{x_1(h)}^{x_2(h)} dx/y + \int_{x_1(h)}^{x_2(h)} y(F/f^2)''dx.$$

Thus

$$2hT'(h) = 2\int_{x_1(h)}^{x_2(h)} y(F/f^2)''dx = \oint_{\gamma(h)} y(F/f^2)''dx.$$

The formula (10) for $T'(h)$ follows immediately by applying Green's theorem (and remembering that $\gamma(h)$ is described clockwise).

Lemma 2. *Let the function f satisfy the hypotheses (*). If $f'''(x) < 0$ for all $x \in (\alpha, \beta)$, then the function $\psi = F/f^2$ satisfies $\psi''(x) > 0$ for all $x \in (\alpha, \beta)$. The same conclusion holds if $f'''(x) = 0$, $f''(x) \neq 0$ for all $x \in (\alpha, \beta)$.*

Proof. Consider first the case $f''' < 0$. From (5) we immediately obtain $\psi''(0) > 0$. If $x \neq 0$, then from $\psi(x) = F(x)/f^2(x)$ we obtain

$$\psi''(x) = N(x)/f^4(x),$$

where

$$N = -2Fff'' - 3(f^2 - 2Ff')f'.$$

Hence

$$N' = -2Fff''' - 5(f^2 - 2Ff')f''.$$

It follows that if $\psi''(x_0) = 0$ for some $x_0 \neq 0$ with $f'(x_0) \neq 0$, then

$$\psi'''(x_0) = (2/3)[5f''(x_0)^2 - 3f'(x_0)f'''(x_0)]F(x_0)/f'(x_0)^3 f'(x_0). \tag{12}$$

Suppose that $f'(x) > 0$ for $0 \leq x < \delta$, where $0 < \delta \leq \beta$. We will show that $\psi''(x) > 0$ for $0 \leq x < \delta$. Assume, on the contrary, that $\psi''(x) = 0$ for some $x \in (0, \delta)$ and let x_0 be the nearest such x to 0. Then $\psi'''(x_0) \leq 0$, since $\psi''(x) > 0$ for $0 \leq x < x_0$. But, since $f(x_0) > 0$, $f'(x_0) > 0$ and $f'''(x_0) < 0$, it follows from (12) that $\psi'''(x_0) > 0$. Thus we have a contradiction.

Similarly we can show that if $f'(x) > 0$ for $\gamma < x \leq 0$, where $\alpha \leq \gamma < 0$, then $\psi''(x) > 0$ for $\gamma < x \leq 0$.

Thus we may now assume that $f'(x) = 0$ for some $x \in (\alpha, \beta)$. Suppose that $f'(x) = 0$ for some $x \in (0, \beta)$. Then we can choose $\delta \in (0, \beta)$ so that $f'(\delta) = 0$ and $f'(x) > 0$ for $0 \leq x < \delta$. This implies $f''(\delta) \leq 0$ and actually $f''(\delta) < 0$, since $f'''(\delta) < 0$. Since $f''' < 0$, it follows that $f''(x) < 0$ for $\delta \leq x < \beta$ and hence $f'(x) < 0$ for $\delta < x < \beta$. Therefore, $(f^2 - 2Ff')(x) > 0$ for $\delta < x < \beta$ and $N(x) > 0$ for $\delta \leq x < \beta$. This proves that $\psi''(x) > 0$ for all $x \in [0, \beta)$.

Similarly we can show that if $f'(x) = 0$ for some $x \in (\alpha, 0)$, then $\psi''(x) > 0$ for all $x \in (\alpha, 0]$.

Consider next the case $f''' \equiv 0$, $f'' \neq 0$. In this case $f(x) = kx(1 + ax)$ for some constants $k > 0$, $a \neq 0$, and without loss of generality we may assume $k = 1$. But then, by direct calculation,

$$\psi''(x) = a^2(5 + 2ax)/3(1 + ax)^4.$$

Since $f(x)/x = 1 + ax > 0$, this shows that $\psi''(x) > 0$ for all $x \in (\alpha, \beta)$. □

From Lemmas 1 and 2 we obtain at once

Proposition 1. *Let the function f satisfy the hypotheses (*). If either $f'''(x) < 0$ for all $x \in (\alpha, \beta)$, or $f'''(x) = 0$, $f''(x) \neq 0$ for all $x \in (\alpha, \beta)$, then the period function $T(h)$ of the differential equation (2) satisfies $T'(h) > 0$ wherever it is defined, i.e., for $0 < h < h^*$, with*

$$h^* = \min\{F(\alpha + 0), F(\beta - 0)\}.$$

Remark. The formula (10) shows that $T'(h) \to \pi(F/f^2)''(0) > 0$ as $h \to +0$.

The interest of the preceding discussion lies more in the derivations than in the results. The formula (10) for $T'(h)$ appears to be new, but it is equivalent to one given by Chicone [1] by a less transparent argument. Furthermore, Proposition 1 is contained in Theorem 1 of Schaaf [10], which has more general hypotheses. However, the hypothesis $f''' < 0$ makes the proof of Proposition 1 very simple and natural.

3. Hamiltonian quadratic systems. We now return to the Hamiltonian system (1), where $H(x, y)$ is a cubic polynomial in x and y. Without loss of generality we can write $H = H_2 + H_3$, where

$$H_2 = (Ax^2 + 2Bxy + Cy^2)/2$$

is a positive definite quadratic form and H_3 is a homogeneous cubic polynomial. By a linear transformation with determinant $+1$ we may suppose $B = 0$ and $A = C$. By a constant magnification of the time scale we may further suppose $A = C = 1$. Thus we may assume

$$H_2 = (x^2 + y^2)/2, \quad H_3 = (1/3)ax^3 + bx^2y + cxy^2 + (1/3)dy^3.$$

Furthermore, by a rotation of the axes we may also assume $d = 0$. The Hamiltonian system (1) now has the form

$$\begin{aligned} dx/dt &= y + bx^2 + 2cxy \\ dy/dt &= -x - ax^2 - 2bxy - cy^2. \end{aligned} \tag{13}$$

A periodic orbit of (13) which surrounds the origin must lie in the half-plane $1 + 2cx > 0$ if $c \neq 0$, since $dx/dt = b/4c^2$ on the bounding line $x = -1/2c$. If (also for $c = 0$) we define a new independent variable τ by setting $dt/d\tau = \varphi(x)$, where

$$\varphi(x) = (1 + 2cx)^{-1/2}, \tag{14}$$

then it is not difficult to verify that (13) is replaced by the second-order differential equation

$$d^2x/d\tau^2 + f(x) = 0, \tag{15}$$

where

$$f(x) = x + ax^2 - 2b^2x^3(1 + 2cx)^{-1} + b^2cx^4(1 + 2cx)^{-2}.$$

(The mysterious choice of φ is explained at the end of the paper.) We can rewrite this expression for f in the form

$$f(x) = \frac{b^2}{16c^3(1 + 2cx)^2} - \frac{b^2}{16c^3} + \frac{b^2x}{4c^2} - \frac{3b^2x^2}{4c} + x + ax^2. \tag{16}$$

Evidently $f(0) = 0$, $f'(0) = 1$ and $f'''(x) = -12b^2/(1 + 2cx)^5$. We note also that

$$F(x) = \int_0^x f(\xi)\, d\xi = \frac{1}{2}x^2 + \frac{1}{3}ax^3 - \frac{b^2x^4}{2(1 + 2cx)}. \tag{17}$$

If γ is a periodic orbit of (13) which surrounds the origin then, since the system is quadratic, the interior of γ is a convex region which contains no critical point of (13) besides the origin (see, e.g., [5]). The critical points of (13) are the points (x_0, y_0), where $f(x_0) = 0$ and $y_0 = -bx_0^2/(1 + 2cx_0)$. On the line $x = x_0$, dx/dt has opposite signs on opposite sides of the critical point (x_0, y_0). It follows that γ cannot intersect the line $x = x_0$. Hence for the system

$$dx/d\tau = y, \quad dy/d\tau = -f(x) \tag{18}$$

equivalent to (15), every periodic orbit which surrounds the origin must lie in a strip $\alpha < x < \beta$, where $\alpha < 0 < \beta$, such that $f(x) \neq 0$ for all $x \in (\alpha, \beta)$ with $x \neq 0$. Thus the function f in (15) satisfies the hypotheses (*). In addition we have $f'''(x) < 0$ for all $x \in (\alpha, \beta)$ if $b \neq 0$. If $b = 0$ and $a \neq 0$, then $f'''(x) = 0$, $f''(x) \neq 0$ for all $x \in (\alpha, \beta)$. Nevertheless, because of the change of time scale, Proposition 1 cannot be applied to the original system (13). Without changing the time scale the system (13) is equivalent to the system

$$\varphi(x)dx/dt = y, \quad \varphi(x)dy/dt = -f(x), \tag{19}$$

where φ is given by (14). Consider the system (19) for any positive, twice continuously differentiable function φ. The systems (18) and (19) have the same phase portraits, but different period functions. Denote the period function of (19) by $\tilde{T}(h)$. Then in the same way that we proved Lemma 1 we can prove

Lemma 3.

$$\tilde{T}'(h) = (1/2h) \iint_{\sigma(h)} [2\varphi' F/f^2 + \varphi(F/f^2)']' dx\, dy.$$

Hence, in order to prove that $\tilde{T}'(h) > 0$ for the system (13), it is sufficient to show that for all $x \in (\alpha, \beta)$

$$[2\varphi' F/f^2 + \varphi(F/f^2)']' > 0,$$

where φ is given by (14). That is, with the notation (4), it is sufficient to show that for all $x \in (\alpha, \beta)$

$$\psi''(x) - 3c(1 + 2cx)^{-1}\psi'(x) + 6c^2(1 + 2cx)^{-2}\psi(x) > 0.$$

Lemma 4. *Let f satisfy the hypotheses (*) and suppose either $f'''(x) < 0$ for all $x \in (\alpha, \beta)$ or $f'''(x) = 0$, $f''(x) \neq 0$ for all $x \in (\alpha, \beta)$. If $\psi = F/f^2$ then, for all $x \in (\alpha, \beta)$ and every real u,*

$$\psi''(x) - (3/2)\psi'(x)u + (3/2)\psi(x)u^2 > 0.$$

Proof. This is certainly true for $u = 0$, by Lemma 2. Consequently we need only show that the discriminant

$$\psi'(x)^2 - (8/3)\psi(x)\psi''(x)$$

is negative for all $x \in (\alpha, \beta)$. In fact, since $\psi(x)\psi''(x) > 0$ for all $x \in (\alpha, \beta)$, it is sufficient to show that

$$\Delta(x) := \psi'(x)^2 - (2/3)\psi(x)\psi''(x)$$

is negative for all $x \in (\alpha, \beta)$.

From the values of ψ and its derivatives at $x = 0$ we obtain

$$\Delta(0) = [3f'(0)f'''(0) - f''(0)^2]/36f'(0)^4 < 0.$$

Hence we will prove that $\Delta(x) < 0$ for all $x \in (\alpha, \beta)$ if we show that $\Delta(x_0) = 0$ implies $\Delta'(x_0) < 0$ if $x_0 \in (0, \beta)$ and $\Delta'(x_0) > 0$ if $x_0 \in (\alpha, 0)$.

From $\psi = F/f^2$, $\psi' = (f^2 - 2Ff')/f^3$ and $\psi'' = N/f^4$, we obtain $\Delta = M/f^5$, where

$$M := (f^2 - 2Ff')f + (4/3)F^2 f''.$$

Consequently

$$M' = (f^2 - 2Ff')f' + (2/3)Fff'' + (4/3)F^2 f'''.$$

If $M(x_0) = 0$ for some $x_0 \neq 0$, then

$$Fff'' = -3(f^2 - 2Ff')f^2/4F$$

and hence

$$M'(x_0) = -(f^2 - 2Ff')^2/2F + (4/3)F^2 f'''.$$

In the case $f''' < 0$ this shows at once that $M'(x_0) < 0$. In the other case we reach the same conclusion, since it is easily verified that $f^2 - 2Ff'$ vanishes only for $x = 0$. □

We can now establish our main result:

Theorem 1. *For any Hamiltonian quadratic system with a center, the period function $\tilde{T}(h)$ associated with the center satisfies $\tilde{T}'(h) > 0$ throughout its interval of definition.*

Proof. We may suppose that the Hamiltonian quadratic system has the form (13). Then the function f, defined by (16), satisfies the hypotheses of Lemma 4 if $b \neq 0$ or if $b = 0$, $a \neq 0$. Thus in these cases the result follows immediately from Lemma 4 and the discussion preceding its statement. In the remaining case $a = b = 0$ the result is obvious from Lemma 3, since $F/f^2 \equiv 1/2$. □

4. Concluding remarks.

A period function $T(h)$ can be associated with a center, not only of any Hamiltonian system, but of any plane autonomous system

$$dx/dt = P(x, y), \quad dy/dt = Q(x, y),$$

where P and Q are, say, holomorphic functions. In this case the parameter h is no longer the energy, but a local parameter on a smooth transversal to the periodic orbits surrounding the center. Such a transversal, and the local parameter on it, are not uniquely determined. However, the number of critical points of the period function $T(h)$ - and in particular its monotonicity - is independent of their choice. A nice result for non-Hamiltonian systems, due to Rothe [9] and Waldvogel [11], states that the period function of the Lotka-Volterra system

$$dx/dt = x(a - by), \quad dy/dt = y(cx - d)$$

is monotonic.

In this connection it is natural to conjecture that Theorem 1 remains valid if the word "Hamiltonian" is omitted from its statement. However, the conjecture is false, as Chicone and Dumortier [2] first showed by an explicit example. Chicone and Jacobs [3] later proved that there exist quadratic systems with centers whose period functions have at least two critical points. Moreover, their work suggests that two may indeed be the maximum number of critical points for the period function associated with the center of any quadratic system. It is natural to distinguish between the different types of center of a quadratic system. Any quadratic system with a center at the origin can, by a non-singular linear transformation of the coordinates and a constant magnification of the time scale, be brought to the form

$$\begin{aligned} dx/dt &= y + bx^2 + (2c + \beta)xy \\ dy/dt &= -x - ax^2 - (2b + \alpha)xy - cy^2, \end{aligned} \tag{20}$$

where one of the following sets of conditions is satisfied:

(i) $\alpha = \beta = 0$;

(ii) $\alpha = b = 0$;

(ii)′ $\alpha b \neq 0$, $\beta/\alpha = (a + c)/b$ and $a - (3b + \alpha)\beta/\alpha + (3c + \beta)\beta^2/\alpha^2 = 0$;

(iii) $a + c = b = 0$;

(iv) $\alpha + 5b = \beta + 5c = a = 0$;

(iv)′ $\alpha + 5b = \beta - 5a = 2a + c = 0$.

These conditions are obtained by setting $d = 0$ in the conditions given in Coppel [5]. However, by a rotation of axes (ii)′ is reduced to (ii) and (iv)′ to (iv).

A general procedure for transforming a quadratic system into a Liénard equation

$$d^2x/d\tau^2 - \tilde{f}(x)dx/dt + f(\tau) = 0$$

by a change of independent variable $dt/d\tau = \varphi(x)$ is described in Coppel [6]. The manner in which we replaced the Hamiltonian system (13), which is case (i) above, by the second-order differential equation (15) was simply an application of this procedure.

If we write $\delta = 2c + \beta$, the corresponding formulae in cases (ii) and (iii) may be given the common form

$$\varphi(x) = (1 + \delta x)^{c/\delta - 1},$$

$$\tilde{f}(x) = -\alpha x\varphi(x),$$

$$f(x) = x(1 + ax)(1 + \delta x)\varphi^2(x).$$

However, in case (ii) we have $\alpha = 0$. Since the example of Chicone and Dumortier [2] belongs to this type, the period function need not be monotonic in case (ii). In case (iv) the corresponding formulae are

$$\begin{aligned}
\varphi(x) &= (1-3cx)^{-4/3}, \\
\tilde{f}(x) &= 5bx(1-2cx)(1-3cx)^{-7/3}, \\
f(x) &= x[(1-3cx)^2 + 3b^2x^2 - 8b^2cx^3](1-3cx)^{-11/3}.
\end{aligned}$$

It is conjectured by Chicone and Jacobs [3], p. 459, that, except for the isochronous system

$$\begin{aligned}
dx/dt &= y - 2cxy \\
dy/dt &= -x + cx^2 - cy^2,
\end{aligned}$$

the period function is strictly monotonic in case (iii). Perhaps this may be established by methods similar to those used in the present paper.

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