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LJUBOMIR GAVRILOV
MOHAMMED OUAZZANI-JAMIL
RÉGIS CABOZ

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BIFURCATION DIAGRAMS AND FOMENKO'S SURGERY ON LIOUVILLE TORI OF THE KOLOSSOFF POTENTIAL $U=\rho+(1/\rho)-k\cos\varphi$

BY LJUBOMIR GAVRILOV, MOHAMMED OUAZZANI-JAMIL AND REGIS CABOZ

ABSTRACT. — By making use of the rich algebraic structure of the problem and Fomenko's theory of surgery on (bifurcations of) Liouville tori, we give a complete description of the topology and bifurcations of the invariant level sets of the Kolossoff system corresponding to the integrable potential $U=\rho+(1/\rho)-k\cos\varphi$.

I. Introduction

Consider the motion of a particle of unit mass on the plane (x, y) in a potential field

$$U=a\rho+\frac{b}{\rho}+c\cos\varphi+d\sin\varphi,\qquad a,b,c,d\in\mathbf{R}$$

where $x=\rho\cos\varphi$, $y=\rho\sin\varphi$. Without loss of generality one may suppose (after a rotation and $\mathbf{R}$-linear change of ρ and U) that

$$U(x, y)=\pm\rho\pm\frac{1}{\rho}-k\cos\varphi, k\in\mathbf{R}$$

The corresponding Hamiltonian function is:

$$H=\frac{1}{2}(p_x^2+p_y^2)+U(x, y)$$

and the energy level sets $\{H=h\}\subset\mathbf{R}^4$ are compact if $U=\rho+(1/\rho)-k\cos\varphi$. The Hamiltonian system

$$(1)\qquad \left\{\begin{array}{ll} x'=\dfrac{dH}{dp_x}, & p_x'=-\dfrac{dH}{dx} \\ y'=\dfrac{dH}{dp_y}, & p_y'=-\dfrac{dH}{dy} \end{array}\right. \quad (')=\frac{d}{dt}$$

where

$$H=\frac{1}{2}(p_x^2+p_y^2)+\rho+\frac{1}{\rho}-k\cos\varphi$$

is integrable and the second integral of motion reads:

$$F=-(k^2+y^2)p_x^2+2y(x-k)p_xp_y-p_y^2(x-k)^2-\frac{2k(x-k)(kx-1)}{\sqrt{x^2+y^2}}$$

The integrability of the system (1) was discovered by Kolossoff [8] who used it to linearize the celebrated Kovalevskaya top.

In the present paper we give a complete description of the topology of the level sets

$$\mathbf{A_R}=\{(x, y, p_x, p_y)\in\mathbf{R}^4 : H=h,\ F=f\}\subset\mathbf{R}^4.$$

For doing that we find first the bifurcation diagram **B** of the problem (1), *i.e.* the set of critical values of the energy-momentum mapping

$$(x, y, p_x, p_y)\to(F, H).$$

It turns out (like in Hénon-Heiles system [5], Gorjatchev-Tchaplygin [4] and Kovalevskaya top [9], [10]) that **B** is exactly the discriminant locus of a certain polynomial whose coefficients are functions in f, h, k. The latter is closely related to the algebraic structure of the complexified system (1). This structure is studied in section 2 where we prove that the complexified generic level set $\{H=h, F=f\}$ is an affine part of an Abelian variety (Theorem 1). Contrary to the most of the known examples [1], the Hamiltonian flows corresponding to H and F do not linearize on this Abelian variety. Thus the system (1) is not algebraically completely integrable in the sense of Adler and van Moerbeke [1]. For non-critical values of F and H the level set $\mathbf{A_R}$ is, according to Liouville theorem, a finite union of two-dimensional tori. Their number is related to the number of ovals of an associated genus two Riemann surface and could be calculated by making use of the results of chapter 2 (*see* Theorem 2 of section 3).At last, in section 4, we describe the structure of singular level sets $\mathbf{A_R}$. According to Fomenko's theory of surgery on (bifurcations of) Liouville tori they turn out to be homeomorphic to a finite list of two-dimensional complexes. To "guess" exactly which bifurcation takes place we use once again the reach algebraic structure of the problem. Namely, each bifurcation of Liouville tori is related to a bifurcation of ovals on a Riemann surface (the last being easily studied). Thus we find all generic bifurcations of Liouville tori as f and h pass through the bifurcation diagram **B** (Theorem 3 and Theorem 4 of section 4).

II. Algebraic structure

Denote by $\mathbf{A_C}$ the complex affine algebraic variety:

$$\mathbf{A_C}=\{(x, y, p_x, p_y, z)\in \mathbf{C}^5 : \mathrm{H}=h,\ \mathrm{F}=f,\ x^2+y^2=z^2,\ z\neq 0\}\subset \mathbf{C}^5,$$

where

$$\mathrm{H}(x, y, p_x, p_y, z)=\frac{1}{2}(p_x^2+p_y^2)+z+\frac{1}{z}-k\frac{x}{z},$$

$$\mathrm{F}(x, y, p_x, p_y, z)=-(k^2+y^2)p_x^2+2y(x-k)p_xp_y-p_y^2(x-k)^2-\frac{2k(x-k)(kx-1)}{z}$$

The variety $\mathbf{A_C}$ is invariant under the (complex) flow of the (complexified) system (1). Consider also the polynomial

$$\varphi(u)=-2(u^3-hu^2+(1-k^2)u-f/2) \tag{2}$$

and the corresponding hyperelliptic curve

$$\mathrm{K}:\ \{w^2=(u^2-k^2)\varphi(u)\}. \tag{3}$$

Remark. – K is precisely the curve used by Kovalevskaya [11] to integrate the Kovalevskaya top.

THEOREM 1. – *If the polynomial* $(u^2-k^2)\varphi(u)$ *has no double roots then the affine algebraic variety* $\mathbf{A_C}$ *is a smooth complex manifold which is biholomorphically equivalent to the complex manifold* $\tilde{\mathbf{A}}_\mathbf{C}\setminus\mathcal{D}$, *where* $\tilde{\mathbf{A}}_\mathbf{C}$ *is a complex algebraic torus (Abelian variety) and* $\mathcal{D}$ *is a divisor.* $\tilde{\mathbf{A}}_\mathbf{C}$ *is a two-sheeted unramified covering of the Jacobi variety* Jac(K) *of the genus algebraic two curve* K. *The trajectories of the Hamiltonian flow generated by* H *on* $\mathbf{A_C}$ *are straight lines on which, however, the motion is non-linear. The trajectories of the Hamiltonian flows generated by* $\mathrm{H}+s\mathrm{F}$, $s\neq 0$ *on* $\mathbf{A_C}$ *are not straight lines.*

Theorem 1 will be proved later in this section. We recall that the Hamilton-Jacobi equation corresponding to (1) separates in the following (λ, μ) coordinates (*see* [8] for details):

$$\begin{cases} x=\dfrac{\lambda\mu}{k}+k \\ y=\dfrac{1}{k}\sqrt{(\lambda^2-k^2)(k^2-\mu^2)} \end{cases} \tag{4}$$

The canonical variables $(p_\lambda, p_\mu, \lambda, \mu)$ on $\mathbf{T}^*\mathbf{R}^2$ are given by

$$\left\{\begin{aligned} p_x &= \frac{(\lambda^2-k^2)\,\mu p_\lambda-(\mu^2-k^2)\,\lambda p_\mu}{k(\lambda^2-\mu^2)}\\ p_y &= \frac{\sqrt{(\lambda^2-k^2)(k^2-\mu^2)}\,(\lambda p_\lambda-\mu p_\mu)}{k(\lambda^2-\mu^2)}\end{aligned}\right. \tag{5}$$

In these new variables the integrals of motion take the form

$$\mathrm{H}=\frac{(\lambda^2-k^2)p_\lambda^2-(\mu^2-k^2)p_\mu^2+2(1-k^2)(\lambda-\mu)+2(\lambda^3-\mu^3)}{2(\lambda^2-\mu^2)},$$

$$\mathrm{F}=\frac{-\mu^2(\lambda^2-k^2)p_\lambda^2+\lambda^2(\mu^2-k^2)p_\mu^2-2\lambda\mu(\lambda\mu+k^2+k^2-1)(\lambda-\mu)}{(\lambda^2-\mu^2)}$$

and hence on each level set $\mathbf{A_C}$ holds

$$p_\lambda=\sqrt{\frac{\varphi(\lambda)}{\lambda^2-k^2}},\qquad p_\mu=\sqrt{\frac{\varphi(\mu)}{\mu^2-k^2}}. \tag{6}$$

For a further use we note also the relation

$$\mathrm{F}=p_\mu^2(\mu^2-k^2)+2\mu^3-2\mu^2\mathrm{H}+2\mu(1-k^2). \tag{7}$$

Denote by d/dt_s the time derivative along the Hamltonian flow of the function $\mathrm{H}_s=\mathrm{H}+s\mathrm{F}$. By making use of the equations

$$\frac{d\lambda}{dt_s}=\frac{\partial\mathrm{H}}{\partial p_\lambda},\qquad \frac{d\mu}{dt_s}=\frac{\partial\mathrm{H}_s}{\partial p_\mu}$$

and (6) one obtains

$$\left\{\begin{aligned} \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2-k^2)}}+\frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2-k^2)}}&=-2s\,dt_s\\ \frac{\lambda^2\,d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2-k^2)}}+\frac{\mu^2\,d\mu}{\sqrt{\varphi(\mu)(\mu^2-k^2)}}&=dt_s\end{aligned}\right. \tag{8}$$

The system (8) can be also written in the following equivalent form

$$\left\{\begin{aligned} \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2-k^2)}}+\frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2-k^2)}}&=-2s\,dt_s\\ \frac{\lambda\,d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2-k^2)}}+\frac{\mu\,d\mu}{\sqrt{\varphi(\mu)(\mu^2-k^2)}}&=\frac{1-2s\lambda\mu}{\lambda+\mu}dt_s\end{aligned}\right. \tag{9}$$

The flow of Kolossoff system (1) corresponds to $s=0$, and obviously $t_s|_{s=0}=t$. The system (9) implies, roughly speaking, that our initial system linearizes on an Jacobian variety after using a "new time"

$$d\tau = \frac{dt}{\lambda+\mu}. \tag{10}$$

The time τ will play an important role and it is exactly the "Kovalevskaya time" (*see* [8] for details).

Define now the Abel-Jacobi map

$$\zeta : S^2 K \to \mathrm{Jac}(K) : (P_1, P_2) \to \left(\int_{P_\infty}^{P_1} \omega_1 + \int_{P_\infty}^{P_2} \omega_1, \int_{P_\infty}^{P_1} \omega_2 + \int_{P_\infty}^{P_1} \omega_2 \right)$$

where

$$\omega_1 = \frac{du}{\sqrt{\varphi(u)(u^2-k^2)}}, \qquad \omega_2 = \frac{u\,du}{\sqrt{\varphi(u)(u^2-k^2)}}$$

P_1, P_2, $\in K$ P_∞ is the "infinite" point on K and S^2 K is the second symetric product of K.

Solving the Jacobi inversion problem (9), we obtain the explicit solutions of our initial problem (1) [2]. Thus x, y, p_x, p_y, $z=\sqrt{x^2+y^2}$ can be expressed in terms of genus two theta functions living on the Jacobi variety Jac(K). These functions however are not single-valued as it can be seen from (4). Indeed to each point on the symetric product S^2 K of the curve K (which is birational to Jac(K) according to Jacobi theorem) correspond two values of (x, y, p_x, p_y). On the other hand these functions do not have branch points on Jac(K) and hence they are root functions (Wurzelfunktionen [14]) on Jac(K).

Consider the Abelian variety $\tilde{\mathbf{A}}_{\mathbf{C}} = \mathbf{C}^2/\mathbf{Z}\{e_1, e_2, e_3, 2e_4\}$ where

$$\mathrm{Jac}(\mathrm{K}) = \mathbf{C}^2/\mathbf{Z}\{e_1, e_2, e_3, e_4\}.$$

If the basis (e_1, e_2, e_3, e_4) of the period lattice is chosen in a proper way then the function x, y, p_x, p_y, z become single-valued on $\tilde{\mathbf{A}}_{\mathbf{C}}$. Let us fix such a basis. The natural projection

$$\pi : \tilde{\mathbf{A}}_{\mathbf{C}} \to \mathrm{Jac}(\mathrm{K}) \tag{11}$$

corresponds to the involution

$$(x, y, p_x, p_y, z) \to (x, -y, p_x, -p_y, z) \tag{12}$$

on $\tilde{\mathbf{A}}_{\mathbf{C}}$. Consider the mapping

$$i : \mathbf{C}^5 \to \mathbf{CP}^7 : (x, y, z, p_x, p_y) \to [f_0, f_1, \ldots, f_7]$$

where

$$(13)\qquad \left|\begin{array}{c} f_0=1 \\ f_1=x \\ f_2=y \\ f_3=z \\ f_4=xp_y-yp_x \\ f_5=f_4^2 \\ f_6=f_3(f_4-kp_y) \\ f_7=(p_x^2-p_y^2)y-2p_xp_yx-2f_2f_3. \end{array}\right.$$

LEMMA 1. — *The functions f_i, $i=0, 1 \ldots, 7$ considered as single-valued meromorphic functions on $\tilde{\mathbf{A}}_{\mathbf{C}}$ provide a smooth embedding of $\tilde{\mathbf{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$.*

Proof of theorem 1 *assuming the above lemma.* — As the functions $f_0, f_1, \ldots, f_7$ provide an embedding of $\tilde{\mathbf{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$ (Lemma 1) then the closure $\overline{i(\mathbf{A}_{\mathbf{C}})}$ of $i(\mathbf{A}_{\mathbf{C}})$ in $\mathbf{CP}^7$ is biholormophically equivalent to $\tilde{\mathbf{A}}_{\mathbf{C}}$. Consider the divisors $\mathscr{D}_\infty$ and $\mathscr{D}'_{2\infty}$ defined by

$$(\lambda\mu)_\infty=2(\zeta(P_\infty)+\zeta(K))=2\mathscr{D}_\infty$$

and

$$(z)_0=(\lambda+\mu)_0=\mathscr{D}'_{2\infty}$$

Obviously $\mathscr{D}'_{2\infty}\sim 2\mathscr{D}_\infty$. It is easily seen that $\mathbf{A}_{\mathbf{C}}$ is biholormopyically equivalent to $\overline{i(\mathbf{A}_{\mathbf{C}})}\setminus\{\mathscr{D}_\infty\cup\mathscr{D}'_{2\infty}\}$. Indeed i is a biholomorphic mapping between some neighbourhood $\mathbf{V}_{\mathbf{A}_{\mathbf{C}}}$ of $\mathbf{A}_{\mathbf{C}}$ in $\mathbf{C}^5\setminus\{z\neq 0\}$ and $i(\mathbf{V}_{\mathbf{A}_{\mathbf{C}}})\subset\mathbf{CP}^7$. To check that it suffice to note that if $(x, y, p_x, p_y, z)\in\mathbf{A}_{\mathbf{C}}$ then

$$\det\left(\frac{\partial(f_1, f_2, f_3, f_4, f_6)}{\partial(x, y, p_x, p_y, z)}\right)=kyz$$

$$\det\left(\frac{\partial(f_1, f_2, f_3, f_5, f_7)}{\partial(x, y, p_x, p_y, z)}\right)=-4p_y(p_xx^2y+p_xy^3-p_yx^3-p_yxy^2)$$

and hence $\operatorname{rank}(i)=5$ (otherwise the equality $y=p_y=0$ implies $\operatorname{disc}((k^2-u^2)\varphi(u))=0$). As $i(\mathbf{A}_{\mathbf{C}})=\tilde{\mathbf{A}}_{\mathbf{C}}\setminus\mathscr{D}_\infty$ is a smooth complex manifold, it is concluded that $\mathbf{A}_{\mathbf{C}}$ is also a smooth complex manifold. $\triangle$

Proof of Lemma 1. — For an arbitrary divisor $\mathscr{D}\subset\tilde{\mathbf{A}}_{\mathbf{C}}$ we denote

$$\mathscr{L}(\mathscr{D})=\{f \text{ meromorphic on } \tilde{\mathbf{A}}_{\mathbf{C}},\ (f)\geqq-\mathscr{D}\}$$

As $\zeta(\mathrm{K})$ defines (1, 1) polarization on $\mathrm{Jac}(\mathrm{K})$ then $\mathscr{D}_\infty=\pi^{-1}\circ\zeta(\mathrm{K})$ defines (1, 2) polarization on $\mathbf{A}_{\mathbf{C}}$. Thus $2\mathscr{D}_\infty$ defines (2, 4) polarization on $\tilde{\mathbf{A}}_{\mathbf{C}}$ and $\dim\mathscr{L}(2\mathscr{D}_\infty)=2\times 4=8$, [7]. To prove lemma 1, it is enough to check that the functions $f_0, f_1, \ldots, f_7$ provide a basis of $\mathscr{L}(2\mathscr{D}_\infty)$. First of all let us note that f_i blow up only along $\mathscr{D}_\infty$. Indeed in λ, μ

coordinates we have

$$f_1 = 1$$

$$f_1 = \frac{\lambda\mu}{k} + k$$

$$f_2 = \frac{1}{k}\sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)}$$

$$f_3 = \lambda + \mu$$

$$f_4 = \frac{1}{(\lambda - \mu)}\{\sqrt{(k^2 - \mu^2)}\sqrt{\varphi(\lambda)} - \sqrt{(\lambda^2 - k^2)}\sqrt{-\varphi(\mu)}\}$$

$$f_5 = f_4^2$$

$$f_6 = \frac{1}{(\lambda - \mu)}\{\mu\sqrt{(k^2 - \mu^2)}\sqrt{\varphi(\lambda)} - \lambda\sqrt{\varphi(\lambda)} - \sqrt{(\lambda^2 - k^2)}\sqrt{-\varphi(\mu)}\}$$

$$f_7 = \frac{1}{k(\lambda - \mu)}\{2(\lambda\mu - k^2)\sqrt{\varphi(\lambda)}\sqrt{-\varphi(\mu)} - \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)}(\varphi(\lambda) + \varphi(\mu))\} - 2f_2 f_3.$$

To prove that $f_i \in \mathscr{L}(2\mathscr{D}_\infty)$ we shall find, following [1], the asymptotic expansions of x, y, z as functions of the time τ (10) in a neighbourhood of a generic point $\tau^0 \in \mathscr{D}_\infty$. Formulae (4) imply that $\lambda + \mu = \sqrt{x^2 + y^2}$ and hence the changing of time in the system (1) is equivalent to multiplying each equation by z. According to (9) and (4) the variables x, y, z are meromorphic in τ and the corresponding Laurent series are:

$$(14)\qquad \begin{cases} x = \sum_{j=0}^{\infty} x_j \tau^{j-2}, & p_x = \sum_{j=0}^{\infty} p_{x_j} \tau^{j-1} \\ y = \sum_{j=0}^{\infty} y_j \tau^{j-2}, & p_y = \sum_{j=0}^{\infty} p_{y_i} \tau^{j-1} \\ z = \sum_{j=0}^{\infty} z_j \tau^{j-2} \end{cases}$$

(here τ stays for $\tau - \tau_0$). After substituting the above series in the Kolossoff system (1) one obtains a recurrent system of linear equations for the coefficients x_j, y_j, z_j. The general solution (14) depends effectively upon three free parameters α, γ, δ:

$$(15)\qquad \begin{cases} x = \dfrac{\alpha}{\tau^2} + \dfrac{(k\beta^3 - 4\gamma\alpha)}{4\beta} + \delta\tau + \dots \\ y = \dfrac{\beta}{\tau^2} - \dfrac{(k\alpha\beta + 4\gamma)}{4} - \dfrac{\alpha\delta}{\beta}\tau + \dots \\ z = \dfrac{-2}{\tau^2} + \dfrac{2\gamma}{\beta} + \dots \end{cases}$$

where $\alpha^2+\beta^2=4$ (for details about the general procedure of finding the series (15) we refer the reader to [1] or [6, 15]). After substituting (15) in (14), we obtain

$$
(16)\qquad \begin{cases} f_0=1 \\ f_1=\dfrac{\alpha}{\tau^2}+\dots \\ f_2=\dfrac{\beta}{\tau^2}+\dots \\ f_3=-\dfrac{2}{\tau^2}+\dots \\ f_4=\dfrac{k\beta}{\tau}+\dots \\ f_5=\dfrac{k^2\beta^2}{\tau^2}+\dots \\ f_6=\dfrac{12\delta}{\beta\tau^2}+\dots \\ f_7=-2\dfrac{(k\alpha\beta+6\gamma)}{\tau^2}+\dots \end{cases}
$$

The complex constants α (or β such that $\alpha^2+\beta^2=4$), γ, δ parametrize the pole divisor $\mathscr{D}_\infty$. Indeed substituting (15) in $\{\mathrm{H}=h,\ \mathrm{F}=f,\ z^2=x^2+y^2\}$ we obtain the genus three curve

$$
(17)\qquad \begin{cases} \gamma=\dfrac{2h\beta-k\alpha\beta}{16}, \\ \delta^2=\dfrac{\beta}{72}(k^3\alpha\beta^3+8k^2\gamma\beta^2-2k(1+k^2)\alpha\beta-32k^2\gamma-2f\beta), \\ \alpha^2+\beta^2=4 \end{cases}
$$

$\mathscr{D}_\infty$ is a double unramified covering of the genus two curve

$$
(18)\qquad \delta^2=\frac{(\alpha^2-4)}{144}(k^3\alpha^3+2hk^2\alpha^2+4k(1-k^2)\alpha+4f)
$$

and obviously this curve (18) coincides with (3) after making the substitution

$$
\alpha\to\frac{2u}{k},\qquad \delta\to\frac{w}{3k}.
$$

Equations (16) and (18) imply that $f_0, f_1, \dots, f_7$ are linearly independent on $\tilde{\mathbf{A}}_{\mathbf{C}}$ which completes the proof of lemma 1. $\triangle$

III. Topology of Regular Level Sets

In this section we shall describe the topological type of $\mathbf{A_R}$ for all generic constants f, h, $k \in \mathbf{R}$. The system (1) will be considered as a real system of differential equations.

According to Theorem 1 $\mathbf{A_R}$ is a smooth real manifold if the polynomial $(k^2 - u^2)\varphi(u)$ has no double roots. Define the bifurcation set

$$\mathbf{B} = \{(f, h, k) \in \mathbf{R}^3 : \operatorname{disc}((u^2 - k^2)\varphi(u)) = 0\}. \tag{19}$$

It is clear that the topological type of $\mathbf{A_R}$ may change only as (f, h, k) passes through $\mathbf{B}$. Thus in each connected component of the set $\mathbf{R}^3 \setminus \mathbf{B}$ the level set $\mathbf{A_R}$ has the same topological type. Note that the bifurcation set $\mathbf{B} \subset \mathbf{R}^3\{f, h, k\}$ is invariant under the involution

$$(f, h, k) \to (f, h, -k)$$

and the topological type of the level set $\mathbf{A_R}$ is one and the same at the points (f, h, k) and $(f, h, -k)$. Thus it is enough to consider $k \geqq 0$.

THEOREM 2. — *The set* $\{\mathbf{R}^3 \setminus \mathbf{B}\} \cap \{k \geqq 0\}$ *consists of* 12 *connected components. The sections of these components with the plane* $\{k = \text{const.}\}$ *are shown on* figure 1. *If* $(f, h, k) \in \mathbf{R}^3 \setminus \mathbf{B}$ *the level set* $\mathbf{A_R}$ *is (diffeomorphic to) a torus, to a disjoint union of two tori, or it is the empty set as it is shown in* table I.

Remark. — The notation 2T in table I means a disjoint union of 2 two-dimensional tori.

Proof of Theorem 2. — The complex conjugation

$$(x, y, z, p_x, p_y) \to (\bar{x}, \bar{y}, \bar{z}, \bar{p}_x, \bar{p}_y) \tag{20}$$

acts as an antiholomorphic involution on $\mathbf{A_C}$. The set of its fixed points is the real part $\Re(\mathbf{A_C})$ of $\mathbf{A_C}$ and $\mathbf{A_R} = \Re(\mathbf{A_C}) \cap \{z > 0\}$. Consider also the natural antiholomorphic involution τ of the Kovalevskaya curve (3) given in (w, u) coordinates by:

$$\tau : \quad (w, u) \to (\bar{w}, \bar{u}).$$

It induces an antiholomorphic involution on the symetric product $S^2 K$ and hence on Jac(K) and $\tilde{\mathbf{A}}_\mathbf{C}$. Formulae (4), (5), (6) imply that this involution coincides with the complex conjugation (20) on $\mathbf{A_C}$. The upshot is that in order to describe $\mathbf{A_R}$ it is enough to study the projection

$$\pi : \quad \mathbf{A_C} \to \operatorname{Jac}(K)$$

and the pair (K, τ).

Remark. — The pair (K, τ) where K is a Riemann surface and τ is an antiholomorphic involution on K is called Klein surface. For the theory of Klein surfaces we refer the reader to [12].

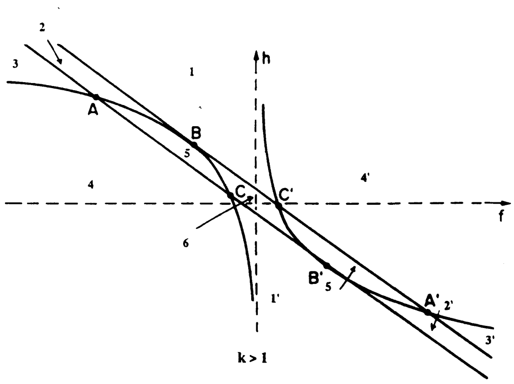

Fig. 1.1

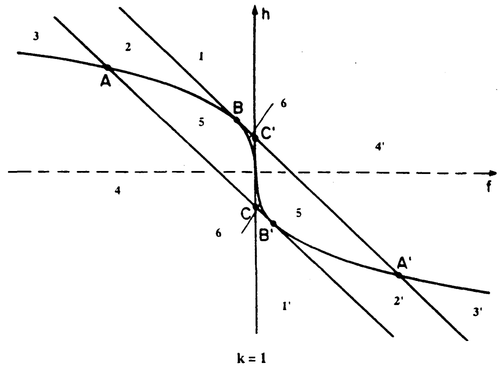

Fig. 1.2

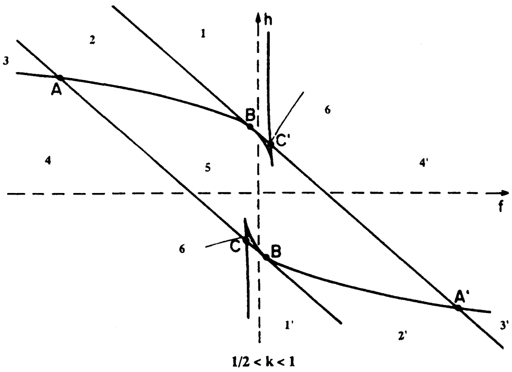

Fig. 1.3

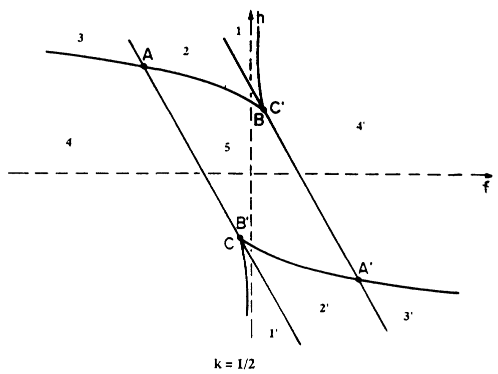

Fig. 1.4

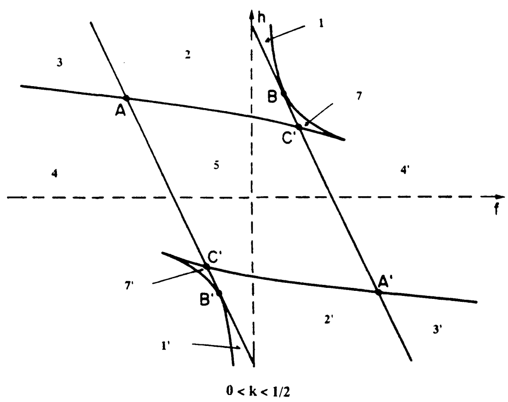

Fig. 1.5

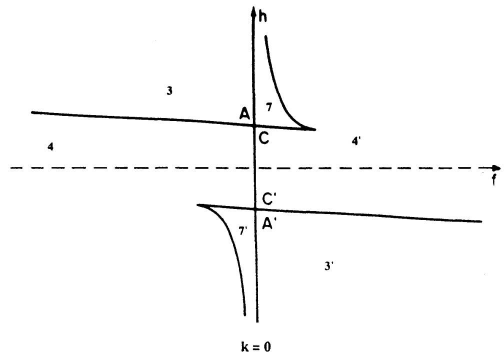

Fig. 1.6

Fig. 1. — The set $\mathbf{B} \cap \{k=\text{const.}\}$ for $k \geqq 0$.

TABLE I

Topological type of $\mathbf{A_R}$ *and real roots of the polynomial* $(u^2-k^2)\,\varphi(u)$ *for* $(f, h, k) \in \mathbf{R}^3 \setminus \mathbf{B}$ (see fig. 1).

Domain	Roots	Topological type
1	$-k<u_1<u_2<k<u_3$	T
1′	$u_1<-k<u_2<u_3<k$	$\varnothing$
2	$-k<u_1<k<u_2<u_3$	T
2′	$u_1<u_2<-k<u_3<k$	$\varnothing$
3	$u_1<-k<k<u_2<u_3$	2 T
3′	$u_1<u_2<-k<k<u_3$	$\varnothing$
4	$u_1<-k<k$	$\varnothing$
4′	$-k<k<u_1$	$\varnothing$
5	$-k<u_1<k$	$\varnothing$
6	$-k<u_1<u_2<u_3<k$	$\varnothing$
7	$-k<k<u_1<u_2<u_3$	$\varnothing$
7′	$u_1<u_2<u_3<-k<k$	$\varnothing$

DEFINITION. — A connected component of the set of fixed points of τ on K is called an oval.

To determine the ovals of K it suffices to study the real roots of the polynomial $(u^2-k^2)\,\varphi(u)$ for different values of f, h and k. These roots are shown on table I. Using the formulae (4), (5), (6) and the condition $(x, y, z, p_x, p_y) \in \mathbf{R}^5$ we obtain that $\mathbf{A_R} \neq \varnothing$ only if (f, h, k) belongs to domain 1, 2 or 3. There we find exactly two "admissible"

TABLE II

Domain	1	2	3
Projection of the "admissible" ovals on z-plane	$\begin{cases}\Delta_1=[u_1, u_2]\\ \Delta_2=[k, u_3]\end{cases}$	$\Delta_1=[u_1, k]$ $\Delta_2=[u_2, u_3]$	$\Delta_1=[-k, k]$ $\Delta_2=[u_2, u_3]$

ovals whose projections on the z-plane are given by the intervals Δ_1 and Δ_2 (*see* table II). The product of the "admissible" ovals in $S^2 K$ [and hence in Jac(K)] gives a Liouville torus. Thus we proved that $\pi(\mathbf{A_R})$ consists of a torus T. There are two possibilities for $\mathbf{A_R}=\pi^{-1}(\mathrm{T})$ (recall that $\mathbf{A_C}$ is a double covering of $\mathrm{Jac}(\mathrm{K}) \setminus \mathscr{D}$ and the projection is given by the map (11)):

— $\mathbf{A_R}$ is a disjoint union of two copies of T;

— $\mathbf{A_R}$ is homeomorphic to a torus two times "longer" than T.

To determine which case arises it suffices to note that when λ (respectively μ) makes one turn around the interval Δ_1 (respectively Δ_2) in a complex domain then the function y does not change in the first case, whereas in the second case it changes the sign [we recall that the projection π corresponds to the involution (20)]. Thus we find that in domain 1 and 2 $\mathbf{A_R}$ is a torus and in domain 3 it is a disjoint union of two tori. $\triangle$

At last we shall find the topological type of the regular energy-level surface $\{H=h\}$

LEMMA 2. — *The bifurcation set* Σ *of the family of surfaces*

$$Q_{h,k}=\left\{\frac{p_x^2+p_y^2}{2}+\rho+\frac{1}{\rho}-k\cos\varphi=h\right\}$$

is given by the union of two lines $\Sigma=\{h=2+k\}\cup\{h=2-k\}\subset\mathbf{R}^2\{h, k\}$. *The set* $\mathbf{R}^2\diagdown\Sigma$ *consists of* 4 *components shown on* figure 2. *The topological type of* $Q_{h,k}$ *in each of these domains is given in table* 3.

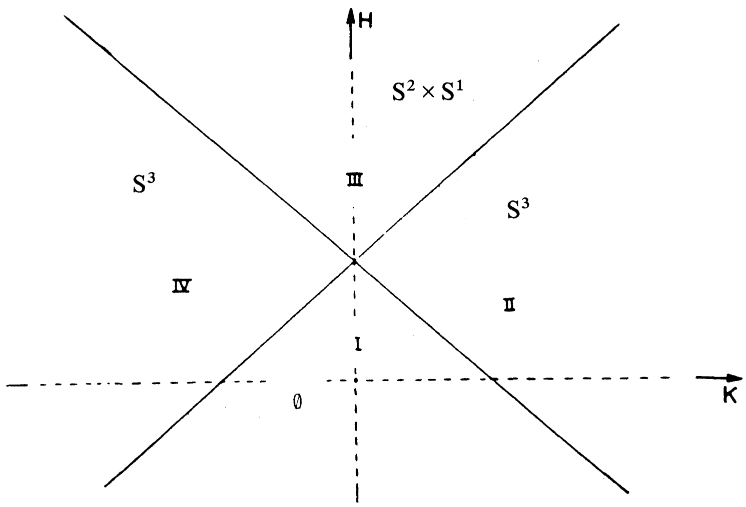

Fig. 2. — The set Σ.

Remarks. — We note that the three dimensional constant-energy surfaces most often met in mathematical physics and theoretical mechanics are: S^3 (the sphere), $\mathbf{R}P^3$ (the projective space), T^3 (the torus) and $S^2\times S^1$ (the direct product), *see* [13] for details.

TABLE III

Topological type of the energy level set $Q_{h,k}=\{H=h\}$ *for* $(h, k)\in\mathbf{R}^2\diagdown\Sigma$ (see fig. 2).

Domain	1	2	3	4
Topological type	$\varnothing$	S^3	$S^2\times S^1$	S^3

Proof of Lemma 2. — The function H has exactly two critical points $p_x=p_y=0$, $y=0$, $x=\pm 1$, for $k\neq 0$ and a critical variety $\{p_x=p_y=0, x^2+y^2=1\}$ for $k=0$ with corresponding critical values $h=2\pm k\,(k\neq 0)$ and $h=2\,(k=0)$. Let us compute the topological type of $Q_{h,k}$. If $k=0$ then

$$H=\frac{p_x^2+p_y^2}{2}+\frac{(\rho-1)^2}{\rho}+2\geqq 2$$

and hence for $h<2$ we have $Q_{h,k}=\varnothing$. This implies that in domain 1 $Q_{h,k}=\varnothing$. Suppose now that $k=0$. On the surface $H=2+\varepsilon$ where ε is small and positive, $\rho-1$ is small together with ε. As.

$$\left\{\frac{p_x^2+p_y^2}{2}+\rho+\frac{1}{\rho}=2+\varepsilon\right\}$$

can be written as

$$\left\{\frac{p_x^2+p_y^2}{2}+\frac{(\rho-1)^2}{(\rho-1+1)}=\varepsilon\right\} \Leftrightarrow \left\{\frac{p_x^2+p_y^2}{2}+(\rho-1)^2-(\rho-1)^3+\ldots=\varepsilon\right\}\sim S^2$$

Then $Q_{2+\varepsilon,0}$ is topologically equivalent to $S^2\times S^1$ and hence in domain 3 the topological type of $Q_{h,k}$ is $S^2\times S^1$. Consider at last $Q_{2,\varepsilon}$, for ε small and positive

$$Q_{2,\varepsilon}=\left\{\frac{p_x^2+p_y^2}{2}+\frac{(\rho-1)^2}{\rho}=\varepsilon\cos\varphi\right\}.$$

The set $Q_{2,\varepsilon}\cap\{\varphi=\text{const.}\}$ is topologically equivalent to S^2 for $\varphi\in((-\pi/2),(\pi/2))$ and to a point for $\varphi=\pm(\pi/2)$. Hence $Q_{2,\varepsilon}$ is topologically equivalent to S^3. This implies that in domain 2 (and 4 by a symmetry) the topological type of $Q_{h,k}$ is S^3. $\triangle$

IV. Topology of Singular Level Sets and Surgery on Liouville Tori

In this section we shall find the topological type of the level set $\mathbf{A_R}$ for generic values $(f, h, k)\in\mathbf{B}$ and thus we shall describe all generic bifurcations of Liouville tori (the non-generic ones are easily found by continuity). For doing that we shall use the Fomenko's classification theorem of bifurcations of (surgery on) Liouville tori [3].

In section 3 we found the topological type of level set $\mathbf{A_R}$ far from the bifurcation diagram. Suppose now that the constants f, h, k are changed in such a way, that (f, h, k) passes through the bifurcation diagram $\mathbf{B}$. Then the topological type of $\mathbf{A_R}$ may change and bifurcations of (surgery on) Liouville tori takes place. Consider the following three types of bifurcations (see *fig.* 3).

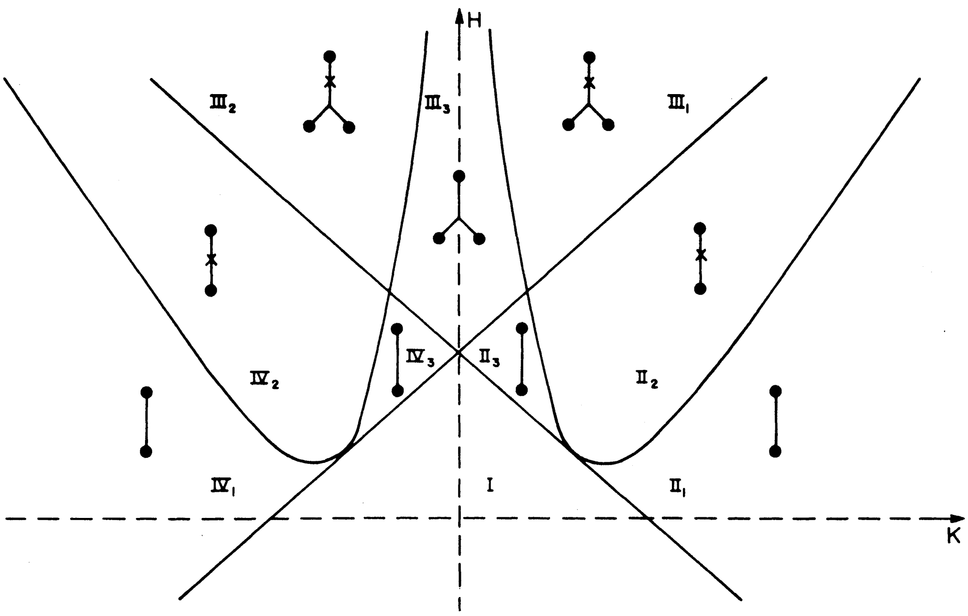

Fig. 3. — Bifurcations of two-dimensional invariant Liouville tori and the corresponding graphs.

1) A (two-dimensional) torus T^2 is contracted to the axial circle S^1 and then vanishes. Denote this surgery as $T \to S^1 \to \varnothing$.

2) A torus T splits into two tori by passing through the complex $S^1 \times \{S^1 \wedge S^1\}$ where $S^1 \wedge S^1$ is a union of two circles having exactly one common point. Denote this bifurcation as $T \to 2T$.

3) A torus T becomes twice "shorter" as it spirals twice round a torus. The last complex is homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \to S^1$, and the corresponding bifurcation will be denoted as $T \to T$.

Following Fomenko [3] we present each of the above bifurcations by a graph shown on figure 3. An ordinary point denotes a non-singular Liouville torus. A black circle stands for a circle and a "branching" point (see *fig.* 3) stands for $\{S^1 \wedge S^1\} \times S^1$. At last asterisk denotes a set homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \to S^1$.

For fixed constants h and k let us consider the energy level surface $Q_{h,k} = \{H = h\}$. As f varies the Liouville tori contained in the level set $\{F = f\}|_{Q_{h,k}}$ may change its topological type. Denote by $\Gamma(Q_{h,k}, F)$ the graph describing the corresponding sequence of bifurcations of Liouville tori. The main result of this section is the following

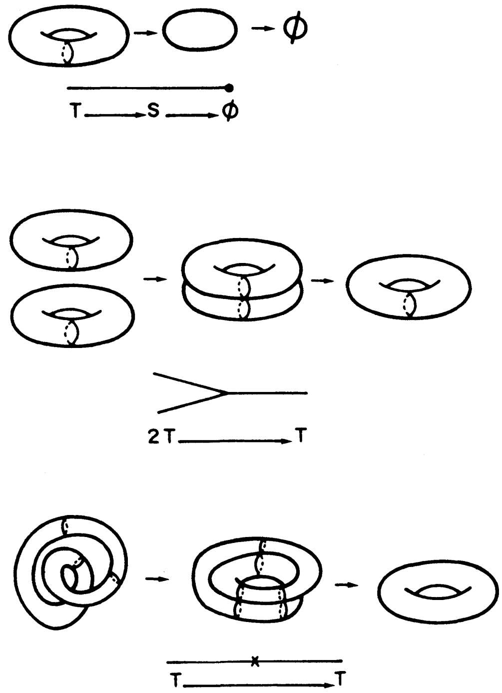

Fig. 4. — The set $\mathscr{D}$ and the graphs $\Gamma(Q_{h,k}, F)$.

THEOREM 3. — *If* (h, k) *belongs to one and the same connected component of the set*

$$\mathscr{D}=\{h\neq 2\pm k\}\cap\left\{h\neq\pm\left(k+\frac{1}{2k}\right)\right\}\subset\mathbf{R}^2\{h, k\}$$

then the graph $\Gamma(Q_{h,k}, F)$ *is the same and it is shown on figure* 4.

Theorem 3 also implies a description of all generic bifurcation of Liouville tori of our initial system (1). Namely, consider a parametrized smooth curve

$$\gamma(s): s\to(f(s), h(s), k(s))\subset\mathbf{R}^3\{f, h, k\}$$

intersecting the bifurcation diagram $\mathbf{B}$ at $s=s_0$.

DEFINITION. — A bifurcation of Liouville tori contained in the level set

$$\mathbf{A}_{\mathbf{R}}\equiv Q_{h(s), k(s)}\cap\{F=f(s)\}$$

as s passes through s_0 is called generic, provided that $\mathbf{B}$ is smooth in a neighbourhood of $(f(s_0), h(s_0), k(s_0))$ and $\gamma(s)$ intersects $\mathbf{B}$ transversally.

THEOREM 4. — *All generic bifurcations of Liouville tori of the system* (1) *are given in table IV.*

TABLE IV

Generic bifurcations of the level set $\mathbf{A}_{\mathbf{R}}$.

$1 \to 2$	$1 \to 6$	$1 \to 4'$	$2 \to 3$	$2 \to 5$	$3 \to 4$
$T \to T$	$T \to \varnothing$	$T \to \varnothing$	$T \to 2T$	$T \to \varnothing$	$T \to \varnothing$ $T \to \varnothing$

Before proving Theorem 3 and Theorem 4 we shall formulate Fomenko's theorem [3] (adapted to our case).

DEFINITION. — A smooth function F on a manifold Q is a Bott function, provided that its critical points form nondegenerate critical smooth submanifolds. A critical submanifold of a smooth function F on a manifold Q is called nondegenerated, provided that the Hessian matrix $d^2 F$ is nondegenerate in normal planes to the submanifold.

Now we may state the Fomenko's classification theorem of bifurcations of two-dimentional Liouville tori.

THEOREM (Fomenko [3]. — *Let* F *be a Bott integral on a non-singular constant energy surface* Q^3 *of an integrable two-degrees of freedom Hamiltonian system. Suppose that each critical manifold of* F *on* Q^3 *is a union of circles. Then each bifurcation of Liouville tori contained in the level set* $\{F=f\}$, *as* f *varies, is a composition of the three bifurcations* $T \to S^1 \to \varnothing$, $T \to 2T$, *and* $T \to T$ *described above.*

Remark. — The condition that each critical manifold of F is a union of circles does not seem to be very restrictive. To our knowledge all studied integrable mechanical systems fall into this case (it may be a conjecture).

In order to apply Fomenko's theorem we need to check that F is a Bott function when restricted on an energy level surface $Q_{h,k}$.

LEMMA 3. — *The second integral* F *is a Bott function on the non-singular energy level surface* $Q_{h,k}=\{H=h\}$ *provided that* $h \neq \pm(k+(1/2k))$.

Proof of Lemma 3. — Suppose that $Q_{h,k}$ is a non-singular compact manifold, *i.e.* $h \neq 2 \pm k$ (Lemma 2). If F has a critical value f on $Q_{h,k}$ then the corresponding level surface $\mathbf{A}_{\mathbf{R}}=\{H=h, F=f\}$ is degenerated and hence the polynomial $(u^2-k^2)\varphi(u)$ has multiple zeros. The condition $h \neq \pm(k+(1/2k))$ means that $(u^2-k^2)\varphi(u)$ has no triple zeros on the boundary of the domains 1, 2 and 3 on figure 1, as the h-coordinates of the points A, B, C′ are $2+k$, $k+(1/2k)$, $2-k$ for $k>0$ and $2-k$, $-k(1/2k)$, $2+k$ for $k<0$. So let us suppose that the level set $\mathbf{A}_{\mathbf{R}}$ is degenerated and consider a degenerated connected component of it. Such a component is parametrized locally by (λ, μ), formulae (5), (6) and (7), at least for $\lambda \neq \mu$. If in addition λ and μ are far from a double root of $(u^2-k^2)\varphi(u)$ then the equations (8) imply that the Hamiltonian flows of H and $H+sF$ are linearly independent and hence dH and dF are linearly independent at such point.

Thus critical points of $F|_{Q_{h,k}}$ correspond only to (λ, μ) such that λ (or μ) is a double root of $(u^2-k^2)\varphi(u)$. This is an one-dimensional analytical set and hence it is a disjoint union of circles. The last follows from the fact that the flow of H on $Q_{h,k}$ has no stationary points and the critical set of F on $Q_{h,k}$ is invariant under the action of this flow.

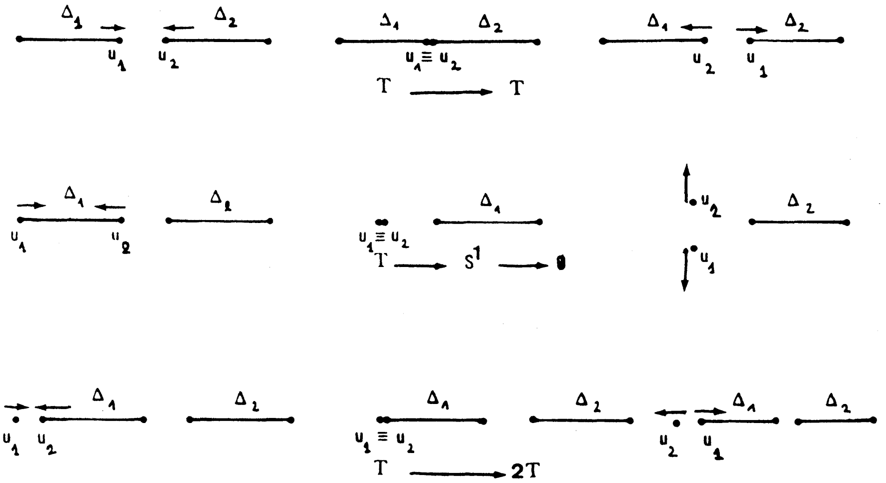

Fig. 5. – Correspondence between bifurcation of roots of the polynomial $(u^2-k^2)(u^3-hu^2+(1-k^2)u-f/2)$ and bifurcations of invariant Liouville tori.

At last let us prove that the hessian matrix of $F|_{Q_{h,k}}$ is non-degenerated of the normal planes to these circles. Let $\mu=\mu_0$ be a double root of $(u^2-k^2)\varphi(u)$. According to (7) we have

$$F|_{Q_{h,k}}=(\mu^2-k^2)p_\mu^2+2\mu^3-2\mu^2 h+2\mu(1-k^2)$$

and a critical circle of the level set $\{F|_{Q_{h,k}}\}=f$ is given by $\mu=\mu_0$, $p_\mu=0$. The normal directions to this circle are given by derivations with respect to μ and p_μ. We have

$$\frac{\partial^2 F}{\partial\mu\,\partial p_\mu}=\begin{pmatrix} 2(\mu_0^2-k^2) & 0 \\ 0 & -\varphi''(\mu_0)\end{pmatrix}$$

and as $\mu_0 \neq \pm k$ then $\operatorname{rank}(d^2(F|_{Q_{h,k}})) \geqq 2$. On the other hand the Hessian $d^2(F|_{Q_{h,k}})$ is degenerated on tangent lines to the critical circle and hence $\operatorname{rank}(d^2(F|_{Q_{h,k}})=2$ which completes the proof of lemma 3. $\triangle$

Proof of Theorem 3. – Let us fix a regular energy level set $Q_{h,k}$ with a Bott integral F on it, and let us consider the corresponding line $h=\text{const.}$ on figure 1 (plane $k=\text{const.}$, $h=\text{const.}$ in the space $\mathbf{R}^3\{f, h, k\}$). As f vary the topological type of $A_{\mathbf{R}}=\{Q_{h,k}\}\cap\{F=f\}$ may change. Using Theorem 2 and the Fomenko's classification theorem we identify several possible bifurcations. For example passing from domain 3 (where $\mathbf{A_R} \sim 2\mathrm{T}$) to domain 2 where ($\mathbf{A_R} \sim \mathrm{T}$) on figure 1 we may have the following surgeries: $2\mathrm{T} \to \mathrm{T}$, or composition of $\mathrm{T} \to \mathrm{T}$ and $\mathrm{T} \to \varnothing$. To make the difference between the two possibilities it suffices to look at the bifurcations of roots of the polynomial $(u^2-k^2)\varphi(u)$, and more specifically the four ends of the "admissible" ovals Δ_1 and Δ_2. The correspondence between bifurcation of roots and tori is shown on figure 5. As the bifurcations of real roots of the polynomial $(u^2-k^2)\varphi(u)$ are easely decribed on table 1 then we obtain a description of the bifurcations of invariant Liouville tori of our initial system (1). By making use of figure 1 we note that if (h, k) is fixed and belongs to one and the same connected component of the set

$$\mathscr{D}=\{h\neq 2\pm k\}\cap\left\{h\neq\pm\left(k+\frac{1}{2k}\right)\right\},$$

then changing f the same bifurcations of roots of the polynomial $(u^2-k^2)\varphi(u)$ take place. This implies that if (h, k) belongs to one and the same connected component of the set $\mathscr{D}$ the corresponding Fomenko's graph $\Gamma(\mathrm{Q}_{h,\,k}, \mathrm{F})$ is the same and it is shown on figure 4. This completes the proof of theorem 3. $\triangle$

DEFINITION. — The straight line $l\subset\mathbf{R}^3\{f, h, k\}$ is generic provided that it intersects $\mathbf{B}$ transversally.

To prove Theorem 4 we note that instead of a generic smooth curve $l\subset\mathbf{R}^3\{f, h, k\}$ it suffice to consider a generic straight line

$$\{c_1 h+c_2 f+c_3=0,\ k=\text{const.}\}\subset\mathbf{R}^3\{f, h, k\}.$$

Then Theorem 4 follows from the following

LEMMA 4. — *Let* $\{c_1 h+c_2 f+c_3=0,\ k=\text{const.}\}$ *be a generic straight line in* $\mathbf{R}^3\{f, h, k\}$. *Then* $\{c_1\mathrm{H}+c_2\mathrm{F}+c_3=0\}\subset\mathbf{R}^4\{x, y, p_x, p_y\}$ *is a smooth surface, and* F *is a Bott integral on it.*

Indeed, instead of H we may take for a Hamiltonian of (1) the function $c_1\mathrm{H}+c_2\mathrm{F}$. The same arguments as in the proof of Theorem 3 imply the desirable result (table IV).

To the end of the paper we shall prove Lemma 4 (wich generalizes Lemma 2 and Lemma 3).

Let $k=k_0$ be fixed, $(f_0, h_0, k_0)\in\mathbf{B}$ be a generic point (*i.e.* in a neighbourhood of it $\mathbf{B}$ is a smooth manifold), and let $q=(x^0, y^0, p_x^0, p_y^0)$ be a point on the level set $\{\mathrm{H}=h_0,\ \mathrm{F}=f_0\}$. We shall prove that if

$$c_1\,\mathrm{grad}(\mathrm{H})|_q+c_2\,\mathrm{grad}(\mathrm{F})|_q=0 \tag{21}$$

then the straight line $\{c_1 h+c_2 f+c_3=0\}$ is tangent to $\mathbf{B}$ (and hence it is not generic). As the equation of a straight line tangent to $\mathbf{B}$ at the point (f_0, h_0, k_0) is given by

$$\{u_0^3-hu_0^2+(1-k^2)u_0-f/2=0\}\subset\mathbf{R}^2\{f, h\}$$

where u_0 is the double root of the polynomial $\mathrm{P}(u)=(u^2-k^2)\varphi(u)$ then it is enough to prove that $c_1/c_2=2u_0^2$. In $(\lambda, \mu, p_\lambda, p_\mu)$ coordinates defined by (4), (5) we have the identity

(7)

$$F = p_\mu^2(\mu^2-k^2)+2\mu^3-2\mu^2 H+2\mu(1-k^2).$$

Then, at least far from the locus we have

(22)
$$\{\lambda=\mu\}\cup\{(\lambda^2-k^2)(\mu^2-k^2)=0\}$$

(23)
$$\left\{\begin{aligned} \frac{\partial F}{\partial \mu} &= 2\mu p_\mu^2-\varphi'(\mu)-2\mu^2\frac{\partial H}{\partial \mu} \\ \frac{\partial F}{\partial p_\mu} &= 2(\mu^2-k^2)p_\mu-2\mu^2\frac{\partial H}{\partial p_\mu} \\ \frac{\partial F}{\partial \lambda} &= -2\mu^2\frac{\partial H}{\partial \lambda} \\ \frac{\partial F}{\partial p_\lambda} &= -2\mu^2\frac{\partial H}{\partial p_\lambda} \end{aligned}\right.$$

As grad(H) and grad(F) are colinear according to (21), then

(24)
$$\left\{\begin{aligned} &2(\mu^2-k^2)p_\mu=0 \\ &2\mu p_\mu^2-\varphi'(\mu)=0 \end{aligned}\right.$$

and hence $p_\mu=0$, $\varphi'(\mu)=0$. Now (6) implies that $\varphi(\mu)=\varphi'(\mu)=0$ and hence μ is a double root of the polynomial $(u^2-k^2)\varphi(\mu)$. Suppose now that $\lambda^0, \mu^0, p_\lambda^0, p_\mu^0)$ belongs to the locus (22) and let $(\lambda, \mu, p_\lambda, p_\mu)$ tends to $(\lambda^0, \mu^0, p_\lambda^0 . p_\mu^0)$. The vectors grad(H) and grad(F) tend to some vectors grad$(H)^0$ and grad$(F)^0$ and let us suppose that these vectors are colinear. Using (6), (23) and (24) we conclude that

$$(\mu^2-k^2)\varphi(\mu)\to 0 \qquad \text{and} \quad 2\mu\frac{\varphi(\mu)}{\mu^2-k^2}-\varphi'(\mu)\to 0$$

and hence μ^0 is a double root of the polynomial $(\mu^2-k^2)\varphi(\mu)$. The upshot is that if $c_1\,\mathrm{grad}(H)+c_2\,\mathrm{grad}(F)=0$ then $c_1/c_2=2\mu_0^2$, where μ_0 is the double root of the polynomial $(\mu^2-k^2)\varphi(\mu)$, and hence the straight line

$$\{c_1 h+c_2 f+c_3=0,\ k=\text{const.}\}$$

is tangent to **B**. This completes the proof of Lemma 4. △

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L. Gavrilov (1)
M. Ouazzani-Jamil (2)
and
R. Caboz,
Laboratoire de Physique Appliquée
Université de Pau et des pays de l'Adour
64000 Pau, France
and
(1) Laboratoire de Topologie et Géométrie
URA CNRS 1408, Université Paul Sabatier
31062 Toulouse, France
(2) Faculté des Sciences Dhar el Mehraz Fès,
Département de Physique, Fès, Morocco.