LIMIT CYCLES AND ZEROES OF ABELIAN INTEGRALS SATISFYING THIRD ORDER PICARD - FUCHS EQUATIONS

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1. INTRODUCTION

The second part of Hilbert 16th problem raises the question about an upper bound c(N) for the maximal number of the limit cycles (i.e. isolated periodic solutions) of a planar polynomial vector field of degree N :

$$P(x,y)dx + Q(x,y)dy = 0 \qquad (1.1)$$

This problem is still open, and it is not known even whether $c(N) < \infty$. The above question splits into a few subquestions, the first of which is: *is the number of limit cycles of a fixed vector field finite*? It was believed almost 60 years that the answer is yes, as claimed Dulac in his memoir "Sur les cycles limites"[8].However, in 1985 Yu. Il'yashenko[12] found a trivial technical gap in the final step of Dulac's proof. The correct proof was announced in the articles of Il'yashenko[21] and Ecalle, Martinet, Moussu, Ramis[9] . An important part of this proof containing the crucial idea appeared recently in [24]

A possible approach to the Hilbert 16th problem is, starting from a known system, to study the possible subsequent bifurcations

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of the vector field [18]. Consider the perturbed integrable system of ordinary differential equations

$$dH(x,y) + \varepsilon.(R_1(x,y)dx + R_2(x,y)dy) = 0 \qquad (1.2)$$

where H, R_1, R_2 are polynomials. If the level sets $\{H = h\} \subset \mathbb{R}^2$ of the integrable system contain a family of ovals then, after perturbing this system some of the ovals sustain the perturbation, i.e. become limit cycles. To determine whether it happens we consider the Poincaré map p(h), corresponding to a given oval $o(h) \subset \{H = h\}$. The first approximation of p(h) - h with respect to ε is given by the Abelian integral

$$I(h) = \oint_{o(h)} R_2.dx - R_1.dy \quad . \qquad (1.3)$$

The maximal number $\tilde{c}(N)$ (deg H ≤ N + 1, deg R_1 ≤ N, deg R_1 ≤ N) of the zeroes of I(h) (including the multiplicities) is an upper bound of the number of the ovals, sustaining the bifurcation.

The next question is: *Determine the number* $\tilde{c}(N)$. It is also known as the "weakened 16th Hilbert problem"[23]. Recently Khovansky[15] and Varchenko[20] proved independently that $\tilde{c}(N) < \infty$.

Consider the simplest case N = 2. The number of zeroes of (1.3) is known for certain fixed Hamiltonian functions H, and certain perturbations [7,13,16,17]. In [7] the authors consider a small one-parameter cubic perturbation of a fixed cubic polynomial H. As the moduli space of all cubic Hamiltonians, whose level sets contain ovals, is a two-dimensional one (see section 2) then these perturbed Hamiltonians represent a co-dimension one "local" family (i.e. defined for "small" values of the parameter) of cubic polynomials.

In the present paper we consider another co-dimension one family of Hamiltonians, namely Hamiltonians with an invariant line. This family is "global", in the sense that it depends upon a parameter taking all real values. We prove the monotonicity of the period functions (which are given by special Abelian integrals) - Theorem 1 of section 2. Our second result - Theorem 2 of section 2, gives an upper bound for the limit cycles of any quadratic perturbation of these Hamiltonians. The method exploited here applies to general cubic Hamiltonians. The case considered in this paper has the only advantage that the computations needed for the proofs are simpler. We hope to return to the general case in another publication.

We use well known methods of algebraic geometry : Picard-Fuchs equations, Picard-Lefschetz theory. The Abelian integral I(h) satisfies a third order Picard-Fuchs equation, and $J(h) = \frac{d^2}{dh^2}I(h)$ satisfies a second order Picard-Fuchs equation. As the coefficients of this equation are rational functions in h, then the standard technique [2,3,5,6,7,10,16,17] implies a bound for the number of zeroes of J(h), and hence of I(h). As we want to find an exact bound for the number of zeroes then we face two problems :

- by differentiating I(h) twice we obviously loose information about the behaviour of I(h)
- we need an estimate of the number of the zeroes of J(h), which is a priori lower than this number for an arbitrary solution of the Picard-Fuchs equation satisfied by J(h).

To overcome the second problem we have to know what is the distinction between J(h) and the remaining solutions of the Picard-Fuchs equation. From a geometrical point of view the answer is surprisingly simple. Namely, consider the corresponding Riccati equation as a two-dimensional autonomous system of ordinary differential equations. *Then the solution corresponding to* J(h) *is a separatrix solution of this system.* To make use of this fact we study the global phase portrait, and apply the "Rolle's theorem for dynamical systems"[14].

To compensate the loss of information after differentiating I(h) we use the "local" information about the values of I(h), $\frac{d}{dh}I(h)$, $\frac{d^2}{dh^2}I(h)$, at the ends of the interval $\Delta \ni h$ on which the ovals are defined. Probably, however, that is not enough to find the exact value of the number of zeroes of I(h).

The paper is organized as follows. In section 2 we formulate our results and give some definitions. Section 3 and Section 4 are more or less routine. There we derive the Picard-Fuchs equations satisfied by our Abelian integrals, and describe their asymptotic behaviour. The reader may skip these two sections, and then to come back when the occasion arises. In section 5 we prove the monotonicity of the period functions corresponding to the Hamiltonian functions under consideration. At last, all these results are used in section 6, where we prove Theorem 2.

Remark. When this text was already prepared we learned that W.A.Coppel [23] has proved that any quadratic system with an invariant line has at most one limit cycle. This implies immediately our Theorem 2. Nevertheless we keep it as the proof contains the main ideas needed to find an upper bound for the limit cycles of quadratic perturbations of the general quadratic Hamiltonian system - namely the second derivative of the Abelian integral (1.3) satisfies a second order Picard-Fuchs equation.

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2. STATEMENT OF THE RESULTS

Let $H(x,y)$ be a polynomial and $\omega = P_1(x,y)dx + P_2(x,y)dy$ be a polynomial one-form, where $\deg(H) = 3$, $\deg(P_1), \deg(P_2) \le 2$, and H, P_1, P_2 have real coefficients. Further we shall suppose that for some $h \in \mathbb{R}$ the set $\{H = h\} \subset \mathbb{R}^2$ contains an oval (i.e. a compact real curve Γ, such that $(H_x, H_y) \neq (0,0)$ on Γ). Let $\Delta \subset \mathbb{R}$ be the set of those $h \in \mathbb{R}$ for which $\{H = h\} \subset \mathbb{R}^2$ contains an oval. Δ is either an open interval or a union of two open intervals.

Let $H = x.(y^2 + (x - g)^2 - 1)$. Define the period function $T(h)$ to be the period of the unique periodic solution lying on the level set $\{H= h\}$, $h \in \Delta$ (see [6]). The central results of the present paper are the following theorems.

<u>Theorem 1.</u> If $h \in \Delta$, then $T'(h) \neq 0$.

<u>Theorem 2.</u> The Abelian integral $I(h) = \int_{\delta(h)} \omega$ either vanishes identically in Δ, or it has no more than three zeroes (including the multiplicities) in Δ. If $|g| \ge 1$, then $I(h)$ has no more than one simple zero in Δ.

If Δ is not empty, the Hamiltonian system corresponding to H has a continuous family of periodic solutions. It is known that in this case the system also has a center [23]. Hence, after suitable $\mathbb{R}$ - linear change of the variables H takes the form

$$H = \frac{x^2 + y^2}{2} + Ax^3 + Bx^2y + Cxy^2 + Dy^3 \qquad (2.1)$$

The reader may check that there always exists a rotation which brings H into the form (2.1) with $D = 0$. If $B \neq 0$ we shall also suppose (without loss of generality) that $B = 1$, or if $B = 0$, we shall also suppose that $A = 1$.Thus we get the following two normal forms

$$H = \frac{x^2 + y^2}{2} + Ax^3 + x^2y + Cxy^2 \qquad (2.2)$$

$$H = \frac{x^2 + y^2}{2} + x^3 + Cxy^2 \tag{2.3}$$

<u>Definition.</u> We shall say that the two polynomials $H_1(x,y)$, $H_2(x,y)$ with real coefficients are equivalent, provided that there exists a $\mathbb{R}$ - linear change of the variables which brings $H_1(x,y)$ into the form $c_1 + c_2.H_2(x,y)$, for some c_1, $c_2 \in \mathbb{R}$, $c_2 \neq 0$.

The constants A, C defined by (2.2), (2.3) provide moduli for the space of all non-equivalent cubic polynomials, whose level sets contain ovals. To obtain a one to one correspondance between the parameters A, C and this space we have to reduce $\mathbb{R}^2\{A,C\}$ and $\mathbb{R}\{C\}$ by identifying the points lying in one and the same orbit of the free action of a finite group, generated by the rotations and the reflections preserving the normal form (2.2) or (2.3) , and the rank two group $(x,y) \to (x_0-x, y_0-y)$, where (x_0,y_0) is the other center (if it exists).The upshot is that the moduli space of all non-equivalent cubic polynomials is a two dimensional one.

<u>Definition.</u> The Hamiltonian H (with real coefficients) is reducible, provided that for some $h \in \mathbb{R}$ the set $\{H = h\} \subset \mathbb{C}^2$ is reducible.

In the case when deg(H) is an odd number, one easely checks that the above definition is equivalent to

<u>Definition.</u> The Hamiltonian H (with real coefficients) is reducible, provided that for some $h \in \mathbb{R}$ holds $H - h = H_1.H_2$, where H_1 and H_2 are polynomials with real coefficients.

<u>Example.</u> The Hamiltonian (2.3) is reducible. Indeed

$$\frac{x^2 + y^2}{2} + x^3 + Cxy^2 - \frac{3}{8} C = (x + \frac{1}{2} C).(y^2 + \frac{3}{2} x - \frac{3}{4} C) \quad .$$

The present paper deals only with a codimension one subspace of reducible cubic polynomials, given by $H = x.(y^2 + (x - g)^2 - 1)$.

Let H be a reducible cubic polynomial with real coefficients. Then without loss of generality $H = H_1.H_2$, where $H_1 = x$ and $\deg(H_2) = 2$, and let

$$H = x.(ax^2 + bxy + cy^2 + dx + ey + f).$$

If $c = 0$ then $\{H = h\} \cap \{x = \text{const.}\}$ is the empty set, or it consists of one point, and hence $\{H = h\}$ does not contain ovals. Thus, without loss of generality we may put $c = 1$ and also $b = e = 0$. Further $\mathbb{R}$ - linear changes show that H is equivalent to one of the following Hamiltonians

$$x.(y^2 \mp (x-g)^2 \mp 1)$$
$$x.(y^2 \mp (x-g)^2)$$
$$x.(y^2 + x \mp 1) \qquad (2.4)$$
$$x.(y^2 \mp 1)$$
$$x.y^2$$

The only reducible Hamiltonians (2.4) contained in the family (2.2) or (2.3) are

$$x.(y^2 + (x-g)^2 \pm 1) \text{ (elliptic case)}$$
$$x.(y^2 - (x-g)^2 \pm 1) \quad \text{(hyperbolic case)}$$
$$x.(y^2 + x - 1) \quad \text{(parabolic case)}$$
$$x.(y^2 \pm (x-1)^2) \quad \text{(linear case)} .$$

In this paper we shall study only one of the above cases.

3. THE PICARD - FUCHS EQUATIONS

Consider the Hamiltonian

$$H = x.(y^2 + (x-g)^2 - 1) \quad . \qquad (3.1)$$

The level sets $\{H = h\} \subset \mathbb{R}$ are shown on fig.1. For any fixed $g \in \mathbb{C}$ there are three critical values of h. They are $h = 0$, $h = h^{\pm}(g)$, where

$$h^{\pm} = 2.(g.(g^2 - 9) \pm (g^2 + 3)^{3/2})/27 \quad .$$

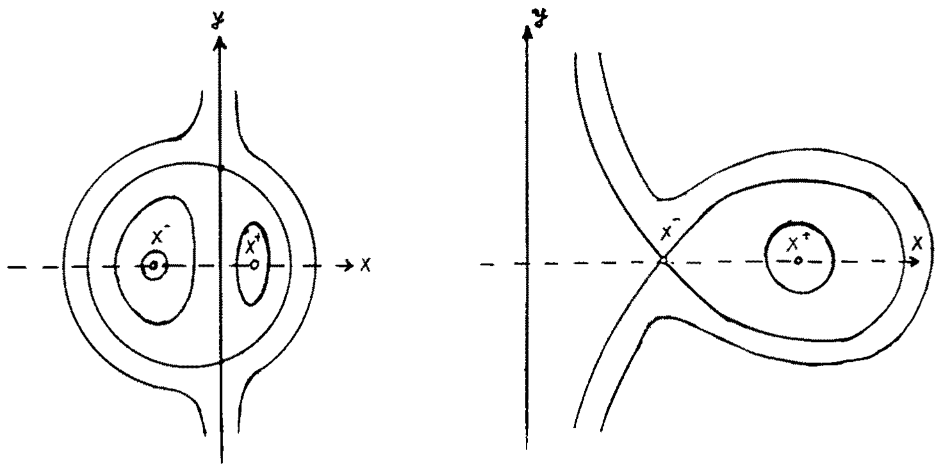
For all non-critical values of h the affine algebraic curve $\Gamma_{g,h} = \{H = h\} \subset \mathbb{C}^2$ is a smooth complex manifold. The union of the curves defined by the equations $h = 0$ and $h = h^{\pm}(g)$ in $\mathbb{C}\{h,g\}$ form the bifurcation diagram. It is given on fig.2. For each fixed g let us define on $\Gamma_{g,h}$ the following differential one-forms

$$\alpha = ydx \ , \ a = \frac{d}{dh}\alpha = \frac{dx}{2xy} \ ,$$

$$\beta = x(x-g).y.dx \ , \ b = \frac{d}{dh}\beta = \frac{(x-g)}{2y}\,dx \ , \qquad (3.2)$$

$$\gamma = xy.dx, \ c = \frac{d}{dh}\gamma = \frac{dx}{2y} \ .$$

Here $\frac{d}{dh}$ is a covariant derivative in the Gauss-Manin connection of



a) $|g| < 1$ b) $|g|>1$

$$x^{\mp} = \frac{2g \mp \sqrt{g^2+3}}{3}$$

Fig.1.Level sets of Hamiltonian (3.1).

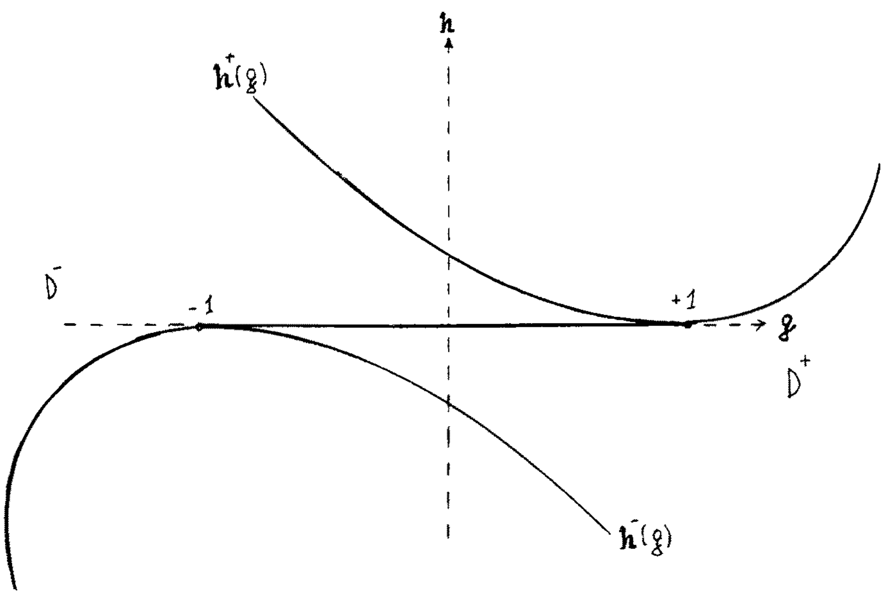


Fig 2. Bifurcation diagram of (3.1).

the bundle $H^1(\Gamma_{g,h},\mathbb{C}) \to h \in \mathbb{C}\setminus\{h^+(g),h^-(g),0\}$, associated with the local trivial bundle $\mathbb{C}^2 \supset \Gamma_{g,h} \to h \in \mathbb{C}\setminus\{h^+(g),h^-(g),0\}$ [1].

Proposition 3.1. For any fixed point (g,h) which does not lie on the bifurcation diagram, the one-forms a, b, c defined by (3.2) form a basis of the three dimensional linear space $H^1(\Gamma_{g,h},\mathbb{R})$.

Proof. It is more convenient to make the computations in the proof of this Proposition using the following new coordinates

$$x \to x,\ x.y \to z \quad .$$

The elliptic curve $\Gamma_{g,h} = \{H = h\} \subset \mathbb{C}^2$ takes the form

$$\{\ z^2 = x.(h - x.(x - g)^2 + x)\}\ . \tag{3.3}$$

One may easely check that the above change of variables provides a biholomorphic mapping between the affine curve (3.3) and $\Gamma_{g,h}$. In the new coordinates we have $a = \frac{dx}{2z}$, $b = \frac{x.(x - g)}{2z}\,dx$, $c = \frac{x}{2z}\,dx$. According to Grothendieck's theorem [11] the polynomial one-forms generate a basis of the first de Rham cohomology group of the affine algebraic curve (3.3). Hence it is enough to prove that each polynomial one-form restricted to (3.3) is equal, up to an addition of an exact form, to a linear combination of the forms a, b, c. Namely, each polynomial one-form equals on (3.3) to a sum of one forms $\frac{x^s}{z}\,dx$, s = 0, 1, 2. The degrees of the one forms $\frac{x^s}{z}\,dx$ is reduced with the help of the identity

$$d(x^{s-3}.z) = \frac{2(s-3)x^{s-4}.P(x) + x^{s-3}.P'(x)}{2z}\,dx$$

where $P(x) = x.(h - x(x - g)^2 + x)$, $P'(x) = \frac{d}{dx}P(x)$. The leading term in the coefficient of the above one-form is $x^s.(2(s-3)+4)$, and hence $\frac{x^s}{z}\,dx$, $s \geq 3$, is equivalent on (3.3) to a linear combination of forms $\frac{x^r}{z}\,dx$, r = 0, 1, ..., s-1.∎

Proposition 3.2 For each fixed (g,h) the following identities hold on $\Gamma_{g,h}$

$$\alpha = \frac{3}{2}.h.a + g.b + c + \frac{1}{2}.d(xy)$$

$$\beta = \frac{(3-2g^2).h}{24}.a + \frac{(9h + 5g - 2g^3)}{12}.b + \frac{1}{4}.c + d\left(\frac{xy(6x^2-10gx + 2g^2-3)}{24}\right)$$

$$\gamma = \frac{gh}{6}.a + \frac{2+g^2}{3}.b + (h+g).c + d\left(\frac{xy(2x-g)}{6}\right) \quad . \tag{3.4}$$

The proof is a straightforward computation : we use (3.2) and the identity $y = ((h - x.(x - g)^2 + x)/x)^{1/2}$.

Let $\delta(h) \to h$ be a locally constant section of the bundle $H_1(\Gamma_{g,h},\mathbb{C}) \to h \in \mathbb{C}\setminus\{h^+(g),h^-(g),0\}$. Suppose that $\delta(h)$ are represented by (homological) ovals on $\Gamma_{g,h}$. Thus we can associate the above section with a continuous family of ovals on $\Gamma_{g,h}$. We shall denote these ovals again by $\delta(h)$.

<u>Proposition</u> <u>3.3.</u> The Abelian integrals $A = \int_{\delta(h)} a$, $B = \int_{\delta(h)} b$, $C = \int_{\delta(h)} c$, satisfy the following Picard-Fuchs system

$$d(h).\frac{d}{dh}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} h(8g + 9h) & ,2(2g^2+3gh-2) & ,0 \\ h(2g-2g^3+3h-2g^2h) & ,-h(8g+9h) & ,0 \\ h(2g^2-2+3gh) & , 2h(g^2+3) & ,0 \end{pmatrix}.\begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{3.5}$$

where

$$d(h) = -27h(h - h^+(g))(h - h^-(g)) = h(4g^4+4-8g^2+4hg^3-36hg-27h^2).$$

<u>Proof.</u> We integrate (3.4) along $\delta(h)$ and then differentiate with respect to h . As $\frac{d}{dh}\alpha = a$, $\frac{d}{dh}\beta = b$, $\frac{d}{dh}\gamma = c$, then we obtain a linear system for A, B, C, $\frac{d}{dh}A$, $\frac{d}{dh}B$, $\frac{d}{dh}C$. Solving this system in $\frac{d}{dh}A$, $\frac{d}{dh}B$, $\frac{d}{dh}C$ (which in view of Proposition 3.1. is always possible), we obtain (3.5).■

It is easy to check that the one forms a, b have no residues on $\Gamma_{g,h}$. Hence they represent elements of $H^1(\overline{\Gamma}_{g,h},\mathbb{R})$, where $\overline{\Gamma}_{g,h}$ is the compactification of $\Gamma_{g,h}$.As $\dim(H^1(\overline{\Gamma}_{g,h},\mathbb{Z})) = \dim(H_1(\overline{\Gamma}_{g,h},\mathbb{Z}) = 2$ ($\overline{\Gamma}_{g,h}$ is an elliptic curve), then A , B, satisfy a second order Picard-Fuchs equation (see (3.5)).

Let $\delta(h)$, $\theta(h)$ form a basis of $H_1(\overline{\Gamma}_{g,h},\mathbb{Z})$. For arbitrary rational one-forms ω_1, ω_2 , which do not possess residues, consider the Wronskian

$$W(\omega_1,\omega_2) = \det\begin{pmatrix} \int_{\delta(h)}\omega_1 & , & \int_{\delta(h)}\omega_2 \\ \int_{\theta(h)}\omega_1 & , & \int_{\theta(h)}\omega_2 \end{pmatrix} .$$

It is a rational function in g, h [1]. Now the Liouville theorem, applied to the Picard-Fuchs system satisfied by A, B, implies that W(a,b) does not depend upon h, as the trace of this (linear) system is equal to zero. On the other hand Proposition 3.1. implies that $W(a,b) \neq 0$, if $d(h) \neq 0$, and hence W(a,b) is equal to a onstant $\rho\neq 0$.

$A'=\frac{d}{dh}A$, $C'=\frac{d}{dh}C$ satisfy the following Picard-Fuchs system

$$\frac{d}{dh}\begin{pmatrix} C' \\ A' \end{pmatrix}=\begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix}.\begin{pmatrix} C' \\ A' \end{pmatrix} \tag{3.6}$$

where $a_{11} = W(c'',a')/W(c',a')$, $a_{12} = -W(c'',c')/W(c',a')$, $a_{21} = W(a'',a')/W(c',a')$, $a_{22} = -W(a'',c')/W(c',a')$. $A = A(h)$ satisfies the following Picard-Fuchs equation

$$p.A'' + q.A' + r.A = 0 \tag{3.7}$$

where $p = W(a',a).d^2$, $q = -W(a'',a).d^2$, $r = W(a'',a').d^2$, $d = d(h)$. Below we shall compute the above Wronskians explicitely. For that purpose we shall use (3.5). For example to compute $W(c',a')$ we note that

$$d(h).\frac{d}{dh}\begin{pmatrix} C \\ A \end{pmatrix} = M(h).\begin{pmatrix} A \\ B \end{pmatrix}$$

where the matrix $M(h)$ is given by (3.5). Then $W(c',a') = W(a,b).\det(M(h)/d^2 = \rho.\det(M(h)/d^2$. However, one can prove along the same lines as in [13] that $W(c',a')$ has only simple poles. It means that $d = d(h)$ divides the polynomial $\det(M(h))$. The same cancelations occur for the remaining Wronskians, and the explicite expressions are rather simple. The direct computation gives

<u>Proposition</u> <u>3.4.</u> Let $d(h)$ be as in Proposition 3.3. Then

$$W(a',a)/\rho = 2.(2 - 2g^2 - 3gh)/d(h) \quad ,$$

$$W(c',a')/\rho = 2/d(h) \quad ,$$

$$W(c'',a')/\rho = 6h.(2g^2h + 15h + 8g)/d(h)^2 \quad ,$$

$$W(c'',c')/\rho = 2h.(2g^3h + 3gh + 2 - 2g^2)/d(h)^2 \quad ,$$

$$W(a'',c')/\rho = 4.\{h^2.(3g^2 - 18) + h.(4g^3 - 24g) + 2.(g^2 - 1)^2\}/d(h)^2,$$

$$W(a'',a')/\rho = 12.(3gh^2 + 4g^2h - 5h + 2g.(g^2 - 1))/d(h)^2 \quad ,$$

$$W(a'',a)/\rho = 4.\{-81gh^3 + h^2.(6g^4 - 135g^2 + 81) + h.(8g^5 - 80g^3 + 72g) + 4.(g^2 - 1)^3\}/d(h)^2 \quad , \rho = \text{const.} \neq 0.$$

4. ASYMPTOTIC BEHAVIOUR OF ABELIAN INTEGRALS

Consider the open subsets of $\mathbb{R}^2$

$$D^+ = \{(g,h) \in \mathbb{R}^2: h^-(g) < h < h^+(g),\ g \geq 1\} \cup \{ (g,h) \in \mathbb{R}^2: h^-(g) < h < 0,\ |g| < 1 \}$$

and

$$D^- = \{ (g,h) \in \mathbb{R}^2: h^-(g) < h < h^+(g),\ g \leq -1 \} \cup \{ (g,h) \in \mathbb{R}^2: 0 < h < h^+(g),\ |g| < 1 \} ,$$

shown on fig.2. For each $(g,h) \in D^+ \cup D^-$ the set $\{H = h\} \subset \mathbb{R}^2$

contains an oval $\delta(h)$ (see fig.1). Note that for $|g| > 1$, $h = 0$ the complex curve $\Gamma_{g,h}$ is singular (i.e. (g,h) is a bifurcation point). Nevertheless it still contains an oval. For all the remaining bifurcation points $\Gamma_{g,h}$ does not contain an oval.

Consider now the open set $\Delta_g = \{h \in \mathbb{R} : (g,h) \in D^+ \cup D^-\}$. For each fixed $g \in \mathbb{R}$ it coincides with the set Δ of Theorem 1. Let us define on Δ_g the following two meromorphic functions

$$\zeta_g(h) = \int_{\delta(h)} c \Big/ \int_{\delta(h)} a, \quad \xi_g(h) = \int_{\delta(h)} \frac{d}{dh} c \Big/ \int_{\delta(h)} \frac{d}{dh} a \quad ,$$

where the one forms a, b, c are defined by (3.2). The importance of these functions were suggested to us by [5,7]. In this section we shall study the asymptotic behaviour of $\zeta_g(h)$, $\xi_g(h)$, when h tends to a point on the boundary $\overline{\Delta_g} \setminus \Delta_g$ of Δ_g. Notice the obvious relations :

$$(g,h) \in D^+ \Leftrightarrow (-g,-h) \in D^-, \ \zeta_g(h) = -\zeta_{-g}(-h), \ \xi_g(h) = -\xi_{-g}(-h).$$

<u>Proposition</u> <u>4.1.</u> $\zeta_g(h)$ has the following asymptotic behaviour

i) $\lim_{h \downarrow h^-(g)} \zeta_g(h) > \lim_{h \uparrow h^+(g)} \zeta_g(h)$, for $g \geq 1$

ii) $\lim_{h \downarrow h^-(g)} \zeta_g(h) > \lim_{h \uparrow 0} \zeta_g(h) = 0 = \lim_{h \downarrow 0} \zeta_g(h) > \lim_{h \uparrow h^+(g)} \zeta_g(h)$

for $-1 < g < 1$

iii) $\lim_{h \downarrow h^-(g)} \frac{d}{dh} \zeta_g(h) < 0$, for $g > -1$

iv) $\lim_{h \uparrow 0} \frac{d}{dh} \zeta_g(h) = \lim_{h \downarrow 0} \frac{d}{dh} \zeta_g(h) = -\infty$, for $-1 < g < 1$

v) $\lim_{h \uparrow h^+(g)} \frac{d}{dh} \zeta_g(h) = -\infty$, for $g \geq 1$.

<u>Proposition</u> <u>4.2.</u> $\xi_g(h)$ has the following asymptotic behaviour

i) $\lim_{h \uparrow h^+(g)} \xi_g(h) = \lim_{h \uparrow h^+(g)} \zeta_g(h)$, for $g \geq 1$

ii) $\lim_{h \uparrow 0} \xi_g(h) = \lim_{h \uparrow 0} \zeta_g(h) = 0$, for $-1 \leq g \leq 1$

iii) $\lim_{h \uparrow h^+(g)} \frac{d}{dh} \xi_g(h) = \infty$, for $g \geq 1$

iv) $\lim_{h \uparrow 0} \frac{d}{dh} \xi_g(h) = \infty$, for $-1 \leq g \leq 1$.

v) $\lim_{h \downarrow h^-(g)} \frac{d}{dh} \xi_g(h) < 0$, for $g > 1$.

Like in section 3 we can associate the oval $\delta(h)$ with the corresponding homological cycle in $H_1(\Gamma_{g,h},\mathbb{Z})$, which we shall denote again by $\delta(h)$. Now $\delta(h)$ can be defined also for complex values $h \in \mathbb{C}\setminus\{h^+(g),h^-(g),0\}$, in such a way that $\delta(h) \to h$ is a locally constant section of the bundle $H_1(\Gamma_{g,h},\mathbb{Z}) \to h \in \mathbb{C}\setminus\{h^+(g),h^-(g),0\}$. Thus each Abelian integral $\int_{\delta(h)} \omega$ (and hence $\zeta_g(h)$, $\xi_g(h)$) becomes a multivalued meromorphic function on $\mathbb{C}\setminus\{h^+(g),h^-(g),0\}$, with branch points at $h^+(g)$, $h^-(g)$, and 0. To prove Proposition 4.1. and Proposition 4.2. we shall need a formula for the branching of the Abelian integrals at these points. It can be derived from the Picard-Lefschetz formula [1].

For any fixed g the affine curve $\Gamma_{g,h}$ is singular only if $h = 0$, or $h = h^{\pm}(g)$. Suppose that $g \neq \pm 1$ is a fixed real number. If $h = 0$, the level set $\Gamma_{g,h}$ orresponding to the critical value $h = 0$ contains two Morse critical points ($x = 0$, $y = \pm\sqrt{1 - g^2}$) of H. Let us denote the corresponding vanishing cycles by $\theta^{\pm}$. Without loss of generality we may suppose that θ^+ and θ^- are homological and let us denote them by $\theta \equiv \theta^+ \sim \theta^-$. If $h = h^{\pm}(g)$, $\Gamma_{g,h}$ has one double point at $x = x^{\mp} = \dfrac{2g \mp \sqrt{g^2+3}}{3}$, $y = 0$, which is a Morse critical point of H with critical value $H = h^{\pm}(g)$. Let us denote the corresponding vanishing cycle by $\delta^{\pm}$. Denote $D_\rho = \{|z| < \rho\}\setminus\{h^-(g),\ 0,\ h^+(g)\} \subset \mathbb{C}$, where ρ is a sufficiently big fixed real number. Let z_0 be a point on the boundary $|z| = \rho$ of D_ρ. Any loop $\ell \in \pi_1(D_\rho,z_0)$ induces an isomorphism ℓ_* (monodromy) in the first homology group

$$\ell_* : H_1(\Gamma_{g,h},\mathbb{Z}) \to H_1(\Gamma_{g,h},\mathbb{Z}) .$$

Let ℓ^0, $\ell^{\pm} \in \pi_1(D_\rho,z_0)$ be loops around 0 and $h^{\pm}(g)$ respectively.The (generalized) Picard-Lefschetz formula [4,1] reads

$$\begin{aligned} \ell_*^0(\delta) &= \delta + (\theta^+\circ\delta)\theta^+ + (\theta^-\circ\delta)\theta^- = \delta + 2(\theta\circ\delta)\theta \\ \ell_*^{\pm}(\delta) &= \delta + (\delta^{\pm}\circ\delta)\delta^{\pm} \end{aligned} \tag{4.1}$$

where $(\delta^{\pm}\circ\delta)$, $(\theta^{\pm}\circ\delta)$, $(\theta\circ\delta)$ are the intersection indexes of the corresponding cycles.

Consider an arbitrary Abelian integral $I(h) = \int_{\delta(h)} \omega$, where ω is a meromorphic one-form without residues on $\Gamma_{g,h}$. The Picard-Lefschetz formula (4.1) implies [1] that in a neighbourhood of $h = 0$ holds

$$I(h) = \frac{\log(h)}{\pi i} \cdot \int_{\theta(h)} \omega + P(h) \quad , \tag{4.2}$$

and in a neighbourhood of $h = h^{\pm}(g)$ holds

$$I(h) = \frac{\log(h - h^{\pm}(g))}{2\pi i} \cdot \int_{\delta^{\pm}(h)} \omega + Q(h - h^{\pm}(g)) \quad , \tag{4.3}$$

where P,Q are meromorphic functions.

Consider now the case $g = \pm 1$. The Milnor number [1] of the critical point $x = 0$, $y = 0$ of H is two, and hence the Picard-Lefschetz formula (4.1) can not be directly applied. Denote

$$D_\rho = \{|z| < \rho\}\setminus\{h^-(g),\ h^+(g)\} \subset \mathbb{C}$$

where ρ is a sufficiently big fixed real number, and let z_0 be a fixed point on the boundary $|z| = \rho$ of D_ρ. Let ℓ be a loop around (1,0) lying in the complex plane $\{g = 1\}\mathrm{x}\mathbb{C}\{h\}$, and let $\ell_1, \ell_2 \in \pi_1(D_\rho, z_0)$ be loops around $(1+\varepsilon, h^+(1+\varepsilon))$ and $(1+\varepsilon, 0)$ in the complex plane $\{g = 1+\varepsilon\}\mathrm{x}\mathbb{C}\{h\}$, where $\varepsilon > 0$ is a sufficiently small number (fig. 3).

Denote by M_ℓ, M_{ℓ_1}, M_{ℓ_2}, the corresponding monodromy matrices acting upon $H_1(\Gamma_{g,h}, \mathbb{Z})$. It is well known that $M_\ell = M_{\ell_1} \circ M_{\ell_2}$. Hence to ompute the monodromymatrix M_ℓ acting upon $H_1(\Gamma_{g,h}, \mathbb{Z})$ it is enough to compute M_{ℓ_1} and M_{ℓ_2}. The Picard-Lefschetz formula implies that in θ, δ^+, δ^- coordinates (having suitable orientations) holds

$$M_{l_1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} , \quad M_{l_2} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} , \Rightarrow M_{l_1} \circ M_{l_2} = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{pmatrix} .$$

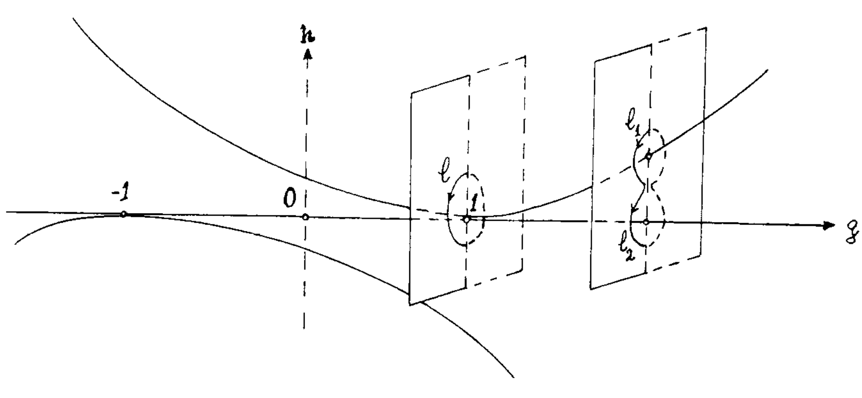


Fig.3. Deformation of the loop l into the loop $l_1 \circ l_2$.

As the eigenvalues of the last matrix are 1, $\pm i$, then it is concluded that the Abelian integral $I(h)$ is a meromorphic function in a neighbourhood of $h = 0$, with respect to $h^{1/4}$ [1].

Proof of Proposition 4.1.

i) As $\delta(h)$ vanishes at $h = h^-(g)$, $g \neq \mp 1$, into a Morse critical point of H, then $\int_{\delta(h)} \gamma$, $\int_{\delta(h)} \alpha$ are holomorphic functions in h, in a sufficiently small neighbourhood of $h = h^+(g)$. As

$$\lim_{h\downarrow h^-(g)} \int_{\delta(h)} \gamma = \lim_{h\downarrow h^-(g)} \int_{\delta(h)} \alpha = 0, \text{ then}$$

$$\lim_{h\downarrow h^-(g)} \int_{\delta(h)} \gamma \Big/ \int_{\delta(h)} \alpha = \lim_{h\downarrow h^-(g)} \int_{\delta(h)} c \Big/ \int_{\delta(h)} a = \lim_{h\downarrow h^-(g)} \zeta_g(h) =$$

$$= \operatorname{Res}\Big|_{x=x^+}\left(\frac{x.dx}{\sqrt{x.[h^-+x.(1-(x-g)^2)]}}\right)\Big/\operatorname{Res}\Big|_{x=x^+}\left(\frac{dx}{\sqrt{x.[h^-+x.(1-(x-g)^2)]}}\right)=$$

$= x^+$, where $x^+ = \dfrac{2g + \sqrt{g^2 + 3}}{3}$ (see fig.1). In quite a similar way one computes $\lim_{h\uparrow h^+(g)} \int_{\delta(h)} \gamma \Big/ \int_{\delta(h)} \alpha = \lim_{h\uparrow h^+(g)} \zeta_g(h) = x^-$.Obviously $x^- < x^+$ (see fig.1) and hence to prove i) it remains to consider the case $g = 1$. The two cycles θ_1, θ_2 vanish simultaneously when h 0. For $h < 0$, $h \in \mathbb{R}$, the projections of these two cycles, and the cycle $\delta(h)$ on the $\mathbb{C}\{x\}$-plane are shown in fig.4.

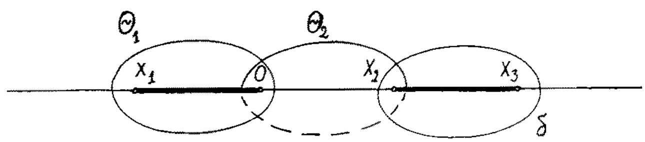


Fig.4. The cycles θ_1, θ_2, δ in the complex x - plane.

$x_1(h), x_2(h), x_3(h)$, are roots of the polynomial $h-x.(x-g)^2+x$, and $x_1(0) = x_2(0) = 0$. Fig.4 implies that $\delta(h)$ is homological in $H_1(\Gamma_{g,h},\mathbb{Z})$ to $\theta_1 \pm \theta_3$ where θ_3 is a cycle represented by a loop around one of the "infinite" points on $\Gamma_{g,h}$.Hence

$$\int_{\delta(h)} c = \int_{\theta_1(h)} c \pm 2\pi i.\operatorname{Res}\Big|_{x=\infty}\left(\frac{dx}{2.\sqrt{(1-(x-g)^2}}\right) = \pi + \int_{\theta_1(h)} c \quad .$$

(the sign of π is fixed by the condition $\int_{\delta(h)} c > 0$), and also

$$\int_{\delta(h)} a = \int_{\theta_1(h)} a \pm 2\pi i.\mathrm{Res}\Big|_{x=\infty} \left(\frac{dx}{2x.\sqrt{(1-(x-g)^2}} \right) = \int_{\theta_1(h)} a \quad .$$

The Picard-Lefschetz formula implies that the integrals $\int_{\delta(h)} c$ and $\int_{\delta(h)} a$ are holomorphic functions in $h^{1/4}$ (in a neighbourhood of $h = 0$). Changing the variable x as $x \to z.\sqrt{-h}$, where $\sqrt{-h} > 0$ for $h < 0$ we obtain

$$2.\int_{\delta(h)} c = (-h)^{1/4}. \int_{\theta_1(h)} \frac{zdz}{\sqrt{z(-1 + 2z^2 - z^3.\sqrt{-h}\ )}} + \pi =$$

$$= \pi + (-h)^{1/4}.(2.\int_{-1/\sqrt{2}}^{0} \sqrt{\frac{z}{(-1 + 2z^2)}}\ dz + O((-h)^{1/4})), \text{ and}$$

$$2.\int_{\delta(h)} a = (-h)^{-1/4}. \int_{\theta_1(h)} \frac{dz}{\sqrt{z(-1 + 2z^2 - z^3.\sqrt{-h}\ )}} =$$

$$= (-h)^{-1/4}.(2.\int_{-1/\sqrt{2}}^{0} \frac{dz}{\sqrt{z(-1 + 2z^2)}} + O((-h)^{1/4})) .$$

Hence in a neighbourhood of $h = 0$ we have

$$\zeta_g(h) = c_1.(-h)^{1/4} + O((-h)^{1/2}), \quad c_1 > 0 \tag{4.4}$$

which implies $\lim_{h \downarrow h^-(1)} \zeta_1(h) > \lim_{h \uparrow h^+(1)} \zeta_1(h)$.

ii) Suppose that $-1 < g < 1$, let $\theta(h)$ be a vanishing cycle at $h = 0$, and $(\theta(h)\circ\delta(h)) = 1$.The Picard - Lefshetz formula implies

$$\zeta_g(h) = (- \frac{\log(h)}{\pi i} \int_{\theta(h)} c + P(h))/(- \frac{\log(h)}{\pi i} \int_{\theta(h)} a + Q(h)) \tag{4.5}$$

where $P(h)$ and $Q(h)$ are holomorphic functions in a neighbourhood of $h = 0$.

$$\text{As} \quad \int_{\theta(0)} c = \mathrm{Res}\Big|_{x=0} \left(\frac{dx}{2.\sqrt{(1-(x-g)^2}} \right) = 0,$$

$$\int_{\theta(0)} a = \mathrm{Res}\Big|_{x=0} \left(\frac{dx}{2x.\sqrt{(1-(x-g)^2}} \right) \neq 0,$$

we conclude that $\lim_{h\uparrow 0} \zeta_g(h) = \lim_{h\downarrow 0} \zeta_g(h) = 0$. As $x^+ > 0$, then ii) is proved.

iii) $\delta(h)$ vanishes at $h = h^-(g)$ and hence

$$\lim_{h\downarrow h^-(g)} \frac{d}{dh} \zeta_g(h) = \lim_{h\downarrow h^-(g)} \frac{d}{dh} \left(\int_{\delta(h)} c \Big/ \int_{\delta(h)} a \right) =$$

$$\lim_{h\downarrow h^-(g)} \{ (\frac{d}{dh} \int_{\delta(h)} c) . \int_{\delta(h)} a - (\frac{d}{dh} \int_{\delta(h)} a) . \int_{\delta(h)} c \} / (\int_{\delta(h)} a)^2 .$$

All integrals above are holomorphic for h in a sufficiently small neighbourhood of $h^-(g)$ and their limits are equal to the corresponding residues at $x = x^+$. After computing these residues we obtain

$$\lim_{h\downarrow h^-(g)} \frac{d}{dh} \zeta_g(h) = \frac{3x^- - 5x^+}{6.x^+.(x^+ - x^-)} < 0 .$$

iv) Let $\theta(h)$ is defined as in ii). We note that in (4.5) $\int_{\theta(0)} c = 0$, $\int_{\theta(0)} a \neq 0$. Also as $\int_{\delta(h)} c = - \frac{\log(h)}{\pi i} \int_{\theta(h)} c + P(h)$, then taking the limit $h\uparrow 0$ we conclude that

$$P(0) = \lim_{h\uparrow 0} \left(\int_{\delta(h)} c + \frac{\log(h)}{\pi i} \int_{\theta(h)} c \right) = \lim_{h\uparrow 0} \int_{\delta(h)} c =$$

$$= \int_0^{1+g} \frac{dx}{\sqrt{1 - (x - g)^2}} = \arccos(-g) \neq 0.$$

Differentiating (4.5) and using the above relations we obtain $\lim_{h\uparrow 0} \frac{d}{dh} \zeta_g(h) = - \infty$.

v) If $g = 1$ then (4.4) implies $\lim_{h\uparrow 0} \frac{d}{dh} \zeta_g(h) = - \infty$. Suppose that $g > 1$ and let $\theta(h)$ be a vanishing cycle at $h = h^+(g)$. The Picard - Lefshetz formula implies that in a neighbourhood of $h = h^+(g)$ holds $\zeta_g(h) - x^- =$

$$\left(- \frac{\log(h - h^+(g))}{2\pi i} P_1(h) + P_2(h) \right) \Big/ \left(- \frac{\log(h - h^+(g))}{2\pi i} Q_1(h) + Q_2(h) \right)$$

where $P_1(h)$, $P_2(h)$, $Q_1(h)$, $Q_2(h)$ are holomorphic functions, and $P_1(h) = \int_{\theta(h)} (c - a.x^-)$, $Q_1(h) = \int_{\theta(h)} a$ .Differentiating the above identity we obtain

$$\frac{d}{dh} \zeta_g(h) = \frac{2\pi i}{\{h - h^+(g)\}.\{\log(h - h^+(g))^2\}} . \frac{P_1.Q_2 - P_2.Q_1}{Q_1^2} .(1+O(h)) .$$

As $P_1(0) = 0$ (see i)) then it is enough to prove that

$\lim\limits_{h\uparrow h^+(g)} \dfrac{2\pi i.P_2(h)}{Q_1(h)} < 0$. The formula

$$\int_{\delta(h)} c - a.x^- = - \frac{\log(h - h^+(g))}{2\pi i} P_1(h) + P_2(h)$$

implies $P_2(h^+(g)) = \lim\limits_{h\uparrow h^+(g)} \int_{\delta(h)} c - a.x^- = \lim\limits_{h\uparrow h^+(g)} \int_{\delta(h)} \dfrac{x - x^-}{2xy} dx =$

$= \displaystyle\int_{x^-}^{2g-2x^-} \frac{dx}{\sqrt{x.(-x - 2x^- + 2g)}} \neq 0$. Also $Q_1(h^+(g)) = \lim\limits_{h\uparrow h^+(g)} \int_{\theta(h)} a =$

$= 2\pi i.\mathrm{Res}\Big|_{x=x^-} \left(\dfrac{dx}{2.\sqrt{x.[h^+ + x.(1-(x-g)^2)]}} \right) \neq 0$.Thus we have proved that $\lim\limits_{h\uparrow h^+(g)} \frac{d}{dh} \zeta_g(h) = \pm\infty$. At last we note that for $h < h^+(g)$ $\zeta_g(h) - x^- = \displaystyle\int_{\delta(h)} \frac{x - x^-}{y} dx \Big/ \int_{\delta(h)} \frac{dx}{y} > 0$, and if h is in a sufficiently small neighbourhood of $h^+(g)$, then

$\zeta_g(h) - x^- = \dfrac{2\pi i.P_2(h)}{Q_1(h).\log(h-h^+(g))}.(1+O(h))$. It implies that

$\lim\limits_{h\uparrow h^+(g)} \dfrac{2\pi i.P_2(h)}{Q_1(h)} < 0$, and hence Proposition 4.1. is proved.■

<u>Proof of Proposition 4.2.</u>

The parts i) and ii) of Proposition 4.2. are proved directly, after applying the Picard-Lefschetz formula. For example if $g > 1$ then $\xi_g(h) =$

$$\frac{d}{dh}\left(- \frac{\log(h - h^+(g))}{2\pi i} \int_{\theta(h)} c + P(h)\right) \Big/ \frac{d}{dh}\left(- \frac{\log(h - h^+(g))}{2\pi i} \int_{\theta(h)} a + Q(h)\right)$$

which , after differentiating and taking the limits, implies $\lim\limits_{h\uparrow h^+(g)} \xi_g(h) = \lim\limits_{h\uparrow h^+(g)} \zeta_g(h) = x^-$. If $g = 1$ then $\lim\limits_{h\uparrow h^+(g)} \xi_g(h) = \lim\limits_{h\uparrow h^+(g)} \zeta_g(h) = 0$, as may be seen from the asymptotics of the Abelian integrals , derived in the proof of Proposition 4.1. i).

iii) Differentiating the above formula for $\xi_g(h)$ we compute that for h sufficiently close to $h^+(g)$ holds

$$\frac{d}{dh} \xi_g(h) = \log(h - h^+(g)).\frac{d}{dh}\left(\int_{\theta(h)} c \Big/ \int_{\theta(h)} a\right).(1+O(h)) \qquad (4.6)$$

As in the proof of Proposition 4.1. iii) we compute that

$$\lim_{h\uparrow h^+(g)} \frac{d}{dh} \left(\int_{\theta(h)} c \Big/ \int_{\theta(h)} a \right) = \frac{3x^+ - 5x^-}{6.x^-.(x^- - x^+)} < 0 ,$$

and hence $\lim_{h\uparrow h^+(g)} \frac{d}{dh}\xi_g(h) = \infty$.

iv) If $g = 1$, then the formulae derived in the proof of Propsition 4.1. imply that in a neighbourhood of $h = 0$ in $\mathbb{C}$ holds $\frac{d}{dh}\xi_g(h) = c.(-h)^{-1/2} + \ldots$, where $c < 0$, and hence $\lim_{h\uparrow 0} \frac{d}{dh} \xi_g(h) = \infty$. At last consider the case $-1 < g < 1$. The Picard - Lefschetz formula implies that in a neighbourhood of $h = 0$ holds

$$\frac{d}{dh} \xi_g(h) = \log(h).\frac{d}{dh} \left(\int_{\theta(h)} c \Big/ \int_{\theta(h)} a \right).(1+O(h)) \quad . \qquad (4.7)$$

As $\int_{\theta(0)} c = 0$ then we obtain

$$\frac{d}{dh} \left(\int_{\theta(h)} c \Big/ \int_{\theta(h)} a \right)\Big|_{h=0} =$$

$$= \left[\left(\frac{d}{dh} \int_{\theta(h)} c\right) \Big/ \left(\int_{\theta(h)} a \right) \right]\Big|_{h=0} =$$

$$\{\mathrm{Res}\big|_{x=0} [\frac{d}{dh} (\frac{x.dx}{\sqrt{x.[h + x.(1-(x-g)^2)]}})]\,|_{h=0})\}/\{\mathrm{Res}\big|_{x=0} (\frac{dx}{x.\sqrt{1-(x-g)^2}})\}$$

$= - \dfrac{1}{2.(1 - g^2)} < 0$, and hence $\lim_{h\uparrow 0} \frac{d}{dh} \xi_g(h) = \infty$.

v) As $\delta(h)$ vanishes at $h = h^-(g)$ then we have

$$\lim_{h\downarrow h^-(g)} \frac{d}{dh} \xi_g(h) = \lim_{h\downarrow h^-(g)} \frac{d}{dh} \left(\frac{d}{dh} \int_{\delta(h)} c \Big/ \frac{d}{dh} \int_{\delta(h)} a \right) =$$

$$\lim_{h\downarrow h^-(g)} \{ (\frac{d^2}{dh^2} \int_{\delta(h)} c).\frac{d}{dh} \int_{\delta(h)} a - (\frac{d^2}{dh^2} \int_{\delta(h)} a).\frac{d}{dh} \int_{\delta(h)} c\}/(\frac{d}{dh} \int_{\delta(h)} a)^2.$$

All integrals above are holomorphic functions for h in a sufficiently small neighbourhood of $h^-(g)$ and their limits are equal to the corresponding residues at $x = x^+$. After computing these residues we obtain

$$\lim_{h\downarrow h^-(g)} \frac{d}{dh} \xi_g(h) = - \frac{80.(g.(16g^2 + 27) + (20g^3 - 3).\sqrt{g^2 + 3})}{9.R(x^+)^2.x^+.\{R(x^+)^2 - 2.R(x^+).x^+ + 5(x^+)^2\}^2} < 0$$

for $g > 1$, and hence Proposition 4.2. is proved.∎

5. MONOTONICITY OF THE PERIOD

In this section we prove Theorem 1.

As $\frac{dx}{dt} = \partial H/\partial y$ then the period T(h) of the only periodic solution contained in the level set {H = h} is equal to $\oint dt = \oint \frac{dx}{\partial H/\partial y} = \int_{\delta(h)} \frac{dx}{2xy} = \int_{\delta(h)} a$. Introduce the following notation (see fig. 2)

$$\Delta_g^+ = \{h \in \mathbb{R} : (g,h) \in D^+\},\ \Delta_g^- = \{h \in \mathbb{R} : (g,h) \in D^-\} .$$

Δ_g^+ and Δ_g^- are open intervals and $\Delta = \Delta_g = \Delta_g^+ \cup \Delta_g^-$. As T(h) takes the same values at (g,h) and at (-g,-h), it is enough to prove the theorem for $h \in \Delta_g^+$. In section 3 we derived the Picard-Fuchs equation (3.7) satisfied by $A(h) = \int_{\delta(h)} a$ (= T(h))

$$p.A'' + q.A' + r.A = 0 \qquad ' = \frac{d}{dh}$$

where p, q, r are polynomials. As a is the holomorphic one form on the compact elliptic curve $\overline{\Gamma}_{g,h}$, and $A(h) \in \mathbb{R}$, then (without loss of generality) $A(h) > 0$ for $h \in \Delta_g^+$, and hence the function A'/A takes only finite values in Δ_g^+. We shall prove that the equation A'/A = 0 does not possess solutions in Δ_g^+. The function A'/A satisfies the following Riccati equation

$$p.x' + q.x + p.x^2 + r = 0 \quad .$$

Consider the autonomous system

$$\begin{cases} \overset{\circ}{x} = -q.x - p.x^2 - r \quad , & x = x(t), \quad {}^{\circ} = \frac{d}{dt} \ , \\ \overset{\circ}{h} = p \quad , & h = h(t) \quad . \end{cases} \tag{5.1}$$

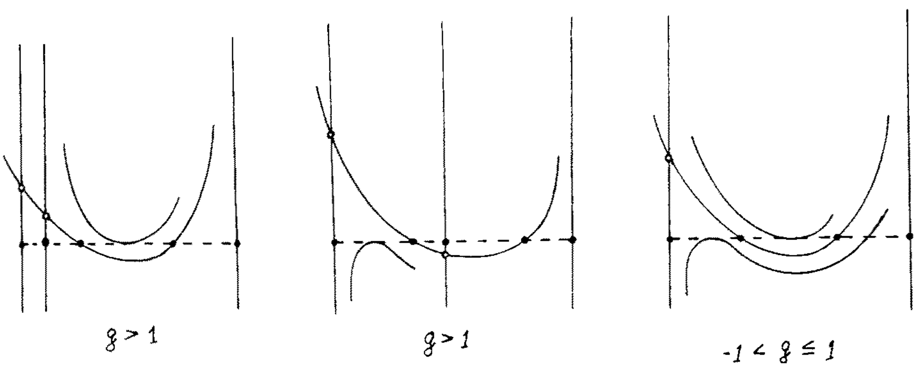


Fig.5. Phase portrait of system (5.1) .

Suppose that $-1 < g < 1$. Then $\Delta_g^+ = (h^-(g),0)$ and $\{h = h^-(g)\}$, $\{h = 0\}$, $\{x = A'(h)/A(h)\}$ are invariant sets of (5.1). Straightforward computation gives

$$\lim_{h\downarrow h^-(g)} A'(h)/A(h) = -\frac{3((R(x^+)^2 + 4(x^+)^2)}{16(x^+)^2.R^3(x^+)} \tag{5.2}$$

where $x.(h-x.(x-g)^2+x)\Big|_{h=h^-(g)} = x.(x - x^+)^2.R(x)$. One easely computes $R(x) = -x - 2x^+ + 2g$, $R(x^+) = -\sqrt{g^2+3} < 0$, and hence

$$\lim_{h\downarrow h^-(g)} A'(h)/A(h) > 0 .$$

For h in a sufficiently small neighbourhood of h = 0 in the complex domain, the Picard-Lefschetz formula implies $A(h) = \frac{\log(h)}{\pi i}.\int_{\theta(h)} a + P(h)$, where $\theta(h)$ is a vanishing cycle at h = 0 and P(h) is a holomorphic function (see section 4).This implies $A'(h)/A(h) = (1+O(h))/\{h.\log(h)\}$, and hence $\lim_{h\uparrow 0} A'(h)/A(h) = +\infty$ .Suppose now that the phase curve $x = A'(h)/A(h)$ intersects the line x = 0 . Then it intersects x = 0 at least twice (fig.5).Denote these points by P_2, P_3, and put $P_1 = (0,h^-(g))$. It is easely seen that on the line $h = h^-(g)$ there is only one equilibrium point, which is a saddle. As $p = p(h) \neq 0$ in Δ_g^+, then the direction of the vector field at the points P_1,P_2,P_3, implies that there exist at least two points on the interval $x = 0$, $h^-(g) < h < 0$, and the vector field (5.1) is tangent to the line x = 0 at these points. In other words the polynomial $r(h) = r_g(h) = 12.(3gh^2+h.(4g^2-5)+2g.(g^2-1))$ has two zeroes on the interval $(h^-(g),0)$. This is, however, impossible as it can be seen after some tedious but straightforward computations (see fig.6, where the level set $\{(g,h) \in \mathbb{R}^2 : r_g(h) = 0\}$ is pictured).

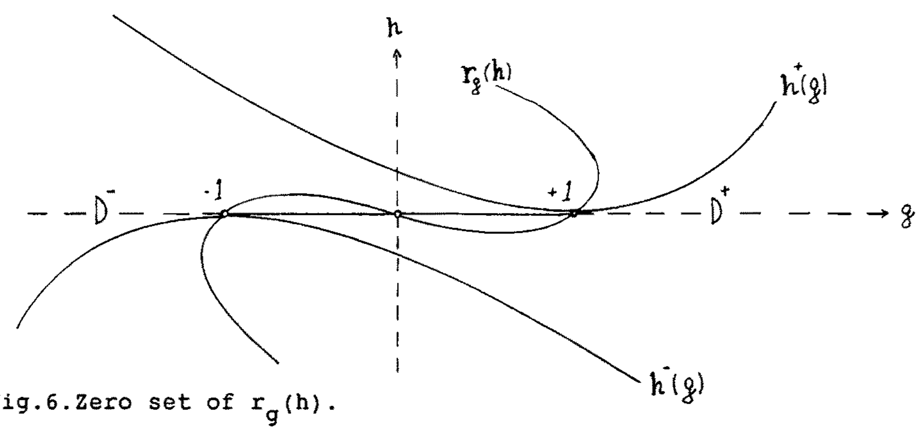


Fig.6.Zero set of $r_g(h)$.

The case $g > 1$ can be studied along the same lines : formula (5.2) holds for any $g > -1$, and $\lim_{h\uparrow h^{+}(g)} A'(h)/A(h) = +\infty$. The polynomial $p = p(h)$ vanishes exactly once in the interval Δ_g^+, and hence $r_g(h)$ has at least one zero in Δ_g^+ (fig.5). This is ,however , a contradiction (fig.6) .At last if $g = 1$ then the asymptotic expansion for $\int_{\delta(h)} a$ around $h = 0$, derived in the proof of Proposition 4.2. i), implies that $A'(h)/A(h) = -(1+O(h))/4h$, and hence $\lim_{h\uparrow 0} A'(h)/A(h) = +\infty$. This completes the proof of Theorem 1.■

Remark. In the above proof we used that there is only one equilibrium point of (5.1) on the line $h = h^-(g)$, and that it is a saddle. Of course it can be checked directly in (5.1). However, we do not need to make the precise computations. Generically this equilibrium is a saddle, or a node. It is a saddle iff $\delta(h)$ vanishes at $h = h^-(g)$, and a node if it is not so (for example the "infinite"point on the line $h = h^-(g)$ is a node). In the non-generic cases the equilibrium point is a standard saddle-node [12]. Indeed, all phase curves of the system (5.1) are given by

$$x_{\alpha,\beta}=(\alpha.\int_{\delta(h)} \mathring{a}+\beta.\int_{\theta(h)} \mathring{a})/(\alpha.\int_{\delta(h)} a+\beta.\int_{\theta(h)} a) \ , \ "\circ"=\frac{d}{dh} ,$$

where $\delta(h),\theta(h)$ form a basis of $H_1(\overline{\Gamma}_{g,h},\mathbb{Z})$, and $\overline{\Gamma}_{g,h}$ is the compactification of $\Gamma_{g,h}$. Let $\delta(h)$ vanishes at $h = h^-(g)$. Applying the Picard-Lefshetz formula we obtain

$$\lim_{h\downarrow h^-(g)} x_{\alpha,\beta} = \begin{cases} -\dfrac{3((R(x^+)^2 + 4(x^+)^2)}{16(x^+)^2.R^3(x^+)} & , \ \beta = 0 \\ -\infty & , \ \beta \neq 0 \end{cases}$$

which implies that the "finite" equilibrium point is a saddle, and the "infinite" one is a node.

6. ZEROES OF THE ABELIAN INTEGRAL

In this section we prove Theorem 2.

Definition (Petrov[16]). We say that the polynomial one-forms ω_1 and ω_2 are equivalent, provided that $\omega_1 - \omega_2 = P_1 dH + dP_2$ where P_1 and P_2 are polynomials.

Let Ω_2 be the factor space of all polynomial one forms with real coefficients $\omega = P_1dx + P_2dy$ where $\deg(P_1) \leq 2$ modulo the above equivalency. Then $\dim(\Omega_2) = 3$ and

$$\Omega_2 = \mathbb{R}^3\{ydx,\ xydx,\ y^2dx\} \quad .$$

It is enough to prove Theorem 2 for $I(h) = \int_{\delta(h)} \omega$, where $\omega \in \Omega_2$, and $h \in \Delta_g$.

Remark If $g = 0$, it is known ([7] Lemma 3.9.) that $I(h)$ has no more than one zero in Δ_g. One can also prove that for all $g \in (-1,1)$ $\frac{d}{dh}\,\zeta_g(h)$ has no zeroes in a neighbourhood of $h = h^-(g),\ h^+(g),\ 0$ $(h \in \Delta_g)$. That is why we conjecture that $I(h)$ has no more than one zero in Δ_g for all values of g.

As $I(h)$ vanishes at $h = h^-(g)$, $g > -1$, and at $h = h^+(g)$, $g < 1$, then the number of the zeroes of $I(h)$ does not exceed the number of the zeroes of $\frac{d}{dh}\, I(h)$ in Δ_g.For an arbitrary one form $\omega \in \Omega_2$ consider the Abelian integral

$$I(h) = \int_{\delta(h)} \omega = p.\Big(\int_{\delta(h)} xydx \Big) + q.\Big(\int_{\delta(h)} ydx \Big) + r.\Big(\int_{\delta(h)} y^2dx \Big)\ ,\ p,q,r \in \mathbb{R}$$

and

$$\frac{d}{dh}\, I(h) = p.\Big(\int_{\delta(h)} \frac{dx}{2y} \Big) + q.\Big(\int_{\delta(h)} \frac{dx}{2xy} \Big) + r.\Big(\int_{\delta(h)} \frac{dx}{x} \Big) =$$

$$= p.\Big(\int_{\delta(h)} \frac{dx}{2y} \Big) + q.\Big(\int_{\delta(h)} \frac{dx}{2xy} \Big) = p.\int_{\delta(h)} c + q.\int_{\delta(h)} a \quad .$$

As $\int_{\delta(h)} \frac{dx}{2xy} \neq 0$ in Δ_g we shall prove that the equation $\zeta_g(h) = \big(\int_{\delta(h)} c\big)/\big(\int_{\delta(h)} a\big) = \text{const.}$ has no more than three solutions in Δ_g, for $|g| < 1$, and no more than one solution in Δ_g, for $|g| \geq 1$.

Proposition 4.1. implies certain restrictions on the possible graphics of $\zeta_g(h)$ (fig.7). Namely, if there exists $c_1 = \text{const.}$ and $\zeta_g(h) = c_1$ has more than three (one) solutions in Δ_g, then there exists a constant c_2, and $\zeta_g(h) = c_2$ has at least five (three) solutions in Δ_g.

As $\int_{\delta(h)} a \neq 0$, $\frac{d}{dh}\big(\int_{\delta(h)} a\big) \neq 0$ for $h \in \Delta_g$ (see section 5), then $\frac{d}{dh}\,\zeta_g(h) = 0 \Leftrightarrow \xi_g(h) = \zeta_g(h)$. If the equation $\zeta_g(h) = c_2$, $|g| < 1$, has at least five solutions in Δ_g, then Proposition 4.2. implies that there exists a constant c_3 such that the equation $\xi_g(h) = c_3$ has at least three solutions in Δ_g^+ or in Δ_g^-. As $\xi_g(h) = -\,\xi_{-g}(-h)$, then we may suppose that the latter equation has *at least three solutions in* Δ_g^+.

Let us suppose that the equation $\zeta_g(h) = c_2$, $|g| \geq 1$, has at least three solutions in Δ_g. Now the same reasonings, together with Proposition 4.2. v) imply that there exists a constant c_4, and the equation $\xi_g(h) = c_4$ has *at least three solutions in* $\Delta_g = \Delta_g^+$.

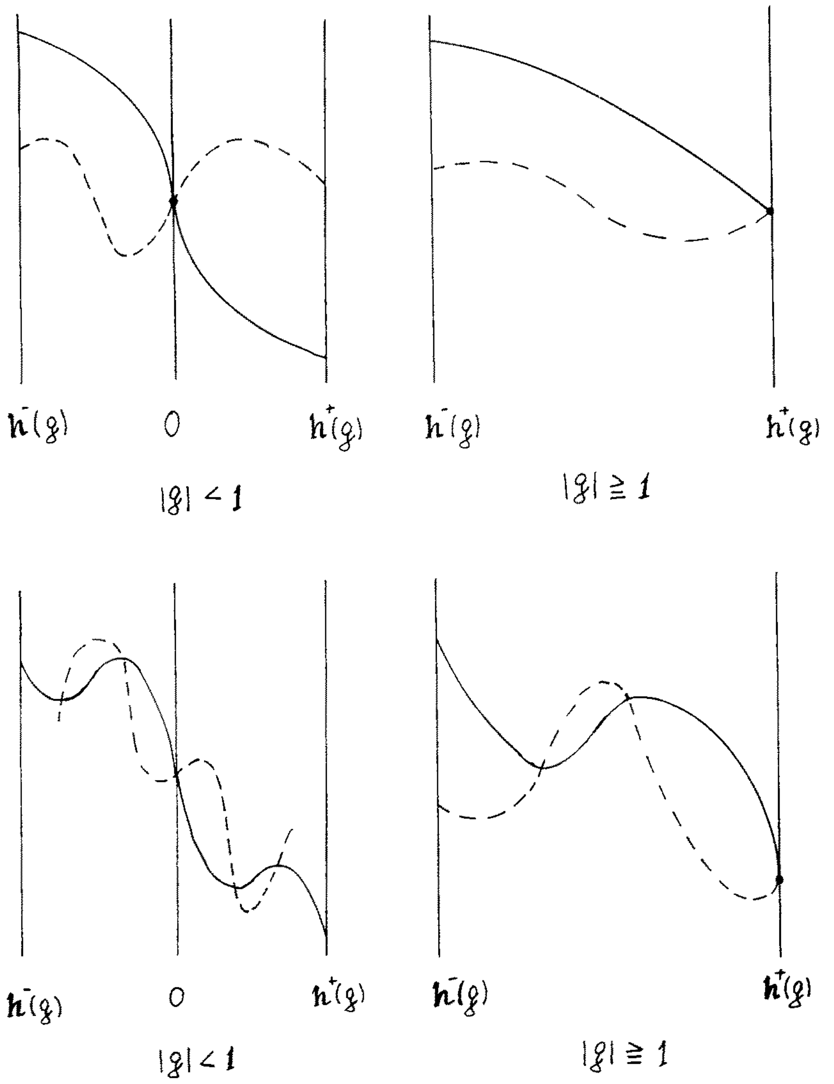


Fig.7.Possible graphics of $\zeta_g(h)$ (bold line) and $\xi_g(h)$ (dottedline) in the interval Δ_g .

The next lemma shows that it is impossible

Lemma 6.1. The equation $\xi_g(h)$ = const. has no more than two solutions in Δ_g^+.

We arrived at the desirable contradiction. To the end of this section we shall prove Lemma 6.1.

The one-forms $\frac{d}{dh}$ c, $\frac{d}{dh}$ a, satisfy a second order Picard-Fuchs system (3.6), derived in section 3. The Riccati equation, satisfied by $\xi_g(h)$ has the form

$$p.x' + q.x + r.x^2 + s = 0$$

where p is a cubic, and q, r, s, are quadratic polynomials in h. Now Rolle's theorem for dynamical systems [14,15] implies that, if x(h) is a solution of the Riccati equation defined for all $h \in \Delta_g^+$, then for any $x_o \in \mathbb{R}$, the function $x(h) - x_o$ has no more than three zeroes in Δ_g^+. We need, however, a stronger statement.To prove Lemma 6.1. we shall study (as in the proof of Theorem 1) the global phase portrait of the system

$$\begin{cases} \mathring{x} = -q.x - r.x^2 - s \quad , \quad x = x(t), \quad \mathring{} = \frac{d}{dt} \quad , \\ \mathring{h} = p \quad , \quad h = h(t) \quad . \end{cases} \qquad (6.1)$$

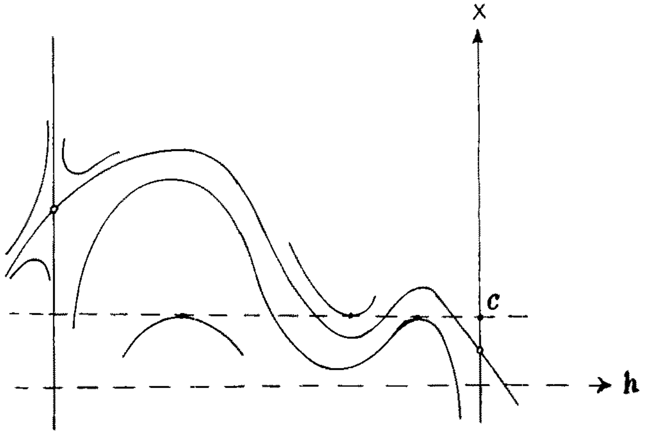


Fig.8. Phase portrait of system (6.1).

All phase curves of the system (6.1) are given by

$$x_{\alpha,\beta}(h)=(\alpha.\int_{\delta(h)}\mathring{c}+\beta.\int_{\theta(h)}\mathring{c})/(\alpha.\int_{\delta(h)}\mathring{a}+\beta.\int_{\theta(h)}\mathring{a}),\ "\circ"=\frac{d}{dh},\ \alpha,\ \beta\in\mathbb{R},$$

where $\delta(h),\theta(h)$ form a basis of $H_1(\overline{\Gamma}_{g,h},\mathbb{Z})$, and $\overline{\Gamma}_{g,h}$ is the compactification of $\Gamma_{g,h}$. Let $\delta(h)$ vanishes at $h = h^-(g)$. Applying the Picard-Lefshetz formula we obtain

$$\lim_{h\downarrow h^-(g)} x_{\alpha,\beta} = \begin{cases} 4.x^+.R(x^+).(3.x^+-R(x^+))/\{3.((R(x^+)-x^+)^2+4(x^+)^2)\} + \\ + x^+ \quad , \quad \beta = 0 \\ x^+ \quad , \quad \beta \neq 0 \end{cases}$$

where $R(x)$ is defined in section 5. As

$$R(x^+).(3.x^+-R(x^+)) = - \ 2.(g \ + \sqrt{g^2+3}\ ).\sqrt{g^2+3} < \ 0,$$

then there are exactly two (different) equilibrium points on the line $h = h^-(g)$, which are a node (with coordinates $(h^-(g),x^+)$) and a saddle (see fig.8). Notice that $\xi_g(h) \equiv x_{1,0}(h)$. Also, as we have proved in Proposition 4.2.,

$$\lim_{h\uparrow h^+(g)} \xi_g(h) = x^- \text{ for } g > 1, \text{ and } \lim_{h\uparrow 0} \xi_g(h) = 0 \text{ for } |g| < 1.$$

As $\lim_{h\downarrow h^-(g)} \xi_g(h) = x^+ > \max(x^-,0)$, then there always exists a constant c, and the vector field (6.1) is tangent to the line $x = c$ at three points (at least) - see fig.7. In other words the quadratic polynomial $P(h) = q.c + r.c^2 + s$ has three roots in Δ_g^+ which is a contradiction. Thus Lemma 6.1., and hence Theorem 2, is proved.■

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REFERENCES

1. V.I. Arnol'd, A.N. Varchenko, S.M. Gusein-Zade, Singularities of differentiable maps, Vol.2, Basel, Birkhauser Verlag, 1988 .

2. J. Carr, S.-N. Chow, J.K. Hale, Abelian integrals and bifurcation theory, Journal Diff. Eqns. 59 (1985), 413-437.
3. J. Carr, S.A. van Gils, J.A. Sanders, Nonresonant bifurcations with symmetry , SIAM J. Math. Anal., 18 (1987), No 3, 579-591.
4.C.H. Jr. Clemens, Picard-Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities, Trans. Am. Math. Soc. 136 (1969) 93-108.
5. R.Cushman, J.A.Sanders, A codimension two bifurcation with a third order Picard-Fuchs equation, Journal Diff. Eqns, 59 (1985),243-256.
6. S.N. Chow, J.A. Sanders, On the number of the critical points of the period, Journal. Diff. Eqns, 64 (1986) 51-66.
7. B. Drachman, S.A. van Gils, Zhang Zhi-fen, Abelian integrals for quadratic vector fields, J. reine angew. Math.382 (1987), 165-180.
8. H. Dulac, Sur les cycles limites, Bull. Soc. Math. France, 51 (1923), 45-188.
9. J. Ecalle, J. Martinet, R. Moussu, J.P. Ramis, Compt. Rendu de l'Acad. sci.(Paris), 304, Série 1, (1987), 375-378, 431 - 434.
10. S.A. van Gils, E. Horozov, Uniqueness of limit cycles in planar vector fields which leave the axes invariant, Contemporary Mathematics 56 (1986), 117-129.
11 A. Grothendieck, On the de Rham cohomology of algebraic varieties, Publ. Math. Inst. HES, 29 (1966), 351-359.
12. Yu. Il'yashenko, Dulac's memoir "Sur les cycles limites" and related problems of the local theory of differential equations, Russian Math. Surveys, 40 (1985), 1-49.
13. Yu. Il'yashenko, Zeroes of special Abelian integrals in a real domain, Funct. Anal. Appl. 11 (1977), 309-311.
14. A.G. Khovansky, Cycles of dynamical systems on a plane and Rolle's theorem , Siberia Math. Journal, 25 (1984), n°3, 198-203.
15. A.G. Khovansky, Real analytic manifolds with finiteness properties and complex abelian integrals, Funct. Anal. Appl. 18 (1984), 119-128.
16. G.S. Petrov, Number of zeroes of complete elliptic integrals Funct. Anal. Appl.18 (1984), 148-150.
17. G.S. Petrov, Elliptic integrals and their nonoscillations, Funct. Anal. Appl.20 (1986), 37-40.
18. C. Rousseau, Bifurcation methods in quadratic systems, Canadian Math. Soc., Conference Proc., 8 (1987), 637-653.
19. D. Schlomiuk, J. Guckenheimer, R. Rand, Integrability of plane quadratic vector fields, to appear.
20. A.N. Varchenko, Estimate of the number of zeroes of abelian

integrals depending on parameters and limit cycles, Funct. Anal. Appl., 18 (1984), 98-108.

21. Yu. Il'yashenko, Finiteness theorems for limit cycles, Russian Math. Surveys, 42 (1987), No. 3, p. 223.

22. W. A. Coppel, A survey of quadratic systems, J. Differential Equations, 2 (1966), 293-304.

23. W. A. Coppel, Some quadratic systems with at most one limit cycle, Dynamics Reported, Vol. 2 (1989), 61-88. Ed.U.Kirchgraber & H.O.Walther, John Wiley & Sons Ltd. and B.G. Teubner.

24. Yu. Il'yashenko, Finiteness theorems for limit cycles, Russian Math. Surveys, 45 (1990), No. 2, p. 143-200.