Spectral decompositions and $\mathbb{L}^2$-operator norms of toy hypocoercive semi-groups

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Abstract

For any $a > 0$, consider the hypocoercive generators $y\partial_x + a\partial_y^2 - y\partial_y$ and $y\partial_x - ax\partial_y + \partial_y^2 - y\partial_y$, respectively for $(x, y) \in \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}$ and $(x, y) \in \mathbb{R} \times \mathbb{R}$. The goal of the paper is to obtain exactly the $\mathbb{L}^2(\mu_a)$-operator norms of the corresponding Markov semi-group at any time, where μ_a is the associated invariant measure. The computations are based on the spectral decomposition of the generator and especially on the scalar products of the eigenvectors. The motivation comes from an attempt to find an alternative approach to classical ones developed to obtain hypocoercive bounds for kinetic models.

Keywords: convergence to equilibrium, hypocoercive Markovian semi-groups, spectral decompositions, Ornstein-Ulhenbeck generator, $\mathbb{L}^2$-operator norms, kinetic evolution equations.

MSC2010: first: 35K65, secondary: 35H10, 47A10, 15A18, 15A60, 37A30, 60J60

1 Introduction

Convergence to equilibrium of Markovian semi-groups has been investigated a lot under various coercive assumptions on the generator, such as spectral gap or logarithmic Sobolev inequalities, especially in a reversible framework. Nevertheless, asymptotical exponential convergence to equilibrium is also encountered when the generator satisfies some hypoelliptic type conditions. This phenomenon has been called hypocoercivity (see the book of Villani [21] for the history) and has recently attracted more and more attention, with e.g. the works of Desvillettes and Villani [2], Eckmann and Hairer [5], Rey-Bellet and Thomas [18], Hérau and Nier [11], Hérau [10], Villani [21], Dolbeault, Mouhot and Schmeiser [4], Ottobre, Pavliotis and Pravda-Starov [17] and the references therein. Typically, a hypocoercivity result bounds the convergence to equilibrium (for instance in $\mathbb{L}^2$ or entropy sense) by a term such as $C\exp(-ct)$, where $t \geqslant 0$ is the time and $C, c > 0$ are two constants depending on the problem at hand. But these constants are not easy to describe and generally not optimal in the literature mentioned above. Furthermore, the previous bound gives no information about the behavior of the underlying semi-group at small times, namely how it begins to go toward equilibrium. To try to clarify the situation, we will study in details in this paper two simple models of hypocoercivity, by computing exactly the corresponding distance to equilibrium in the $\mathbb{L}^2$-sense. Despite the scope may seem limited, some features will be intriguing, such as the appearance of discrete binomial and Poisson laws.

The state space of our first toy model is $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ stands for the usual circle. The coordinates of a generical element of $\mathbb{T} \times \mathbb{R}$ are denoted (x, y), where x and y are often interpreted respectively as a position and a speed (i.e. $\mathbb{T} \times \mathbb{R}$ is seen as the tangent bundle of $\mathbb{T}$). For any given $a > 0$, we are interested in the differential operator

$$L_a \quad := \quad y\partial_x + a\partial_y^2 - y\partial_y.$$

and in the generated Markovian semi-group $(P_t^{(a)})_{t \geqslant 0}$. Consider the product probability measure $\mu_a := \lambda \otimes \gamma_a$, where λ is the normalized Lebesgue measure on $\mathbb{T}$ and where γ_a is the normal distribution of mean 0 and of variance a. It is easy to check that μ_a is invariant for L_a: for any smooth function f on $\mathbb{T} \times \mathbb{R}$ with bounded derivatives, $\mu_a[L_a[f]] = 0$. It follows that for any $t \geqslant 0$, $P_t^{(a)}$ can be extended into a continuous operator on $\mathbb{L}^2(\mu_a)$ with operator norm equal to 1. It is furthermore known that μ_a is ergodic for the semi-group, in the sense that $P_t^{(a)}$ converges toward μ_a in $\mathbb{L}^2(\mu_a)$ for t large:

$$\forall\ f \in \mathbb{L}^2(\mu_a), \qquad \lim_{t \to +\infty} \left\| P_t^{(a)}[f] - \mu_a[f] \right\| \quad = \quad 0,$$

where $\|\cdot\|$ designates the $\mathbb{L}^2(\mu_a)$-norm.
Our goal is to recover and to quantify this convergence given by the next result.

Theorem 1 *For any $a > 0$ and $t \geqslant 0$, we have*

$$\| P_t^{(a)} - \mu_a \| \quad = \quad \max\left(\exp(-t), \exp\left[-a\left(t - 2\frac{1-\exp(-t)}{1+\exp(-t)}\right)\right]\right),$$

where $\|\!|\cdot|\!\|$ stands for the operator norm in $\mathbb{L}^2(\mu_a)$.

It is interesting to look at the behaviours of this operator norm for small and large times. As t goes to 0_+,

$$\ln\left(\| P_t^{(a)} - \mu_a \|\right) \quad = \quad -\frac{a}{12}t^3(1 + o(1)). \tag{1}$$

This shows that initially, the operator norm decreases quite slowly as a function of time, the power 3 should be seen as an order of the hypocoercivity of the operator L_a. On the other side, as t goes to $+\infty$,

$$-\ln\left(\left\|\!\left| P_t^{(a)} - \mu_a \right|\!\right\|\right) \quad = \quad \begin{cases} a(t-2+\mathcal{O}(e^{-t})) & \text{, if } a \leqslant 1 \\ t & \text{, if } a > 1, \end{cases}$$

which reflects the exponential convergence to equilibrium of the semi-group $(P_t^{(a)})_{t\geqslant 0}$.
This kind of informations cannot be deduced from the bounds obtained in the literature. Indeed, note that the mapping $\varphi : \mathbb{R}_+ \ni t \mapsto t - 2(1+\exp(-t))^{-1}(1-\exp(-t))$ is strictly convex, so that the bound of Theorem 1 is equivalent to the family of inequalities, parametrized by $s \geqslant 0$,

$$\forall\ t \geqslant 0, \qquad \left\|\!\left| P_t^{(a)} - \mu_a \right|\!\right\| \quad \leqslant \quad \max(\exp(-t), C_s\exp(-c_s t)),$$

where for all $s > 0$, $C_s \coloneqq \exp(-a(\varphi(s) - s\varphi'(s))) > 1$ and $c_s \coloneqq a\varphi'(s) > 0$.

Up to scalings in time and in the speed variable and to a change of direction in position, we deduce immediately from Theorem 1:

Corollary 2 *For any $a, c > 0$ and $b \in \mathbb{R}\backslash\{0\}$, consider the operator*

$$L_{a,b,c} \quad \coloneqq \quad by\partial_x + a\partial_y^2 - cy\partial_y,$$

which admits $\mu_{a/c}$ as invariant probability. We have for the corresponding semi-goup $(P_t^{(a,b,c)})_{t\geqslant 0}$,

$$\forall\ t \geqslant 0, \qquad \left\|\!\left| P_t^{(a,b,c)} - \mu_{a/c} \right|\!\right\|_{\mathbb{L}^2(\mu_{a/c})\circlearrowleft} \quad = \quad \max\left(\exp(-ct), \exp\left[-\frac{ab^2}{c^3}\left(ct - 2\frac{1-\exp(-ct)}{1+\exp(-ct)}\right)\right]\right).$$

In particular the associated asymptotical exponential rate is

$$\lim_{t\to+\infty} -\frac{1}{t}\ln\left(\left\|\!\left| P_t^{(a,b,c)} - \mu_{a/c} \right|\!\right\|_{\mathbb{L}^2(\mu_{a/c})\circlearrowleft}\right) \quad = \quad \min\left(c, \frac{ab^2}{c^2}\right).$$

It is instructive to draw a comparison with the heat semi-group $(Q_t^{(a)})_{t\geqslant 0}$ on $\mathbb{T}$ generated by the operator $K_a \coloneqq a\partial_x^2$, which injects the same amount a of randomness per unit of time as any one of the generators $L_{a,b,c}$, where $b \in \mathbb{R}$ and $c > 0$ are free parameters. Since K_a is self-adjoint in $\mathbb{L}^2(\lambda)$ and admits a as spectral gap, we get

$$\forall\ t \geqslant 0, \qquad \left\|\!\left| Q_t^{(a)} - \lambda \right|\!\right\|_{\mathbb{L}^2(\lambda)\circlearrowleft} \quad = \quad \exp(-at).$$

Thus it appears that if we had to choose between the Monte Carlo procedures $(Q_t^{(a)})_{t\geqslant 0}$ and $(P_t^{(a,b,c)})_{t\geqslant 0}$ to sample according to λ, it would be better to use, with a tuning $c > a$ and $b/c > 1$, the first coordinate for the latter Markov process, namely the primite integral of an Ornstein-Ulhenbeck process. Of course both procedures require the sampling of the trajectory of a Brownian motion, which is more difficult to get than the sampling of a uniform variable on the circle, nevertheless this is another illustration of the paradigm that to go fast to equilibrium, it is better to resort to non-reversible Markov processes (see for instance [3], where this question was studied in the framework of second order finite Markov chains).

Our second toy model has $\mathbb{R}\times\mathbb{R}$ as state space and also depends on a parameter $a > 0$: we are now interested in the differential operator

$$\widetilde{L}_a \quad \coloneqq \quad y\partial_x - ax\partial_y + \partial_y^2 - y\partial_y. \tag{2}$$

It is easy to check that the probability measure $\widetilde{\mu}_a \coloneqq \gamma_{1/a}\otimes\gamma_1$ is invariant for $\widetilde{L}_a$ and we consider the associated semi-group $(\widetilde{P}_t^{(a)})_{t\geqslant 0}$ of Markov operators on $\mathbb{L}^2(\widetilde{\mu}_a)$. As it will be seen in the next section, in the first model, for all $a > 0$, the operator L_a is diagonalizable in $\mathbb{L}^2(\mu_a)$ and its spectrum is real. For the second model, the value $1/4$ is critical with this respect: for $a \in (0, 1/4)$, $\widetilde{L}_a$ is diagonalizable in $\mathbb{L}^2(\widetilde{\mu}_a)$ and its spectrum is real, while for $a \in (1/4, +\infty)$, $\widetilde{L}_a$ is still diagonalizable in $\mathbb{L}^2(\widetilde{\mu}_a)$ (complexified) but most of its eigenvalues are not real. In the critical case $a = 1/4$, $\widetilde{L}_a$ is not diagonalizable in $\mathbb{L}^2(\widetilde{\mu}_a)$ and contains Jordan blocks of all orders. Nevertheless $\widetilde{\mu}_a$ is always ergodic and the next result quantifies the convergence:

Theorem 3 *For any $a > 0$ and $t \geqslant 0$, we have*

$$\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\| \quad = \quad C_a(t) \exp\left(-\frac{1 - \sqrt{(1-4a)_+}}{2} t\right),$$

where $\|\cdot\|$ stands for the operator norm in $\mathbb{L}^2(\widetilde{\mu}_a)$ and where the factor $C_a(t)$ is described as follows: if $a \in (0, 1/4)$, let $\theta := \sqrt{1-4a}$ and define

$$C_a(t) \quad := \quad \sqrt{e^{-\theta t} + \frac{1-\theta^2}{2\theta^2}\left(1 - e^{-\theta t}\right)^2 + \frac{1 - e^{-2\theta t}}{2}\left(1 + \frac{1}{\theta}\sqrt{1 + (\theta^{-2} - 1)\left(\frac{e^{\theta t} - 1}{e^{\theta t} + 1}\right)^2}\right)}.$$

If $a \in (1/4, +\infty)$, let $\theta := \sqrt{4a - 1}i$ and define

$$C_a(t) \quad := \quad \sqrt{1 + \frac{|e^{\theta t} - 1|}{2|\theta|^2}\left(|e^{\theta t} - 1| + \sqrt{|e^{\theta t} - 1|^2 + 4|\theta|^2}\right)}.$$

If $a = 1/4$, define

$$C_a(t) \quad := \quad \sqrt{1 + \frac{t^2}{2} + t\sqrt{1 + \left(\frac{t}{2}\right)^2}}.$$

Again, let us look more precisely at the behaviors of this operator norm for small and large times.
When $t > 0$ goes to zero, we obtain as above a decrease of order t^3: for $a \in (0, 1/4]$, we have

$$\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\| \quad = \quad 1 - \left(\frac{a}{6} + \frac{1-4a}{2}\left(1 - \sqrt{1-4a}\right)\right) t^3 + o(t^3), \tag{3}$$

and for $a \in [1/4, +\infty)$,

$$\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\| \quad = \quad 1 - \frac{a}{6} t^3 + o(t^3). \tag{4}$$

When t goes to infinity, the behavior is different according to the position of a with respect to $1/4$ (with an asymptotic exponential rate varying for $a \in (0, 1/4]$): if $a \in (0, 1/4)$, we have

$$\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\| \quad \sim \quad \frac{1}{\theta} \exp\left(-\frac{1 - \sqrt{(1-4a)_+}}{2} t\right).$$

The factor in front of the exponential explodes with time if $a = 1/4$:

$$\|\widetilde{P}_t^{(1/4)} - \widetilde{\mu}_{1/4}\| \quad \sim \quad t \exp\left(-\frac{t}{2}\right).$$

If $a > 1/4$, since the mapping

$$\mathbb{R}_+ \ni \nu \quad \mapsto \quad 1 + \frac{\nu}{2(4a-1)}\left(\nu + \sqrt{\nu^2 + 4(4a-1)}\right),$$

is increasing, it appears that the factor $\mathbb{R}_+ \ni t \mapsto C_a(t)$ is oscillating between the values 1 and $\sqrt{1+2(1+2\sqrt{a})(4a-1)^{-1}}$ with period $T_a := 2\pi/\sqrt{4a-1}$. These oscillations are sufficiently moderate so that $\mathbb{R}_+ \ni t \mapsto C_a(t)\exp(-t/2)$ is non-increasing, as it is always the case for the $\mathbb{L}^2(\mu)$-operator norms of a Markovian semi-group admitting μ as invariant probability. The above periodicity admits a peculiar consequence: it follows from (4) that $\frac{d}{dt}C_a(t)\exp(-t/2)|_{t=0} = 0$ and in conjunction with

$$\forall\ k \in \mathbb{Z}_+,\ \forall\ t \geqslant 0, \qquad C_a(kT_a+t)\exp(-(kT_a+t)) \quad = \quad \exp(-kT_a)C_a(t)\exp(-t),$$

we get that the time derivative of $\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\|$ vanishes at all times of the form $t = kT_a$, with $k \in \mathbb{Z}_+$.

On the state space $\mathbb{R}\times\mathbb{R}$, we can play with scalings in time, speed and position to deduce from Theorem 3:

Corollary 4 *For any $c, d > 0$ and $a, b \in \mathbb{R}$ with $ab > 0$, consider the operator*

$$\widetilde{L}_{a,b,c,d} \quad := \quad by\partial_x - ax\partial_y + c\partial_y^2 - dy\partial_y$$

which admits $\widetilde{\mu}_{a,b,c,d} := \gamma_{bc/(ad)} \otimes \gamma_{c/d}$ as invariant probability. We have for the corresponding semi-goup $(\widetilde{P}_t^{(a,b,c,d)})_{t\geqslant 0}$, with the notation of Theorem 3,

$$\forall\ t \geqslant 0, \qquad \|\widetilde{P}_t^{(a,b,c,d)} - \widetilde{\mu}_{a,b,c,d}\|_{\mathbb{L}^2(\widetilde{\mu}_{a,b,c,d})\circlearrowleft} \quad = \quad C_{ab/d^2}(dt)\exp\left(-\frac{1-\sqrt{(1-4abd^{-2})_+}}{2}dt\right).$$

It follows that the asymptotic exponential rate of $(\widetilde{P}_t^{(a,b,c,d)})_{t\geqslant 0}$ is $(1-\sqrt{(1-4abd^{-2})_+})/2$. We are led to make a comparison with the semi-group $(\widetilde{Q}_t^{(a,b,c,d)})_{t\geqslant 0}$ on $\mathbb{R}$ generated by $\widetilde{K}_{a,b,c,d} := c\partial_x^2 - \frac{da}{b}x\partial_x$, whose amount of injected randomness is the same as $\widetilde{L}_{a,b,c,d}$ and whose reversible probability is $\gamma_{bc/(ad)}$, the first marginal law of $\widetilde{\mu}_{a,b,c,d}$. Up to scalings of space and time, $\widetilde{K}_{a,b,c,d}$ is an Ornstein-Ulhenbeck generator whose spectral gap is da/b. It follows that the asymptotical exponential rate of $(\widetilde{Q}_t^{(a,b,c,d)})_{t\geqslant 0}$ is da/b. So if

$$\frac{a}{b} \quad < \quad \frac{1}{2}\left(1-\sqrt{\left(1-4\frac{ab}{d^2}\right)_+}\right),$$

(for instance if $4\frac{a}{d}\frac{b}{d} > 1$ and $2\frac{a}{d} < \frac{b}{d}$), it is more efficient to use the first coordinate of $(\widetilde{P}_t^{(a,b,c,d)})_{t\geqslant 0}$ than $(\widetilde{Q}_t^{(a,b,c,d)})_{t\geqslant 0}$ to sample accordingly to $\gamma_{bc/(ad)}$. Hence the remarks for the first model are still valid.

Instead of scaling position and speed variables as in Corollary 4, we could have considered appropriate linear transformations of $\mathbb{R}^2$ and end up with operators associated to certain quadratic symbols. Hypocoercivity of general differential operators with quadratic symbols have been recently investigated by Ottobre, Pavliotis and Pravda-Starov [17], who obtained bounds on $\mathbb{L}^2$-convergence which are relatively precise at the level of the exponential rate (showing that all rates strictly below those obtained above are admissible). But they provide no clue about the behavior of the operator norm for small times, while it would be very interesting to relate the order of hypercoercivity (the power 3 in (1), (3) or (4)) to the number of times one needs to take Lie brackets in order to get the full tangent space in Hörmander's condition [12], even only in the framework of quadratic symbols. One first step in this direction would be to investigate finite chains of nearest-neighbor interacting harmonic oscillators coupled to one heat bath (see e.g. Eckmann and Hairer [6] or Ottobre, Pavliotis and Pravda-Starov [17], despite that these authors were not primarily interested in this situation).

Our approach is completely different from the pseudo-differential techniques of Ottobre, Pavliotis and Pravda-Starov [17]. We begin by studying in details the spectral decomposition of the operators at hand. For the second model, it was already done by Risken [19] (see also the book of Helffer and Nier [9] or an unpublished paper of Kavian [15]). But we don't stop with the knowledge of the eigenvalues and of the eigenvectors, instead we investigate the scalar product of the eigenvectors: due to the fact that the above generators are not reversible, the eigenvectors cannot be all orthogonals. It appears that their geometric structure can be nicely described by $\mathbb{L}^2$ scalar products with respect to classical discrete laws such as Poisson or binomial distributions. This leads to the construction of certain functions which well-behave under the action of the semi-groups and turn out to be the optimal functions for the computation of the operator norms. It should be noted that these optimal functions change with the time at which are computed the operator norm, explaining why the latter cannot have a simple exponential form.

Of course, one can hope for precise spectral decompositions only in a restricted framework of quadratic symbols (but see also Eckmann and Hairer [6], where the spectrum of certain hypoelliptic generators is proven to be contained in a cusp). Nevertheless, our analysis put forward a simple Lie algebra structure associated to the above models which is "almost" shared by kinetic models corresponding to operators of the form $y\partial_x - U'(x)\partial_y + \partial_y^2 - y\partial_y$, say on $\mathbb{T}\times\mathbb{R}$, where the potential $U : \mathbb{T} \to \mathbb{R}$ is a smooth function. We believed the revealed structure could lead to a third order linear ordinary differential equation satisfied by the evolution of the $\mathbb{L}^2$-norm of the semi-group (applied to a generical function of mean zero with respect to the invariant measure), which is sufficiently coercive to imply hypocoercive bounds. Unfortunately this is not true and an idea is still missing with this respect. It was our initial motivation: to find at each time instantaneous informations on the evolution of $\mathbb{L}^2$-norm of the semi-group which locally describe the trend to equilibrium and globally imply hypocoercive bounds. This approach would be very convenient to deal with the time-inhomogeneous evolutions we have in mind (sampling and optimizing hypocoercive random algorithms) and it explains our interest in the small time behavior. The point of view is different from the traditional analytical approach to hypocoercivity, consisting in replacing the natural $\mathbb{L}^2$-norm by a more coercive norm, typically a norm which is comparable to an appropriately weighted $\mathbb{H}^1$-norm. The additional terms are chosen so that when differentiating with respect to time the evolution semi-group, one gets a first order differential inequality for this new norm (see for instance Villani [21]). The kind of estimates we are looking for are not more provided by the probabilistic approach to hypocoercivity through Liapounov functions (see for instance Bakry, Cattiaux and Guillin [1] and the references therein).

The paper is constructed on the following plan: in the next section (respectively Section 4) we investigate the spectral decomposition of the first model (resp. second model), which is used in Section 3 (resp. Section 5) to compute the corresponding operator norms. The last section is devoted to some observations about simple kinetic models and to the motivations sketched in the preceding paragraph.

2 Spectral decomposition of the first model

We compute here the spectral decomposition of a kinetic generator associated to the null potential on $\mathbb{T}$. Despite it is among the simplest case of hypocoercivity, we did not find its detailed treatment in the literature. The manipulations we are to consider will be encountered again in Section 4, under a slightly modified form. Furthermore, a very helpful Poisson distribution will make a mysterious appearance in this continuous setting!

So, for $a > 0$, which is fixed for the whole section, we are interested in the operator

$$L_a \quad \coloneqq \quad y\partial_x + a\partial_y^2 - y\partial_y.$$

2.1 Decomposition of the generator on stable subspaces

A priori it can be seen as an endomorphism on smooth functions defined on $\mathbb{T} \times \mathbb{R}$, but for our purposes, it is better to consider its closure in $\mathbb{L}^2(\mu_a)$, where the invariant measure $\mu_a = \lambda \otimes \gamma_a$ was presented in the introduction. Here we will mainly consider real Hilbert spaces, since a posteriori all the eigenvalues of L_a will be real.
If we were in a totally Gaussian setting, namely if $\mathbb{T}$ was replaced by $\mathbb{R}$ and $y\partial_x$ by $y\partial_x - bx\partial_y$, for some constant $b > 0$, it would be natural to observe the action of the above operator on tensor products of appropriately normalized Hermite polynomials, as it was done by Risken [19] (see also Section 4 below). In the present situation, it is rather tempting to replace the Hermite polynomials in the first variable (position x) by the usual trigonometric functions. For $p \in \mathbb{Z}_+$, denote

$$\forall\ x \in \mathbb{T}, \qquad \begin{cases} \varphi_p(x) & := & \frac{2^p p!}{\sqrt{(2p)!}} \cos(px), \\ \psi_p(x) & := & \frac{2^p p!}{\sqrt{(2p)!}} \sin(px). \end{cases}$$

The factors are such that $(\varphi_p, \psi_{p+1})_{p \in \mathbb{Z}_+}$ is an orthonormal basis of $\mathbb{L}^2(\lambda)$ and they are obtained via Wallis' integrals. For the second variable y, it is natural to use the Hermite polynomials since they can be conveniently associated to the standard Gaussian distribution γ_1. Recall that they are defined by

$$\forall\ q \in \mathbb{Z}_+,\ \forall\ y \in \mathbb{R}, \qquad h_q(y) \ := \ \frac{(-1)^q}{\sqrt{q!}} \exp(y^2/2) \frac{d^q}{dy^q} \exp(-y^2/2), \tag{5}$$

(see for instance the book of Szegő [20], as well as for their basic properties used below). To get the orthonormal polynomials $(h_{q,a})_{q \in \mathbb{Z}_+}$ associated to γ_a, for any fixed $a > 0$, we use the similitude of scale $1/\sqrt{a}$:

$$\forall\ q \in \mathbb{Z}_+,\ \forall\ y \in \mathbb{R}, \qquad h_{q,a}(y) \ := \ h_q(y/\sqrt{a}).$$

The family $(h_{q,a})_{q \in \mathbb{Z}_+}$ is then an orthonormal basis of $\mathbb{L}^2(\gamma_a)$ and then $(\varphi_p \otimes h_{q,a}, \psi_{p+1} \otimes h_{q,a})_{p,q \in \mathbb{Z}_+}$ is an orthonormal basis of $\mathbb{L}^2(\mu_a)$. We compute that:

Lemma 5 *For all $p, q \in \mathbb{Z}_+$, we have*

$$\begin{aligned} L_a[\varphi_p \otimes h_{q,a}] &= -q\varphi_p \otimes h_{q,a} - \sqrt{a}p\sqrt{q}\psi_p \otimes h_{q-1,a} - \sqrt{a}p\sqrt{q+1}\psi_p \otimes h_{q+1,a}, \\ L_a[\psi_p \otimes h_{q,a}] &= -q\psi_p \otimes h_{q,a} + \sqrt{a}p\sqrt{q}\varphi_p \otimes h_{q-1,a} + \sqrt{a}p\sqrt{q+1}\varphi_p \otimes h_{q+1,a}. \end{aligned}$$

Proof
From the relations satisfied by the usual Hermite polynomials, we get that for any $q \in \mathbb{Z}_+$ and $y \in \mathbb{R}$,

$$\begin{aligned} ah''_{q,a}(y) - yh'_{q,a}(y) &= -qh_{q,a}(y), \\ \sqrt{a}\sqrt{q+1}h_{q+1,a}(y) &= yh_{q,a}(y) - \sqrt{a}\sqrt{q}h_{q-1,a}(y). \end{aligned}$$

We deduce that for all $p, q \in \mathbb{Z}_+$ and all $(x, y) \in \mathbb{T} \times \mathbb{R}$,

$$\begin{aligned} L_a[\varphi_p \otimes h_{q,a}](x,y) &= \varphi'_p(x)yh_{q,a}(y) - q\varphi_p(x)h_{q,a}(y) \\ &= -p\sqrt{a}\psi_p(x)(\sqrt{q+1}h_{q+1,a}(y) + \sqrt{q}h_{q-1,a}(y)) - aq\varphi_p(x)h_{q,a}(y) \\ &= -(q\varphi_p \otimes h_{q,a} + \sqrt{a}p\sqrt{q}\psi_p \otimes h_{q-1,a} + \sqrt{a}p\sqrt{q+1}\psi_p \otimes h_{q+1,a})(x,y). \end{aligned}$$

The computation of $L_a[\psi_p \otimes h_{q,a}]$ is similar.

■

From these computations we get, on one hand that for $q \in \mathbb{Z}_+$, $\varphi_0 \otimes h_{q,a}$ is an eigenfunction of L_a associated to the eigenvalue $-q$ and on the other hand that for $p \in \mathbb{N}$, the following vector subspaces $\mathcal{V}_p$ and $\mathcal{W}_p$ are stable by L_a:

$$\begin{aligned}\mathcal{V}_p &:= \mathrm{Cl}(\mathrm{Vect}(\varphi_p \otimes h_{q,a}, \psi_p \otimes h_{q+1} \ :\ q \in 2\mathbb{Z}_+)),\\ \mathcal{W}_p &:= \mathrm{Cl}(\mathrm{Vect}(\psi_p \otimes h_q, \varphi_p \otimes h_{q+1,a} \ :\ q \in 2\mathbb{Z}_+)),\end{aligned}$$

where for any $A \subset \mathbb{L}^2(\mu_a)$, $\mathrm{Cl}(A)$ and $\mathrm{Vect}(A)$ stand respectively for the closure of A in $\mathbb{L}^2(\mu_a)$ and for the vector space generated by A.

2.2 Spectral analysis of L_a on $\mathcal{V}_p$

Since each $\mathcal{V}_p$ and $\mathcal{W}_p$ are stable subspaces of L_a, we must now study the spectral decomposition of the restriction of L_a to the Hilbert subspace $\mathcal{V}_p$ (the same conclusions will also hold for $\mathcal{W}_p$) , where $p \in \mathbb{N}$ is fixed. Consider the orthonormal basis $(e_q)_{q\in\mathbb{Z}_+}$ given by $e_0 := \varphi_p \otimes h_{0,a}$, $e_1 := \psi_p \otimes h_{1,a}$, $e_2 := \varphi_p \otimes h_{2,a}$ etc. This basis enables us to identify $\mathcal{V}_p$ with $l^2(\mathbb{Z}_+)$, $\mathbb{Z}_+$ being endowed with the counting measure. From Lemma 5, the (infinite) tridiagonal matrix M associated to the restriction of L_a to $\mathcal{V}_p$ described with the basis $(e_q)_{q\in\mathbb{Z}_+}$ is

$$M := \begin{pmatrix} 0 & \sqrt{ap} & 0 & 0 & \cdots \\ -\sqrt{ap} & -1 & -\sqrt{2}\sqrt{ap} & 0 & \cdots \\ 0 & \sqrt{2}\sqrt{ap} & -2 & \sqrt{3}\sqrt{ap} & \ddots \\ 0 & 0 & -\sqrt{3}\sqrt{ap} & -3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{6}$$

It appears that this object is only parametrized by $c := \sqrt{ap}$. Let us write $M = D+cS-cS^*$, where D and cS are respectively the diagonal and upper-diagonal part of M, so that $-cS^*$ corresponds to the lower-diagonal of M. Note that if S is interpreted as an (unbounded) operator of $l^2(\mathbb{Z}_+)$, then S^* is (the infinite matrix associated to) its adjoint operator in $l^2(\mathbb{Z}_+)$. In the sequel, we won't make much difference between such matrices and their corresponding operators on $l^2(\mathbb{Z}_+)$, but some preliminaries are needed in order to precisely define their domains. A priori the operators M, D, S and S^* are well-defined on $\mathcal{D}$, the subspace of real sequences $(z(q))_{q\in\mathbb{Z}_+}$ from $l^2(\mathbb{Z}_+)$ which admit only a finite number of non-zero coefficients. It is immediate to check that they are in fact closable and that the domains of their closures are given by

$$\mathcal{D}(S) = \mathcal{D}(S^*) = \mathcal{D}(D) = \mathcal{D}(M) = \{(z(q))_{q\in\mathbb{Z}_+} \in l^2(\mathbb{Z}_+) \ :\ \sum_{q\in\mathbb{Z}_+} qz^2(q) < +\infty\}.$$

It is natural to identify the operators M, D, S and S^* with their respective closures. In particular the spectral decomposition of the restriction of L_a to $\mathcal{V}_p$ is then equivalent to the one of M. Nevertheless, it is more fruitful to look at the operators M, D, S and S^* as endomorphisms of $\mathcal{S}$, the subspace of sequences $(z(q))_{q\in\mathbb{Z}_+}$ from $l^2(\mathbb{Z}_+)$ which are such that for any $r \geqslant 0$, $\sum_{q\in\mathbb{Z}_+} q^r z^2(q) < +\infty$. The advantage of this point of view is that we can compose the above operators without having to take care about their domains.

We can now state the main result of this paragraph wich describes the spectral analysis of M.

Theorem 6 *Let $\xi_0 := (\xi_0(q))_{q\in\mathbb{Z}_+}$ be the element of $\mathcal{S}$ given by*

$$\forall\ q \in \mathbb{Z}_+, \qquad \xi_0(q) \quad := \quad (-1)^{\lfloor \frac{q+1}{2} \rfloor} \frac{c^q}{\sqrt{q!}} \exp(-c^2/2).$$

Consider the elements of $\mathcal{S}$ defined by

$$\forall\ n \in \mathbb{Z}_+, \qquad \xi_n \quad = \quad (cI - S^*)^n \xi_0,$$

where I is the identity operator. Then for any $n \in \mathbb{Z}_+$, ξ_n is an eigenvector of M associated to the eigenvalue $-c^2 - n$. Furthermore $(\xi_n)_{n\in\mathbb{Z}_+}$ is a (Hilbert) basis of $l^2(\mathbb{Z}_+)$.

The proof will be based on the Lie algebra generated by the operators D, S and S^*, whose structure is determined by the following computation:

Lemma 7 *We have that*

$$\begin{aligned} [S, S^*] &= I, \\ [D, S] &= S, \\ [D, S^*] &= -S^*. \end{aligned}$$

Proof

Recall that we interpret the operators as endomorphism of $\mathcal{S}$, so the above brackets are well-defined. For any $q \in \mathbb{Z}_+$, we have that

$$\begin{aligned} SS^*(e_q) &= S((-1)^q \sqrt{q+1} e_{q+1}) \\ &= (-1)^q \sqrt{q+1} S(e_{q+1}) \\ &= (-1)^q \sqrt{q+1} (-1)^q \sqrt{q+1} e_q \\ &= (q+1) e_q. \end{aligned}$$

Similarly, we get that $S^*S(e_q) = qe_q$, so that

$$\begin{aligned} [S, S^*](e_q) &= (SS^* - S^*S)(e_q) \\ &= e_q, \end{aligned}$$

namely $[S, S^*] = I$.
For any $q \in \mathbb{Z}_+$, we also compute, with the convention $e_{-1} = 0$, that

$$\begin{aligned} DS(e_q) &= (-1)^{q+1} \sqrt{q} D(e_{q-1}) &= (-1)^q \sqrt{q}(q-1) e_{q-1}, \\ SD(e_q) &= -qS(e_q) &= (-1)^q q \sqrt{q} e_{q-1}. \end{aligned}$$

It follows that

$$\begin{aligned} [D, S] &= (DS - SD)(e_q) \\ &= (-1)^{q+1} \sqrt{q} e_{q-1} \\ &= S(e_q), \end{aligned}$$

hence $[D, S] = S$. The last relation is an immediate consequence of the previous one, since $D^* = D$:

$$[D, S^*] \quad = \quad -[D^*, (S^*)^*]^* \quad = \quad -[D, S]^* \quad = \quad -S^*$$

■

Let us denote by V the vector subspace of endomorphisms of $\mathcal{S}$ generated by D, S, S^* and I. Since the latter operators are clearly independent, V is 4-dimensional. Furthermore, taking into account that $[I,D]=[I,S]=[I,S^*]=0$, the bracket $[\cdot,\cdot]$ endows V with a Lie algebra structure. This property of V suggests that to get informations about the spectral decomposition of $M=D+cS-cS^*$, it is interesting to first investigate the spectral decomposition of the adjoint operator of M (in the Lie algebra sense, see for instance the book of Hall [8]), which is defined by

$$\mathrm{ad}_M \,:\, V \ni X \quad\mapsto\quad [M,X] \in V.$$

This is the object of the next result:

Lemma 8 *The kernel of the operator* ad_M *is 2-dimensional and is generated by* I *and* M*. There are two other eigenvalues,* 1 *and* -1*, whose corresponding eigenspaces are respectively generated by* $J_+ := cI+S$ *and* $J_- := cI-S^*$*.*

Proof

Indeed, with the help of Lemma 7 we compute that the matrix associated to ad_M in the basis (I,D,S,S^*) is given by

$$\begin{pmatrix} 0 & 0 & c & c \\ 0 & 0 & 0 & 0 \\ 0 & -c & 1 & 0 \\ 0 & -c & 0 & -1 \end{pmatrix}.$$

This matrix is not difficult to diagonalize, its characteristic polynomial is $X^2(X^2-1)$, and the announced results easily follow.

■

The interest of the operators J_+ and J_- is summarized as follows: if $z \in \mathbb{C}\mathcal{S}$ is an eigenvector of M associated to the eigenvalue $l \in \mathbb{C}$, then either $J_+(z)=0$ or $J_+(z)$ is an eigenvector of M associated to the eigenvalue $l+1$. Indeed, the relation $[M,J_+]=J_+$ implies that

$$\begin{aligned} M(J_+(z)) &= J_+(M(z)) + J_+(z) \\ &= (l+1)J_+(z). \end{aligned}$$

Similarly, either $J_-(z)=0$ or $J_-(z)$ is an eigenvector of M associated to the eigenvalue $l-1$.
This observation will be the key to the spectral decomposition of M, but let us first notice that any eigenvalue $l \in \mathbb{C}$ of M has a non-positive real part. To show this assertion, let $z=(z(q))_{q\in\mathbb{Z}_+} \in l^2(\mathbb{Z}_+,\mathbb{C})\backslash\{0\}$ be an associated eigenvector. It is sufficient to write, with $\langle\cdot,\cdot\rangle$ standing for the usual Hermitian scalar product of $l^2(\mathbb{Z}_+,\mathbb{C})$, that

$$\begin{aligned} 2\Re(l)\,\langle z,z\rangle &= \langle Mz,z\rangle + \langle z,Mz\rangle \\ &= \langle Dz,z\rangle + \langle z,Dz\rangle \\ &= -2\sum_{q\in\mathbb{N}} q\,|z(q)|^2 \\ &\leqslant 0 \end{aligned}$$

This argument can be extended to the eigenvalues of any Markovian generator $\mathcal{L}$ in $\mathbb{L}^2(\mu)$, where μ is an invariant probability for $\mathcal{L}$, and in particular to the eigenvalues of L_a in $\mathbb{L}^2(\mu_a)$.
Thus if there exists an eigenvalue $l \in \mathbb{C}$ of M associated to an eigenvector $z \in \mathbb{C}\mathcal{S}\backslash\{0\}$, then necessary we can find $n \in \mathbb{Z}_+$ such that $J_+^n(z)=0$. Because otherwise, we would conclude that for any $n \in \mathbb{Z}_+$, $l+n$ is an eigenvalue of M and thus its real part is non-positive, which is not possible. This is a hint on how we can find some eigenvectors of M: by looking at the kernel of J_+, whose computation is our next task.

Lemma 9 *The kernel of $J_+ : \mathcal{D}(S) \to l^2(\mathbb{Z}_+)$ is generated by the vector ξ_0 appearing in Theorem 6. The kernel of $J_- : \mathcal{D}(S) \to l^2(\mathbb{Z}_+)$ is reduced to $\{0\}$.*

Proof

More generally, let $z = (z(q))_{q\in\mathbb{Z}_+}$ be any sequence from $\mathbb{R}^{\mathbb{Z}_+}$, $J_+(z)$ can be defined as the sequence $(J_+(z)(q))_{q\in\mathbb{Z}_+}$ with

$$\forall\ q \in \mathbb{Z}_+, \qquad J_+(z)(q) \ \coloneqq\ cz(q) + (-1)^q\sqrt{q+1}z(q+1),$$

So the equation $J_+(z) = 0$ is equivalent to

$$\forall\ q \in \mathbb{Z}_+, \qquad z(q+1) \ =\ \frac{(-1)^{q+1}cz(q)}{\sqrt{q+1}}.$$

It appears that such a sequence z is determined by $z(0)$:

$$\forall\ q \in \mathbb{Z}_+, \qquad z(q) \ =\ \frac{(-1)^{\frac{(q+1)q}{2}}c^q}{\sqrt{q!}}z(0).$$

Remarking that for all $q \in \mathbb{Z}_+$, $(-1)^{\frac{(q+1)q}{2}} = (-1)^{\lfloor\frac{q+1}{2}\rfloor}$, we deduce that z is proportional to ξ_0. The first announced result then follows from the fact that $\xi_0 \in \mathcal{D}(S)$.
The kernel of J_- is obtained in a similar way, noting that J_- can also be extended to $\mathbb{R}^{\mathbb{Z}_+}$ via

$$\forall\ z = (z(q))_{q\in\mathbb{Z}_+} \in \mathbb{R}^{\mathbb{Z}_+},\ \forall\ q \in \mathbb{Z}_+, \qquad J_-(z)(q) \ \coloneqq\ cz(q) - (-1)^{q+1}\sqrt{q}z(q-1).$$

Thus, starting with $J_-(z)(0) = cz(0)$, if z is such that $J_-(z) = 0$, we get that $z(0) = 0$ and by iteration we end up with $z = 0$.

■

More precisely, we have $\xi_0 \in \mathcal{S}$ and for any function $f : \mathbb{Z}_+ \to \mathbb{R}_+$, we observe that

$$\sum_{q\in\mathbb{Z}_+} f(q)\xi_0^2(q) \ =\ \mathbb{E}[f(N_{c^2})] \tag{7}$$

where N_{c^2} is a Poisson distribution of parameter c^2. This is why we have chosen the normalization $\xi_0(0) = \exp(-c^2/2)$, which implies that ξ_0 has norm 1 in $l^2(\mathbb{Z}_+)$. It follows another important computational property of ξ_0 with respect to the operator algebra generated by S^*. As a by-product, we check that ξ_0 is an eigenvector of M, as this was suggested by the observations made before Lemma 9 (note that this is also a qualitative consequence of the facts that $J_+(M(\xi_0)) = M(J_+(\xi_0)) - J_+(\xi_0) = 0$ and that $\ker(J_+)$ is one-dimensional).

Lemma 10 *We compute that for any $n \in \mathbb{Z}_+$,*

$$(S^*)^n(\xi_0) \ =\ \frac{1}{c^n}D(D+1)(D+2)\cdots(D+n-1)(\xi_0). \tag{8}$$

It follows from the particular case $n = 1$ that

$$M(\xi_0) \ =\ -c^2\xi_0.$$

Proof

By the usual convention that a void product is equal to 1 or I, for $n = 0$, (8) reduces to $\xi_0 = \xi_0$. Let us check it for $n = 1$, namely that $S^*(\xi_0) = \frac{1}{c}D(\xi_0)$. For any $q \in \mathbb{Z}_+$, we have

$$\begin{aligned} S^*(\xi_0)(q) &= (-1)^{q+1}\sqrt{q}\xi_0(q-1) \\ &= (-1)^{q+1+\lfloor\frac{q}{2}\rfloor}\sqrt{q}\frac{c^{q-1}\exp(-c^2/2)}{\sqrt{(q-1)!}} \\ &= -\frac{q}{c}\xi_0(q) \\ &= \frac{1}{c}D(\xi_0)(q), \end{aligned}$$

where we have used that $(-1)^{q+1+\lfloor\frac{q}{2}\rfloor} = -(-1)^{\lfloor\frac{q+1}{2}\rfloor}$. Since $J_+(\xi_0) = 0$, we deduce directly that $S(\xi_0) = -c\xi_0$. Recalling that $M = D + cS - cS^*$, it follows that

$$M(\xi_0) = D(\xi_0) - c^2\xi_0 - D(\xi_0) = -c^2\xi_0.$$

Next we prove (8) by induction over n. So let us assume it for a given $n \in \mathbb{N}$, we write

$$\begin{aligned} (S^*)^{n+1}(\xi_0) &= (S^*)^n\frac{D}{c}(\xi_0) \\ &= \frac{1}{c}\left([(S^*)^n, D](\xi_0) + D(S^*)^n(\xi_0)\right). \end{aligned}$$

Lemma 7 enables to compute the above bracket:

$$\begin{aligned} [(S^*)^n, D] &= (S^*)^{n-1}[S^*, D] + (S^*)^{n-2}[S^*, D]S^* + \cdots + [S^*, D](S^*)^{n-1} \\ &= (S^*)^{n-1}S^* + (S^*)^{n-2}S^*S^* + \cdots + S^*(S^*)^{n-1} \\ &= n(S^*)^n. \end{aligned}$$

Putting together these computations, we get

$$\begin{aligned} (S^*)^{n+1}(\xi_0) &= \frac{1}{c}(n(S^*)^n(\xi_0) + D(S^*)^n(\xi_0)) \\ &= \frac{1}{c}(D+n)(S^*)^n(\xi_0) \\ &= \frac{1}{c^{n+1}}D(D+1)(D+2)\cdots(D+n-1)(D+n)(\xi_0), \end{aligned}$$

as wanted.

■

Starting with the eigenvector $\xi_0 \in \mathcal{S}$, we construct the sequence of eigenvectors $(\xi_n)_{n\in\mathbb{Z}_+} := (J_-^n(\xi_0))_{n\in\mathbb{Z}_+}$ which are associated to the eigenvalues $(-c^2 - n)_{n\in\mathbb{Z}_+}$, according to the discussion following the proof of Lemma 8. Indeed, none of the vectors $J_-^n(\xi_0)$, for $n \in \mathbb{N}$, vanishes, because we have seen in Lemma 9 that the kernel of J_- is trivial.

Since the elements of the sequence $(\xi_n)_{n\in\mathbb{Z}_+}$ are non-zero and associated to different eigenvalues, it is easy to see that any *finite* family of them is independent in $l^2(\mathbb{Z}_+)$. It is more involved to check that the *whole* sequence $(\xi_n)_{n\in\mathbb{Z}_+}$ is independent in $l^2(\mathbb{Z}_+)$. To go in this direction, we present an isometry which will also play an important role in the next section. It gives a convenient way to deal with the fact that the vectors of the sequence $(\xi_n)_{n\in\mathbb{Z}_+}$ are non-orthogonal.

Let $\mathcal{Q}$ be the subspace of $\mathcal{V}_p$ consisting of vectors z which can be written as a linear combinaison of a finite number of elements of $(\xi_n)_{n\in\mathbb{Z}_+}$:

$$z = \sum_{n\in\mathbb{Z}_+} f(n)\xi_n, \tag{9}$$

where only a finite number of the real coefficients $f(n)$ are non-zero. Due to the above observation, these coefficients are uniquely determined for $z \in \mathcal{Q}$. So we associate to such an element $z \in \mathcal{Q}$ the polynomial

$$F(X) \ \coloneqq \ \sum_{n\in\mathbb{Z}_+} f(n)X^n. \tag{10}$$

We also consider the function G defined on $\mathbb{Z}_+$ by

$$\forall\ n \in \mathbb{Z}_+, \qquad G(n) \ \coloneqq \ \left(1 + \frac{1}{c}\frac{d}{dX}\right)^n F(X)\bigg|_{X=c}, \tag{11}$$

(where the power n corresponds to the composition of differential operators).

Proposition 11 *The mapping $\mathcal{Q} \ni z \mapsto G$ is an isometry with respect to the norms $l^2(\mathbb{Z}_+)$ and $\mathbb{L}^2(\mathcal{P}(c^2))$, where $\mathcal{P}(c^2)$ stands for the Poisson distribution of parameter c^2.*

Proof

By definition and Lemma 10, we have for any $n \in \mathbb{Z}_+$,

$$\begin{aligned}
\xi_n &= (cI - S^*)^n \xi_0 \\
&= \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{n-l}(-1)^l (S^*)^l \xi_0 \\
&= \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{n-l} \left(\frac{-1}{c}\right)^l D(D+1)\cdots(D+l-1)\xi_0 \\
&= c^n \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l}(-D)(-D-1)\cdots(-D-l+1)\xi_0.
\end{aligned} \tag{12}$$

We deduce that for any $n, m \in \mathbb{Z}_+$,

$$\begin{aligned}
&\langle \xi_n, \xi_m \rangle \\
&= c^{n+m} \sum_{l\in[\![0,n]\!],\, k\in[\![0,m]\!]} \binom{n}{l}\binom{m}{k} c^{-2(l+k)} \langle (-D)(-D-1)\cdots(-D-l+1)\xi_0, \\
&\qquad\qquad (-D)(-D-1)\cdots(-D-k+1)\xi_0 \rangle \\
&= c^{n+m} \sum_{l\in[\![0,n]\!],\, k\in[\![0,m]\!]} \binom{n}{l}\binom{m}{k} c^{-2(l+k)} \sum_{q\in\mathbb{Z}_+} q(q-1)\cdots(q-l+1)q(q-1)\cdots(q-k+1)\xi_0^2(q) \\
&= c^{n+m} \sum_{l\in[\![0,n]\!],\, k\in[\![0,m]\!]} \binom{n}{l}\binom{m}{k} c^{-2(l+k)} \mathbb{E}[N(N-1)\cdots(N-l+1)N(N-1)\cdots(N-k+1)],
\end{aligned}$$

where N is a Poisson random variable of parameter c^2 (recall (7)). It follows that if $z = \sum_{n\in\mathbb{Z}_+} f(n)\xi_n$ belongs to $\mathcal{Q}$, then

$$\begin{aligned}
\langle z, z \rangle &= \sum_{n,m\in\mathbb{Z}_+} f(n)f(m)\langle \xi_n, \xi_m \rangle \\
&= \sum_{n,m\in\mathbb{Z}_+} f(n)f(m)c^{n+m} \sum_{l\in[\![0,n]\!],k\in[\![0,m]\!]} \binom{n}{l}\binom{m}{k} c^{-2(l+k)} \\
&\qquad \mathbb{E}[N(N-1)\cdots(N-l+1)N(N-1)\cdots(N-k+1)],
\end{aligned}$$

where only a finite number of terms are non-zero. Note that we have for any $l \in \mathbb{Z}_+$, we have

$$\left.\frac{d^l}{dX^l}F(X)\right|_{X=c} = \sum_{n\in\mathbb{Z}_+} f(n)n(n-1)\cdots(n-l+1)c^{n-l}.$$

Henw rewritting terms $\binom{n}{l}N(N-1)\cdots(N-l+1)$ under the form $\binom{N}{l}n(n-1)\cdots(n-l+1)$, we get

$$\begin{aligned}
\langle z,z\rangle &= \sum_{l,k\in\mathbb{Z}_+} \left.\frac{d^l}{dX^l}F(X)\right|_{X=c} \left.\frac{d^k}{dX^k}F(X)\right|_{X=c} c^{-(l+k)}\mathbb{E}\left[\binom{N}{l}\binom{N}{k}\right]\\
&= \mathbb{E}\left[\left.\left(1+c^{-1}\frac{d}{dX}\right)^N F(X)\right|_{X=c} \left.\left(1+c^{-1}\frac{d}{dX}\right)^N F(X)\right|_{X=c}\right]\\
&= \mathbb{E}[G^2(N)],
\end{aligned}$$

which is the wanted isometry relation.

■

In order to prove the independence of the family $(\xi_n)_{n\in\mathbb{Z}_+}$, we need to control the mapping associating F to G, this is the goal of next result.

Lemma 12 *Using the notations introduced in (9), (10) and (11), we have*

$$\forall\ n\in\mathbb{Z}_+,\qquad |f(n)| \leqslant c^n\exp\left((4c^2+2+c^{-2})/2\right)\sqrt{\mathbb{E}[G^2(N_{c^2})]},$$

where N_{c^2} is a Poisson random variable of parameter c^2.

Proof

By definition, we have for any $n\in\mathbb{Z}_+$,

$$\begin{aligned}
G(n) &= \sum_{m\in[\![0,n]\!]} \binom{n}{m}\frac{F^{(m)}(c)}{c^m} \qquad (13)\\
&= \sum_{m\in\mathbb{Z}_+} n(n-1)\cdots(n-m+1)\frac{F^{(m)}(c)}{m!\,c^m}.
\end{aligned}$$

For any real $x>c$, denote by H_x the density of a Poisson distribution of parameter $(x-c)c$ with respect to a Poisson distribution of parameter c^2:

$$\begin{aligned}
\forall\ n\in\mathbb{Z}_+,\qquad H_x(n) &= \left(\frac{(x-c)c}{c^2}\right)^n\exp(-(x-c)c+c^2) \qquad (14)\\
&= \left(\frac{x-c}{c}\right)^n\exp(-cx+2c^2).
\end{aligned}$$

Its interest is that for any $m\in\mathbb{Z}_+$, we have

$$\begin{aligned}
\mathbb{E}[N_{c^2}(N_{c^2}-1)\cdots(N_{c^2}-m+1)H_x(N_{c^2})] &= \mathbb{E}[N_{c(x-c)}(N_{c(x-c)}-1)\cdots(N_{c(x-c)}-m+1)]\\
&= (x-c)^m c^m,
\end{aligned}$$

where N_r stands for a Poisson random variable of parameter r, for any $r>0$.
Putting together the above relations, we get that

$$\begin{aligned}
\mathbb{E}[G(N_{c^2})H_x(N_{c^2})] &= \sum_{m\in\mathbb{Z}_+}\frac{F^{(m)}(c)}{m!}(x-c)^m\\
&= F(x).
\end{aligned}$$

By analytic extension, this identity holds for any $x \in \mathbb{C}$, since both sides are easily seen to be holomorphic functions of x. In particular for any $\theta \in [0, 2\pi)$ and $n \in \mathbb{Z}_+$, we get

$$\left(\frac{2}{c\exp(i\theta)}\right)^n F\left(\frac{c\exp(i\theta)}{2}\right) = \mathbb{E}[G(N_{c^2})(2/c)^n \exp(-in\theta) H_{c\exp(i\theta)/2}(N_{c^2})].$$

An integration with respect to $\mathbb{1}_{[0,2\pi)}(\theta)d\theta/(2\pi)$ yields

$$\begin{aligned} f(n) &= \int_{[0,2\pi)} \left(\frac{2}{c\exp(i\theta)}\right)^n F\left(\frac{c\exp(i\theta)}{2}\right) \frac{d\theta}{2\pi} \\ &= \mathbb{E}[G(N_{c^2}) J_n(N_{c^2})], \end{aligned}$$

where for any $m \in \mathbb{Z}_+$, we have

$$\begin{aligned} J_n(m) &= \int_{[0,2\pi)} (2/c)^n \exp(-in\theta) H_{c\exp(i\theta)/2}(m) \frac{d\theta}{2\pi} \qquad (15) \\ &= \frac{1}{n!} \frac{d^n}{dX^n} H_X(m)\Big|_{X=0} \\ &= \frac{\exp(2c^2)}{c^m n!} \frac{d^n}{dX^n} (X-c)^m \exp(-cX)\Big|_{X=0} \\ &= \frac{\exp(2c^2)}{c^m n!} \sum_{p \in [\![0,n]\!]} \binom{n}{p} \frac{d^p}{dX^p}(X-c)^m \frac{d^{n-p}}{dX^{n-p}} \exp(-cX)\Big|_{X=0} \\ &= \frac{\exp(2c^2)}{c^m n!} \sum_{p \in [\![0,n]\!]} \binom{n}{p} m(m-1)\cdots(m-p+1)(X-c)^{m-p}(-c)^{n-p} \exp(-cX)\Big|_{X=0} \\ &= \frac{\exp(2c^2)}{c^m n!} \sum_{p \in [\![0,n]\!]} \binom{n}{p} m(m-1)\cdots(m-p+1)(-c)^{m+n-2p} \\ &= \frac{(-1)^{m+n} c^n \exp(2c^2)}{n!} \sum_{p \in [\![0,n]\!]} \binom{n}{p} m(m-1)\cdots(m-p+1) c^{-2p}. \end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain for any $n \in \mathbb{Z}_+$,

$$|f(n)| \leqslant \sqrt{\mathbb{E}[J_n^2(N_{c^2})]}\sqrt{\mathbb{E}[G^2(N_{c^2})]}.$$

To bound the first factor, we write that for any $m \in \mathbb{Z}_+$,

$$\begin{aligned} \exp(-2c^2)\,|J_n(m)| &= \frac{c^n}{n!} \sum_{p \in \mathbb{Z}_+} \binom{n}{p} m(m-1)\cdots(m-p+1) c^{-2p} \\ &= \frac{c^n}{n!} \sum_{p \in \mathbb{Z}_+} \binom{m}{p} n(n-1)\cdots(n-p+1) c^{-2p} \qquad (16) \\ &\leqslant c^n \sum_{p \in \mathbb{Z}_+} \binom{m}{p} c^{-2p} = c^n \left(1 + c^{-2}\right)^m. \end{aligned}$$

Thus we get as announced that

$$\mathbb{E}[J_n^2(N_{c^2})] \leqslant c^{2n} \exp(4c^2) \mathbb{E}\left[\left(1 + \frac{1}{c^2}\right)^{2N_{c^2}}\right] = c^{2n} \exp(4c^2) \exp\left(c^2\left(1 + \frac{1}{c^2}\right)^2 - c^2\right).$$

■

The independence in $l^2(\mathbb{Z}_+)$ of the family $(\xi_n)_{n\in\mathbb{Z}_+}$ now follows without difficulty: it is equivalent to the fact that if $\sum_{n\in\mathbb{Z}_+} f(n)\xi_n = 0$, where the sum in l.h.s. is converging in $l^2(\mathbb{Z}_+)$, then $f(n) = 0$ for all $n \in \mathbb{Z}_+$. But for $n \in \mathbb{Z}_+$, consider $z_n := \sum_{m\in[\![0,n]\!]} f(m)\xi_m \in \mathcal{Q}$, vector to which we associate the function G_n as in (11). The convergence of the sequence $(z_n)_{n\in\mathbb{Z}_+}$ to zero in $l^2(\mathbb{Z}_+)$ is equivalent to

$$\lim_{n\to\infty} \mathbb{E}[G_n^2(N_{c^2})] \quad = \quad 0.$$

According to Lemma 12, we have for any $m \leqslant n$,

$$|f(m)| \quad \leqslant \quad c^m \exp\left((4c^2 + 2 + c^{-2})/2\right) \sqrt{\mathbb{E}[G_n^2(N_{c^2})]},$$

so letting n going to infinity, we get $f(m) = 0$, for any given $m \in \mathbb{Z}_+$, as required.

Remark 13 Denote by ϕ the map that associates to any $G \in \mathbb{L}^2(\mathcal{P}_{c^2})$ the formal series $F(X) := \sum_{n\in\mathbb{Z}_+} f(n)X^n$, where

$$\begin{aligned} \forall\, n, m \in \mathbb{Z}_+, \qquad f(n) &:= \mathbb{E}[G(N_{c^2})J_n(N_{c^2})] \\ J_n(m) &:= \frac{(-1)^{m+n}c^n\exp(2c^2)}{n!} \sum_{p\in[\![0,n]\!]} \binom{n}{p} m(m-1)\cdots(m-p+1)c^{-2p}. \end{aligned}$$

The previous proof shows that the bound of Lemma 12 is valid in this context, so the convergence radius of F is at least 1. But the above arguments can be improved to get that F define in fact a holomorphic function in the whole plane. More precisely, in (16), we can rather use the bound

$$\begin{aligned} \frac{n(n-1)\cdots(n-p+1)}{n!} &= \frac{1}{(n-p)!} \\ &\leqslant \frac{1}{(n-m)!}, \end{aligned}$$

with the convention that $(n-m)! = 1$ if $m \geqslant n$. Consequently we have

$$\forall\, n, m \in \mathbb{Z}_+, \qquad |J_n(m)| \quad \leqslant \quad \exp(2c^2)\frac{c^n}{(n-m)!}\left(1+\frac{1}{c^2}\right)^m.$$

It follows that, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}[J_n^2(N_{c^2})] &\leqslant c^{2n}\exp(4c^2)\mathbb{E}\left[\frac{1}{((n-N_{c^2})!)^2}\left(1+\frac{1}{c^2}\right)^{2N_{c^2}}\right] \\ &\leqslant c^{2n}\exp(4c^2)\mathbb{E}\left[\frac{1}{(n-N_{c^2})!}\left(1+\frac{1}{c^2}\right)^{2N_{c^2}}\right] \\ &\leqslant c^{2n}\frac{\exp(4c^2)}{n!}\mathbb{E}\left[n^{N_{c^2}}\left(1+\frac{1}{c^2}\right)^{2N_{c^2}}\right] \\ &= c^{2n}\frac{\exp(4c^2)}{n!}\exp\left(c^2\left(1+\frac{1}{c^2}\right)^2 n - c^2\right). \end{aligned}$$

We conclude that

$$\forall\, n \in \mathbb{N}, \qquad |f(n)| \quad \leqslant \quad c^n\frac{\exp(2c^2)}{\sqrt{n!}}\exp\left(\left(\left(c+\frac{1}{c}\right)^2 n - c^2\right)/2\right)\sqrt{\mathbb{E}[G^2(N_{c^2})]},$$

and this bound is sufficient to insure that $F \in \mathcal{H}(\mathbb{C})$. Note that for $n = 0$, the above computations have to be slightly modified, starting with $J_0(m) = (-1)^m\exp(2c^2)$ and ending with $|f(0)| \leqslant$

$\exp(2c^2)\sqrt{\mathbb{E}[G^2(N_{c^2})]}$.

One can go further and check that (13) holds for any $G \in \mathbb{L}^2(\mathcal{P}_{c^2})$. Indeed, first consider the holomorphic function R defined by

$$\forall\, x \in \mathbb{C}, \qquad R(x) \;\coloneqq\; \mathbb{E}[G(N_{c^2})H_x(N_{c^2})] - F(x).$$

By definition of F, see also (15), we have

$$\forall\, n \in \mathbb{Z}_+, \qquad \frac{1}{2\pi}\int_{\mathcal{C}(0,c/2)} x^{-n}R(x)\,dx \;=\; 0,$$

where $\mathcal{C}(0, c/2)$ is the circle of radius $c/2$ centered at 0. By holomorphy, this implies that

$$\forall\, x \in \mathbb{C}, \qquad F(x) \;=\; \mathbb{E}[G(N_{c^2})H_x(N_{c^2})]. \tag{17}$$

Next we compute that for any $n, n' \in \mathbb{Z}_+$ (recall (14)),

$$\begin{aligned}
&\sum_{m\in[\![0,n]\!]}\binom{n}{m}\frac{1}{c^m}\,\frac{d^m}{dx^m}H_x(n')\Big|_{x=c}\\
&= \exp(c^2)\sum_{m\in[\![0,n]\!]}\binom{n}{m}\frac{1}{c^m}\sum_{p\in[\![0,m]\!]}\binom{m}{p}\frac{d^p}{dx^p}\left(\frac{x-c}{c}\right)^{n'}\Big|_{x=c}\frac{d^{m-p}}{dx^{m-p}}\exp(-c(x-c))\Big|_{x=c}\\
&= \mathbb{1}_{[\![0,n]\!]}(n')\exp(c^2)\sum_{m\in[\![0,n]\!]}\binom{n}{m}\frac{1}{c^{m+n'}}\binom{m}{n'}n'!(-c)^{m-n'}\\
&= \mathbb{1}_{[\![0,n]\!]}(n')\exp(c^2)c^{-2n'}\sum_{m\in[\![n',n]\!]}\frac{n(n-1)\cdots(n-m+1)}{(m-n')!}(-1)^{m-n'}\\
&= \mathbb{1}_{[\![0,n]\!]}(n')\exp(c^2)\frac{n(n-1)\cdots(n-n'+1)}{c^{2n'}}\sum_{l\in[\![0,n-n']\!]}\binom{n-n'}{l}(-1)^l\\
&= \delta_n(n')\,n!c^{-2n}\exp(c^2)\\
&= (\mathcal{P}(c^2)[n])^{-1}\delta_n(n').
\end{aligned}$$

Thus (13) is obtained by applying the operator $\sum_{m\in[\![0,n]\!]}\binom{n}{m}\frac{1}{c^m}\,\frac{d^m}{dx^m}\big|_{x=c}$ to (17).

□

Remark 14 Due to the independence of $(\xi_n)_{n\in\mathbb{Z}_+}$ in $l^2(\mathbb{Z}_+)$, the linear morphism $\mathcal{Q} \ni z \mapsto G$ can be extended to the closure $\mathrm{Cl}(\mathcal{Q})$ of $\mathcal{Q}$ in $l^2(\mathbb{Z}_+)$, let us call ψ this mapping. It is an isometry between $\mathrm{Cl}(\mathcal{Q})$ and the closure of $\psi(\mathcal{Q})$ in $\mathbb{L}^2(\mathcal{P}(c^2))$. We deduce from (13) that the image of $\mathcal{Q}$ by ψ is the space of the restrictions to $\mathbb{Z}_+$ of polynomial mappings, which is well-known to be dense in $\mathbb{L}^2(\mathcal{P}(c^2))$. Thus ψ is an isometry between $\mathrm{Cl}(\mathcal{Q})$ and $\mathbb{L}^2(\mathcal{P}(c^2))$. It appears that the inverse of ψ is $\varphi \circ \phi$, where ϕ is defined at the beginning of Remark 13 and where φ associates to any series $\sum_{n\in\mathbb{Z}_+} f(n)X^n$ from $\phi(\mathbb{L}^2(\mathcal{P}(c^2)))$ the element $\sum_{n\in\mathbb{Z}_+} f(n)\xi_n$ of $\mathrm{Cl}(\mathcal{Q})$.

□

It is time to check that $\mathrm{Cl}(\mathcal{Q}) = l^2(\mathbb{Z}_+)$, this will end the

Proof of Theorem 6

Indeed, by the above results, the density of $\mathcal{Q}$ in $l^2(\mathbb{Z}_+)$ will enable us to conclude that $(\xi_n)_{n\in\mathbb{Z}_+}$ is a Hilbert basis of $l^2(\mathbb{Z}_+)$. Thus it remains to show that if $z \in l^2(\mathbb{Z}_+)$ is such that $\langle z, \xi_n\rangle = 0$ for all $n \in \mathbb{Z}_+$, then $z = 0$. So let $z = (z(q))_{q\in\mathbb{Z}_+}$ be such an element. Since $\xi_n = J_-^n(\xi_0)$ and $J_- = cI - S^*$, this vector z also satisfies

$$\forall\, n \in \mathbb{Z}_+, \qquad \langle z, S^{*n}(\xi_0)\rangle \;=\; 0,$$

and according to Lemma 10, this is also equivalent to

$$\forall\ n \in \mathbb{Z}_+, \qquad \langle z, D^n(\xi_0)\rangle \ = \ 0,$$

or

$$\forall\ n \in \mathbb{Z}_+, \qquad \sum_{q\in\mathbb{Z}_+} q^n z(q)(-1)^{\lfloor\frac{q+1}{2}\rfloor}\frac{c^q}{\sqrt{q!}}\exp(-c^2/2) \ = \ 0. \tag{18}$$

Let us denote $m = (m(q))_{q\in\mathbb{Z}_+}$ the signed measure on $\mathbb{Z}_+$ with

$$\forall\ q \in \mathbb{Z}_+, \qquad m(q) \ := \ z(q)(-1)^{\lfloor\frac{q+1}{2}\rfloor}\frac{c^q}{\sqrt{q!}}\exp(-c^2/2).$$

Let m_+ and m_- stand respectively for the non-negative and non-positive parts of m, so that $m = m_+ - m_-$. From (7) and for all $r \geqslant 0$, we have

$$\begin{aligned}
\sum_{q\in\mathbb{Z}_+} \exp(rq)\,|m(q)| \ &\leqslant\ \sqrt{\sum_{q\in\mathbb{Z}_+} \exp(2rq)\xi_0^2(q)}\sqrt{\sum_{q\in\mathbb{Z}_+} z^2(q)} \\
&= \ \exp(c^2(\exp(2r)-1)/2)\sqrt{\langle z,z\rangle} \ < \ +\infty
\end{aligned}$$

and thus m_+ and m_- are non-negative measures admitting exponential moments of all order.

Furthermore (18) shows that all the usual moments of m_+ and m_- coincide, so we can apply the moment characterizing theorem (see for instance the section XV4 of the book of Feller [7]) to get that $m_+ = m_-$, namely $m = 0$. It follows that $z = 0$ as wanted.

■

2.3 Eigenvectors properties

Let us now compute more explicitely the eigenvectors ξ_n, for $n \in \mathbb{Z}_+$,

Proposition 15 *For any $n \in \mathbb{Z}_+$, the mapping ξ_n defined in Theorem 6 is given, as a function of $(x,y) \in \mathbb{T}\times\mathbb{R}$, almost everywhere by*

$$\begin{aligned}
\xi_n(x,y) \ &= \ \frac{2^p p!}{\sqrt{(2p)!}}\sqrt{n!}\Re\left(i^n h_n\left(\frac{y}{\sqrt{a}} - 2ic\right)\exp(ip(x+y))\right) \\
&= \ \frac{2^p p!}{\sqrt{(2p)!}}\sum_{l\in[\![0,n]\!]}\binom{n}{l}(2c)^{n-l}\sqrt{l!}h_{l,a}(y)\Re(i^l\exp(ip(x+y))).
\end{aligned}$$

Thus $\xi_n(x,y)$ is an appropriate linear combination of terms of the types $y^m\cos(p(x+y))$ and $y^m\sin(p(x+y))$ for $m \in [\![0,n]\!]$.

Proof

For any given $n \in \mathbb{Z}_+$, let us write

$$\xi_n \ = \ \sum_{q\in\mathbb{Z}_+} \xi_n(q)e_q,$$

with

$$\forall\ q \in \mathbb{Z}_+, \qquad \xi_n(q) \ := \ ((cI - S^*)^n\xi_0)(q).$$

Taking into account (12) and the definition of ξ_0 given in Theorem 6, we get that for any $q \in \mathbb{Z}_+$,

$$\begin{aligned}
\xi_n(q) &= c^n \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l}((-D)(-D-1)\cdots(-D-l+1)\xi_0)(q) \\
&= c^n \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} q(q-1)\cdots(q-l+1)(-1)^{\lfloor \frac{q+1}{2} \rfloor} \frac{c^q}{\sqrt{q!}} \exp(-c^2/2).
\end{aligned}$$

Denoting $k_p := \frac{2^p p!}{\sqrt{(2p)!}}$, it is not difficult to check from the definition of the orthonormal basis $(e_q)_{q\in\mathbb{Z}_+}$ that

$$\forall\ q \in \mathbb{Z}_+,\ \forall\ (x,y) \in \mathbb{T}\times\mathbb{R}, \qquad e_q(x,y) \;=\; k_p \Re((-1)^{\lfloor \frac{q+1}{2} \rfloor} i^q \exp(ipx) h_{q,a}(y)). \tag{19}$$

Rigorously speaking, such equalities have to be understood almost everywhere in $(x,y) \in \mathbb{T}\times\mathbb{R}$, since we are dealing with functions from $\mathbb{L}^2(\mu_a)$. Putting these expansions together, it appears that

$$\begin{aligned}
&\xi_n(x,y) \\
&= \exp(-c^2/2) k_p c^n \Re\left(\sum_{q\in\mathbb{Z}_+} \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} q(q-1)\cdots(q-l+1)\frac{c^q}{\sqrt{q!}} i^q \exp(ipx) h_{q,a}(y)\right).
\end{aligned} \tag{20}$$

Interpreting again the term and $\mathcal{W}_p$ $q(q-1)\cdots(q-l+1)c^{q-2l}$ as $c^{-l}\ \frac{d^l}{dX^l} X^q\Big|_{X=c}$, we have

$$\sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} q(q-1)\cdots(q-l+1)c^q \;=\; \left(1+\frac{1}{c}\frac{d}{dX}\right)^n X^q\Big|_{X=c},$$

so that

$$\sum_{q\in\mathbb{Z}_+} \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} q(q-1)\cdots(q-l+1)\frac{c^q}{\sqrt{q!}} i^q h_{q,a}(y) \;=\; \left(1+\frac{1}{c}\frac{d}{dX}\right)^n \sum_{q\in\mathbb{Z}_+} \frac{X^q}{\sqrt{q!}} i^q h_{q,a}(y)\Bigg|_{X=c}.$$

To go further, recall that Hermite polynomials satisfy

$$\forall\ r \in \mathbb{C},\ \forall\ y \in \mathbb{R}, \qquad \sum_{q\in\mathbb{Z}_+} r^q \frac{h_q(y)}{\sqrt{q!}} \;=\; \exp(ry - r^2/2). \tag{21}$$

Thus we deduce that

$$\begin{aligned}
\sum_{q\in\mathbb{Z}_+} \frac{X^q}{\sqrt{q!}} i^q h_{q,a}(y) &= \exp\left(\frac{iXy}{\sqrt{a}} + \frac{X^2}{2}\right) \\
&= \exp\left(-\frac{1}{2}\left(iX - \frac{y}{\sqrt{a}}\right)^2 + \frac{y^2}{2a}\right).
\end{aligned}$$

Recalling the definition (5) of the Hermite polynomials, the previous formulation leads to

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}_+}\sum_{l\in[\![0,n]\!]}\binom{n}{l}c^{-2l}q(q-1)\cdots(q-l+1)\frac{c^q}{\sqrt{q!}}i^q h_{q,a}(y)\\
&= \left(1+\frac{i}{c}\frac{d}{d(iX)}\right)^n \exp\left(-\frac{1}{2}\left(iX-\frac{y}{\sqrt{a}}\right)^2+\frac{y^2}{2a}\right)\Bigg|_{X=c}\\
&= \sum_{l\in[\![0,n]\!]}\binom{n}{l}i^l c^{-l}\frac{d^l}{d(iX)^l}\exp\left(-\frac{1}{2}\left(iX-\frac{y}{\sqrt{a}}\right)^2+\frac{y^2}{2a}\right)\Bigg|_{X=c}\\
&= \sum_{l\in[\![0,n]\!]}\binom{n}{l}i^l c^{-l}(-1)^l\sqrt{l!}h_l\left(iX-\frac{y}{\sqrt{a}}\right)\exp\left(-\frac{1}{2}\left(iX-\frac{y}{\sqrt{a}}\right)^2+\frac{y^2}{2a}\right)\Bigg|_{X=c}\\
&= \sum_{l\in[\![0,n]\!]}\binom{n}{l}(ci)^{-l}\sqrt{l!}h_l\left(ic-\frac{y}{\sqrt{a}}\right)\exp\left(\frac{icy}{\sqrt{a}}+\frac{c^2}{2}\right)\\
&= (ci)^{-n}\sqrt{n!}h_n\left(2ic-\frac{y}{\sqrt{a}}\right)\exp\left(\frac{icy}{\sqrt{a}}+\frac{c^2}{2}\right),
\end{aligned}$$

where we have used another property of Hermite polynomials:

$$\forall\ n\in\mathbb{Z}_+,\ \forall\ r,s\in\mathbb{C},\qquad \sqrt{n!}h_n(r+s) = \sum_{l\in[\![0,n]\!]}\binom{n}{l}s^{n-l}\sqrt{l!}h_l(r).$$

This relation, parity properties of the Hermite polynomials and equation (20) lead immediately to the announced expressions.

■

Remark 16 A posteriori, the last assertion of Proposition 15, as well as the spectrum of the restriction of L_a to $\mathcal{V}_p$, could have been obtained in the following way. Consider the change of variables $\mathbb{T}\times\mathbb{R}\ni(x,y)\mapsto(z,y)\in\mathbb{T}\times\mathbb{R}$ with $z=x+y$ (in $\mathbb{T}$). Acting on functions of the form $g(z,y)$, the generator L_a can be rewritten under the form

$$\widehat{L}_a \;:=\; a\partial_z^2+2a\partial_z\partial_y+a\partial_y^2-y\partial_y.$$

Consider next functions g of product type $g_1\otimes g_2$, with

$$g_1\,:\,\mathbb{T}\ni z \;\mapsto\; \exp(\alpha z)\in\mathbb{T},$$

where $\alpha\in i\mathbb{Z}$. The relation $\partial_z g_1(z)=\alpha g_1(z)$ implies that $\widehat{L}_a[g_1\otimes g_2]=g_1\otimes\widehat{L}_{a,\alpha}[g_2]$, where $\widehat{L}_{a,\alpha}$ is the Sturm-Liouville differential operator acting on functions h of the real variable y through

$$\widehat{L}_{a,\alpha}[h](y) \;:=\; ah''(y)-(y-2a\alpha)h'(y)+a\alpha^2h(y).$$

It is not difficult to check that this operator admits a family $(p_q)_{q\in\mathbb{Z}_+}$ of polynomials with complex coefficients such that: for any $q\in\mathbb{Z}_+$, p_q is of degree q and $\widehat{L}_{a,\alpha}[p_q]=(a\alpha^2-q)p_q$ (the factor $a\alpha^2-q$ is imposed by the coefficient of highest degree of $\widehat{L}_{a,\alpha}[p_q]$).
Thus we easily recover all the spectal information contained in Theorem 6 and Proposition 15. But relations such as those described in Lemma 7 will be encountered again in Sections 4 and 6, indeed, they are the starting point of all our developments.

□

2.4 Spectral analysis of L_a on $\mathcal{W}_p$

The spectral decomposition of the restriction of L_a to $\mathcal{W}_p$, for fixed $p \in \mathbb{N}$, is similar. This is due to the fact that the restriction of L_a to $\mathcal{W}_p$ is conjugate to the restriction of L_a to $\mathcal{V}_p$. More precisely, consider the basis $(e'_q)_{q\in\mathbb{Z}_+}$ of $\mathcal{W}_p$ given by

$$\begin{array}{rclcrcl} e'_0 & := & \psi_p \otimes h_{0,a} & , & e'_1, & := & -\varphi_p \otimes h_{1,a}, \\ e'_2 & := & \psi_p \otimes h_{2,a} & , & e'_3, & := & -\varphi_p \otimes h_{3,a}, \\ & \vdots & & & & \vdots & \end{array}$$

Then the matrix of the restriction of L_a to $\mathcal{W}_p$ in the basis $(e'_q)_{q\in\mathbb{Z}_+}$ is also given by (6). Thus Theorem 6 and Proposition 15 are still valid, after obvious modifications (note for instance that (19) remains true is we replace e_q by e'_q and the real part $\Re$ by the imaginary part $\Im$):

Proposition 17 *For $n \in \mathbb{Z}_+$, consider $\xi'_n := (cI - S^{*\prime})^n \xi'_0$, where $S^{*\prime}$ and ξ'_0 have the same coefficients as S^* and ξ_0, introduced in Theorem 6, but in the basis $(e'_q)_{q\in\mathbb{Z}_+}$ instead of $(e_q)_{q\in\mathbb{Z}_+}$. Then $(\xi_n)_{n\in\mathbb{Z}_+}$ is a Hilbert basis of $\mathcal{W}_p$ consisting of eigenvectors associated respectively to the eigenvalues $(-c^2 - n)_{n\in\mathbb{N}}$ of the restriction of L_a to $\mathcal{W}_p$. Coming back to functional notations, we have for all $n \in \mathbb{N}$,*

$$\forall\ (x,y) \in \mathbb{T} \times \mathbb{R}, \qquad \xi'_n(x,y) \quad = \quad \frac{2^p p!}{\sqrt{(2p)!}} \sqrt{n!}\,\Im\left(i^n h_n\left(\frac{y}{\sqrt{a}} - 2ic\right) \exp(ip(x+y))\right).$$

This result completes the spectral decomposition of L_a in $\mathbb{L}^2(\mu_a)$. This operator is diagonalizable, the set of its eigenvalues is

$$\Lambda_a \quad := \quad \{-c^2 - n \ : \ p, n \in \mathbb{Z}_+\},$$

and the multiplicity of any $l \in \Lambda_a$ is $\mathbb{1}_{\mathbb{Z}_+}(l) + 2\mathrm{card}(\{(p,n) \in \mathbb{N}\times\mathbb{Z}_+ \ : \ l = -(c^2+n)\})$ (in particular if a is not rational, the multiplicity of $l \in \Lambda_a$ is 1 or 2, according to $l \in \mathbb{Z}_+$ or not).

Remark 18 The above conclusions do not extend to the Gaussian framework, where one is rather interested in the (closure in $\mathbb{L}^2(\gamma_{b^{-1}} \otimes \gamma_a)$ of the) operator

$$\widetilde{L}_{a,b} \quad := \quad y\partial_x - bx\partial_y + a\partial_y^2 - y\partial_y,$$

where $a, b > 0$. As it will be seen in Section 4 (considering the scalings $x \mapsto x/\sqrt{a}$ and $y \mapsto y/\sqrt{a}$), $\widetilde{L}_{a,b}$ is diagonalizable only if $b \neq 1/4$ (for $b = 1/4$, Jordan blocks of all orders appear), while for $b > 1/4$, some of the eigenvalues are not real. In some sense, the appearance of complex eigenvalues facilitates the convergence to equilibrium (see the end of Section 5) and here we are far from this situation, if we look at L_a as an ersatz of $\widetilde{L}_{1,b}$ as $b \to 0_+$.

□

2.5 Link with hypocoercivity

The above spectral decomposition of L_a is not sufficient to deduce its hypocoercivity. More precisely, let $(P_t^{(a)})_{t\geqslant 0}$ be the Markovian semi-group associated to L_a in $\mathbb{L}^2(\mu_a)$, according to Hille-Yosida's theory [23]. Formally, we have for all $t \geqslant 0$, $P_t^{(a)} = \exp(tL_a)$, which corresponds to the evolution equation

$$\partial_t P_t^{(a)}(f) \quad = \quad P_t^{(a)}(L_a(f)),$$

valid at least for all f in the domain $\mathcal{D}(L_a)$ of L_a in $\mathbb{L}^2(\mu_a)$. Probability theory provides a regular version of this semi-group. Consider the stochastic differential equation in $\mathbb{T}\times\mathbb{R}$

$$\left\{\begin{array}{rcl} dX_t & = & Y_t\,dt \\ dY_t & = & -Y_t\,dt+\sqrt{2}dB_t, \end{array}\right. \tag{22}$$

where $(B_t)_{t\geqslant 0}$ is a standard real Brownian motion. Assume that initially (X_0,Y_0) takes the deterministic value $(x,y)\in\mathbb{T}\times\mathbb{R}$. It is well-known that the above stochastic differential equation admits a solution (which is almost surely (a.s.) unique with respect to the law of the Brownian motion, see for instance the book of Ikeda and Watanabe [13]). Then for any $t\geqslant 0$ and $f\in\mathbb{L}^2(\mu_a)$, we have μ_a-a.s. in $(x,y)\in\mathbb{T}\times\mathbb{R}$,

$$P_t^{(a)}(f)(x,y) \quad = \quad \mathbb{E}_{x,y}[f(X_t,Y_t)],$$

where the subscript x,y of the expectation indicates that we started with $(X_0,Y_0)=(x,y)$.
As already alluded to, hypocoercivity concerns the exponentially fast convergence of $(P_t^{(a)})_{t\geqslant 0}$ toward its equilibrium μ_a, here in $\mathbb{L}^2(\mu_a)$. It was proven that given $f\in\mathbb{L}^2(\mu_a)$, one can find two numbers $C(f)\geqslant 0$ and $\alpha>0$ such that

$$\forall\ t\geqslant 0,\qquad \left\|P_t^{(a)}(f)-\mu_a(f)\right\| \ \leqslant\ C(f)\exp(-\alpha t), \tag{23}$$

where $\|\cdot\|$ stands for the $\mathbb{L}^2(\mu_a)$ norm. The constant α depends on a but not on f, see for instance Villani [21] or Dolbeault, Mouhot and Schmeiser [4] for this kind of hypocoercive bounds.

A straightforward consequence of the spectral analysis of our simple model is that it is sufficient to study hypocoercivity on $\mathcal{V}_p$, for $p\in\mathbb{Z}_+$. Indeed, for $q\in\mathbb{Z}_+$, denote $\mathcal{U}_q$ the line in $\mathbb{L}^2(\mu_a)$ generated by $\varphi_0\otimes h_{q,a}$. The subspaces $\mathcal{U}_q$, $\mathcal{V}_p$, $\mathcal{W}_{p'}$, for $q\in\mathbb{Z}_+$ and $p,p'\in\mathbb{N}$ are mutually orthogonal and their Hilbert sum is equal to whole space $\mathbb{L}^2(\mu_a)$. If A is one of these subspaces, let $\Pi^{(A)}$ be the orthogonal projection on A and remark that $\Pi^{(A)}$ commutes with the elements of the semi-group. Denote by $(P_t^{(a,A)})_{t\geqslant 0}$ the semi-group generated by the restriction of L_a on A, we have for all $t\geqslant 0$, $P_t^{(a,A)}=P_t^{(a)}\Pi^{(A)}=\Pi^{(A)}P_t^{(a)}\Pi^{(A)}$. It follows that for any $t\geqslant 0$ and for any $f\in\mathbb{L}^2(\mu_a)$ with $\mu_a(f)=0$,

$$\left\|P_t^{(a)}(f)\right\|^2 \quad = \quad \sum_{q\in\mathbb{Z}_+}\left\|P_t^{(a,\mathcal{U}_q)}(f)\right\|^2+\sum_{p\in\mathbb{N}}\left\|P_t^{(a,\mathcal{V}_p)}(f)\right\|^2+\sum_{p'\in\mathbb{N}}\left\|P_t^{(a,\mathcal{W}_{p'})}(f)\right\|^2.$$

Since $\mu_a(f)=0$, we have $\Pi^{(\mathcal{U}_0)}(f)=0$. The other terms of the first sum are also easy to estimate:

$$\forall\ t\geqslant 0,\ \forall\ q\in\mathbb{N},\qquad P_t^{(a,\mathcal{U}_q)}(f) \quad = \quad \exp(-qt)\Pi^{(\mathcal{U}_q)}(f). \tag{24}$$

We deduce that for all $t\geqslant 0$,

$$\sum_{q\in\mathbb{Z}_+}\left\|P_t^{(a,\mathcal{U}_q)}(f)\right\|^2 \ \leqslant\ \exp(-t)\sum_{q\in\mathbb{N}}\left\|\Pi^{(\mathcal{U}_q)}(f)\right\|^2.$$

If we were able to estimate $\left\|P_t^{(a,\mathcal{V}_p)}(f)\right\|^2$, for $p\in\mathbb{N}$, then a similar bound would also be valid for $\left\|P_t^{(a,\mathcal{W}_p)}(f)\right\|^2$, because the action of $P_t^{(a,\mathcal{W}_p)}$ is isometrically conjugate to that of $P_t^{(a,\mathcal{V}_p)}$. Thus to deduce bounds such as (23), it is enough to know how to deal with the quantity $\left\|P_t^{(a,\mathcal{V}_p)}(f)\right\|^2$, for $p\in\mathbb{N}$ and $t\geqslant 0$.
This is not obvious, because the eigenvectors $(\xi_n)_{n\in\mathbb{Z}_+}$ of $P_t^{(a,\mathcal{V}_p)}$ (described in Theorem 6 and

Proposition 15) are not orthogonal. Indeed, we computed their scalar products in the proof of Proposition 11: for any $m, n \in \mathbb{Z}_+$,

$$\langle \xi_n, \xi_m \rangle \quad = \quad c^{n+m} \sum_{l \in [\![0,n]\!],\, k \in [\![0,m]\!]} \binom{n}{l}\binom{m}{k} c^{-2(l+k)} \mathbb{E}\left[N_{c^2}^{(l)} N_{c^2}^{(k)}\right] \ > \ 0, \tag{25}$$

where we recall that $c = \sqrt{ap}$, that N_{c^2} is a Poisson variable of parameter c^2 and where we used the notation

$$\forall\ n \in \mathbb{Z}_+, \forall\ m \in \mathbb{Z}_+, \qquad n^{(m)} \quad \coloneqq \quad n(n-1)\cdots(n-m+1). \tag{26}$$

To any function $f \in \mathcal{V}_p$, we can associate a sequence of coefficients $(f(n))_{n \in \mathbb{Z}_+}$ so that $f = \sum_{n \in \mathbb{Z}_+} f(n)\xi_n$ in $\mathbb{L}^2(\mu_a)$. Their interest is that for all $t \geqslant 0$,

$$P_t^{(a,\mathcal{V}_p)}(f) \quad = \quad \exp(-c^2 t) \sum_{n \in \mathbb{Z}_+} \exp(-nt) f(n) \xi_n,$$

and computations similar to those of the proof of Proposition 11 lead to

$$\begin{aligned} \left\| P_t^{(a,\mathcal{V}_p)}(f) \right\|^2 \quad &= \quad \exp(-2c^2 t) \sum_{m,n \in \mathbb{Z}_+} f(n) f(m) \exp(-(n+m)t) \langle \xi_n, \xi_m \rangle \\ &= \quad \exp(-2c^2 t) \mathbb{E}[G_t^2(N_{c^2})], \end{aligned}$$

where

$$\forall\ t \geqslant 0,\, n \in \mathbb{Z}_+, \qquad G_t(n) \quad \coloneqq \quad \left(1 + \frac{1}{c\exp(t)} \frac{d}{dX}\right)^n \sum_{m \in \mathbb{Z}_+} f(m) X^m \Bigg|_{X = c\exp(-t)}.$$

Unfortunately we have not been able to directly relate this quantity and the same expression at time $t = 0$. This is why we develop another approach in the next section, where the important role will rather be played by "Poisson distributions with negative parameters".

3 Computation of $\mathbb{L}^2$-operator norms

Our purpose here is to prove Theorem 1. From the considerations of the end of last section, this requires to compute the operator norm of $P_t^{(a,\mathcal{V}_p)}$ in $\mathbb{L}^2(\mu_a)$, for any given $a > 0$, $p \in \mathbb{N}$ and $t \geqslant 0$.

Indeed, remark that

Lemma 19 *For any $a > 0$ and $t \geqslant 0$, we have*

$$|\!|\!| P_t^{(a)} - \mu_a |\!|\!| \quad = \quad \max\left(\exp(-t), \max_{p \in \mathbb{N}} |\!|\!| P_t^{(a,\mathcal{V}_p)} |\!|\!| \right).$$

Proof

From the orthogonality of the subspaces $\mathcal{U}_q$, $\mathcal{V}_p$, $\mathcal{W}_{p'}$, for $q \in \mathbb{Z}_+$ and $p, p' \in \mathbb{N}$, and from their stability by the operators $P_t^{(a)}$, for all $t \geqslant 0$, we get

$$\begin{aligned} |\!|\!| P_t^{(a)} - \mu_a |\!|\!| \quad &= \quad |\!|\!| P_t^{(a)} - P_t^{(a,\mathcal{U}_0)} |\!|\!| \\ &= \quad \max\left(\sup_{q \in \mathbb{N}} |\!|\!| P_t^{(a,\mathcal{U}_q)} |\!|\!|, \sup_{p \in \mathbb{N}} |\!|\!| P_t^{(a,\mathcal{V}_p)} |\!|\!|, \sup_{p' \in \mathbb{N}} |\!|\!| P_t^{(a,\mathcal{W}_{p'})} |\!|\!| \right), \end{aligned}$$

where we also used that $\mu_a = \Pi^{(\mathcal{U}_0)} = P_t^{(a,\mathcal{U}_0)}$ where μ_a is seen as an endomorphism of $\mathbb{L}^2(\mu_a)$. From (24), we deduce that

$$\forall\ t \geqslant 0,\ \forall\ q \in \mathbb{N}, \qquad |||P_t^{(a,\mathcal{U}_q)}||| \quad = \quad \exp(-qt),$$

and by conjugacy we have

$$\forall\ t \geqslant 0,\ \forall\ p \in \mathbb{N}, \qquad |||P_t^{(a,\mathcal{V}_p)}||| \quad = \quad |||P_t^{(a,\mathcal{W}_p)}|||.$$

■

3.1 Lower bound of $|||P_t^{(a,\mathcal{V}_p)}|||$

So let $a > 0$ and $p \in \mathbb{N}$ be fixed and denote again $c \coloneqq \sqrt{a}p$. By isometry, for any $t > 0$, the operator norm of $P_t^{(a,\mathcal{V}_p)}$ in $\mathbb{L}^2(\mu_a)$ coincides with that of $\exp(tM)$ in $l^2(\mathbb{Z}_+)$, where M is defined in (6). We have seen in Theorem 6 that the spectrum of M consists of the sequence $(-c^2 - n)_{n\in\mathbb{Z}_+}$ and that a corresponding Hilbert basis of eigenvectors is $(\xi_n)_{n\in\mathbb{Z}_+}$, where

$$\forall\ n \in \mathbb{Z}_+,\ \forall\ q \in \mathbb{Z}_+, \qquad \xi_n(q) \quad = \quad c^n \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} q^{(l)} (-1)^{\lfloor \frac{q+1}{2} \rfloor} \frac{c^q}{\sqrt{q!}} \exp(-c^2/2),$$

(see (12)). We have already computed their scalar product in (25), but let us give another expression which will be more convenient.

Lemma 20 *We have for all $n, m \in \mathbb{Z}_+$,*

$$\langle \xi_n, \xi_m \rangle \quad = \quad (2c)^{n+m} \exp((4c^2)^{-1}) \mathbb{E}[n^{(N_{1/(4c^2)})} m^{(N_{1/(4c^2)})}]$$

where $N_{1/(4c^2)}$ is a Poisson random variable of parameter $1/(4c^2)$.

Proof

We have seen in (25) that for all $n, m \in \mathbb{Z}_+$,

$$\langle \xi_n, \xi_m \rangle \quad = \quad c^{n+m} \sum_{p\in\mathbb{Z}_+} \sum_{l\in[\![0,n]\!],\, k\in[\![0,m]\!]} \binom{n}{l} \binom{m}{k} p^{(l)} p^{(k)} \frac{c^{-2(l+k)+2p}}{p!} \exp(-c^2).$$

To go further, let us introduce two free variables X and Y and interpret in the above formula, for $p, l, k \in \mathbb{Z}_+$,

$$p^{(l)} \;=\; \left.\frac{d^l}{dX^l} X^p \right|_{X=1} \qquad \text{and} \qquad p^{(k)} \;=\; \left.\frac{d^k}{dY^k} Y^p \right|_{Y=1}.$$

We are thus lead to compute at $X = 1 = Y$ the expression

$$\begin{aligned}
& c^{n+m}\exp(-c^2) \sum_{l\in[\![0,n]\!],\, k\in[\![0,m]\!]} \binom{n}{l}\binom{m}{k} c^{-2(l+k)} \frac{d^{l+k}}{dX^l dY^k} \sum_{p\in\mathbb{Z}_+} X^p Y^p \frac{c^{2p}}{p!} \\
= \quad & c^{n+m}\exp(-c^2) \sum_{l\in[\![0,n]\!],\, k\in[\![0,m]\!]} \binom{n}{l}\binom{m}{k} c^{-2(l+k)} \frac{d^{l+k}}{dX^l dY^k} \exp(c^2XY) \\
= \quad & c^{n+m}\exp(-c^2) \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} \frac{d^l}{dX^l} \left(1 + \frac{1}{c^2}\frac{d}{dY}\right)^m \exp(c^2XY) \\
= \quad & c^{n+m}\exp(-c^2) \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} \frac{d^l}{dX^l} (1+X)^m \exp(c^2XY) \\
= \quad & c^{n+m}\exp(-c^2) \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} \sum_{q\in[\![0,l]\!]} \binom{l}{q} \left(\frac{d^q}{dX^q}(1+X)^m\right)\left(\frac{d^{l-q}}{dX^{l-q}}\exp(c^2XY)\right) \\
= \quad & c^{n+m}\exp(-c^2) \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} \sum_{q\in[\![0,l]\!]} \binom{l}{q} m^{(q)}(1+X)^{m-q}(c^2Y)^{l-q}\exp(c^2XY).
\end{aligned}$$

For $X = 1 = Y$, we get

$$\begin{aligned}
\langle \xi_n, \xi_m\rangle \quad = \quad & c^{n+m} \sum_{l\in[\![0,n]\!]} \binom{n}{l} c^{-2l} \sum_{q\in[\![0,l]\!]} \binom{l}{q} m^{(q)} 2^{m-q} c^{2(l-q)} \\
= \quad & 2^m c^{n+m} \sum_{l,q\in\mathbb{Z}_+} \binom{n}{l}\binom{l}{q} (2c^2)^{-q} m^{(q)}.
\end{aligned}$$

Interpreting again $m^{(q)}$ as $\frac{d^q}{dX^q}X^m\big|_{X=1}$, we have

$$\begin{aligned}
\sum_{q\in\mathbb{Z}_+} \binom{l}{q} (2c^2)^{-q} m^{(q)} \quad = \quad & \sum_{q\in\mathbb{Z}_+} \binom{l}{q} (2c^2)^{-q} \frac{d^q}{dX^q} X^m \Big|_{X=1} \\
= \quad & \left(1 + \frac{1}{2c^2}\frac{d}{dX}\right)^l X^m \Big|_{X=1},
\end{aligned}$$

so that

$$\begin{aligned}
\langle \xi_n, \xi_m\rangle \quad = \quad & 2^m c^{n+m} \sum_{l\in\mathbb{Z}_+} \binom{n}{l} \left(1 + \frac{1}{2c^2}\frac{d}{dX}\right)^l X^m \Big|_{X=1} \\
= \quad & 2^m c^{n+m} \left(2 + \frac{1}{2c^2}\frac{d}{dX}\right)^n X^m \Big|_{X=1} \\
= \quad & (2c)^{n+m} \left(1 + \frac{1}{4c^2}\frac{d}{dX}\right)^n X^m \Big|_{X=1} \\
= \quad & (2c)^{n+m} \sum_{l\in\mathbb{Z}_+} \binom{n}{l} \frac{1}{(4c^2)^l} \frac{d^l}{dX^l} X^m \Big|_{X=1} \\
= \quad & (2c)^{n+m} \sum_{l\in\mathbb{Z}_+} \binom{n}{l} \frac{1}{(4c^2)^l} m^{(l)} \\
= \quad & (2c)^{n+m} \sum_{l\in\mathbb{Z}_+} n^{(l)} m^{(l)} \frac{(4c^2)^{-l}}{l!}.
\end{aligned}$$

This expression can be written under the form given in the above lemma.

■

This formulation enables to compute quite efficiently the norm of $\exp(tM)z$ for z in a dense subspace of $l^2(\mathbb{Z}_+)$. It is based on the remark that $\mathbb{E}[N^{(n)}] = \rho^n$ if $n \in \mathbb{Z}_+$ and N is a Poisson distribution of parameter $\rho > 0$. In fact we will also use the underlying computation with negative ρ:

Lemma 21 *Let $\widetilde{\rho}, \widehat{\rho} \in \mathbb{R}$ be given and consider $\widetilde{z} := \sum_{n\in\mathbb{Z}_+} \widetilde{f}(n)\xi_n$ and $\widehat{z} := \sum_{n\in\mathbb{Z}_+} \widehat{f}(n)\xi_n$, where for all $n \in \mathbb{Z}_+$,*

$$\widetilde{f}(n) \ := \ \frac{\widetilde{\rho}^n}{n!} \qquad \text{and} \qquad \widehat{f}(n) \ := \ \frac{\widehat{\rho}^n}{n!}.$$

Then we have

$$\langle \widetilde{z}, \widehat{z} \rangle \ = \ \exp(\widetilde{\rho}\widehat{\rho} + 2c(\widetilde{\rho} + \widehat{\rho})).$$

Proof

To justify the absolute convergence in the following computations, they should be first considered with $\widetilde{\rho}$ and $\widehat{\rho}$ replaced by $|\widetilde{\rho}|$ and $|\widehat{\rho}|$.
According to Lemma 20, we have

$$\begin{aligned}
\langle \widetilde{z}, \widehat{z} \rangle &= \sum_{n,m\in\mathbb{Z}_+} \widetilde{f}(n)\widehat{f}(m)\langle \xi_n, \xi_m \rangle \\
&= \sum_{n,m\in\mathbb{Z}_+} \widetilde{f}(n)\widehat{f}(m)(2c)^{n+m} \exp((4c^2)^{-1})\mathbb{E}[n^{(N_{1/(4c^2)})} m^{(N_{1/(4c^2)})}] \\
&= \exp((4c^2)^{-1})\mathbb{E}\left[\left(\sum_{n\in\mathbb{Z}_+} (2c)^n \widetilde{f}(n) n^{(N_{1/(4c^2)})}\right)\left(\sum_{m\in\mathbb{Z}_+} (2c)^m \widehat{f}(m) m^{(N_{1/(4c^2)})}\right)\right],
\end{aligned}$$

where $N_{1/(4c^2)}$ is still a Poisson random variable of parameter $1/(4c^2)$.
For any fixed $N \in \mathbb{Z}_+$, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}_+} (2c)^n \widetilde{f}(n) n^{(N)} &= \sum_{n\in\mathbb{Z}_+} (2c)^n n^{(N)} \frac{\widetilde{\rho}^n}{n!} \\
&= \sum_{n\in\mathbb{Z}_+, n\geqslant N} (2c)^n \frac{\widetilde{\rho}^n}{(n-N)!} \\
&= (2c\widetilde{\rho})^N \sum_{n\in\mathbb{Z}_+} \frac{(2c)^n \widetilde{\rho}^n}{n!} \\
&= (2c\widetilde{\rho})^N \exp(2c\widetilde{\rho}).
\end{aligned}$$

Thus it appears that

$$\begin{aligned}
\langle \widetilde{z}, \widehat{z} \rangle &= \exp((4c^2)^{-1}) \exp(2c(\widetilde{\rho} + \widehat{\rho}))\mathbb{E}\left[(4c^2\widetilde{\rho}\widehat{\rho})^{N_{1/(4c^2)}}\right] \\
&= \exp(2c(\widetilde{\rho} + \widehat{\rho})) \exp(\widetilde{\rho}\widehat{\rho}).
\end{aligned}$$

■

In particular, if $z \in l^2(\mathbb{Z}_+)$ is given by $z = \sum_{n\in\mathbb{Z}_+} \frac{\rho^n}{n!}\xi_n$, with $\rho \in \mathbb{R}$, we get

$$\|z\|^2 \ = \ \exp\left(\rho^2 + 4c\rho\right).$$

The interest of this formula is that for such z, we have

$$\forall\ t \geqslant 0, \qquad \exp(tM)z \quad = \quad \exp(-c^2 t)\sum_{n\in\mathbb{Z}_+}\exp(-nt)\frac{\rho^n}{n!}\xi_n, \tag{27}$$

so that

$$\forall\ t \geqslant 0, \qquad \|\exp(tM)z\|^2 \quad = \quad \exp(-2c^2t)\exp\left(\exp(-2t)\rho^2 + 4\exp(-t)c\rho\right).$$

We deduce a lower bound on the operator norm $|||\exp(tM)|||$ in $l^2(\mathbb{Z}_+)$:

Lemma 22 *For any $t \geqslant 0$, we have*

$$|||\exp(tM)||| \quad \geqslant \quad \exp\left(-c^2\left(t - 2\frac{1-\exp(-t)}{1+\exp(-t)}\right)\right).$$

Proof

Since by definition, for any $t \geqslant 0$,

$$|||\exp(tM)||| \quad \coloneqq \quad \sup_{z\in l^2(\mathbb{Z}_+)\setminus\{0\}}\frac{\|\exp(tM)z\|}{\|z\|},$$

we deduce from the above computations that

$$\begin{aligned}
|||\exp(tM)|||^2 \quad &\geqslant \quad \exp(-2c^2t)\sup_{\rho\in\mathbb{R}}\exp\left((\exp(-2t)-1)\rho^2 + 4(\exp(-t)-1)c\rho\right) && (28)\\
&= \quad \exp(-2c^2t)\exp\left(\sup_{\rho\in\mathbb{R}}\left((\exp(-2t)-1)\rho^2 + 4(\exp(-t)-1)c\rho\right)\right)\\
&= \quad \exp(-2c^2t)\exp\left(-\frac{4(\exp(-t)-1)^2c^2}{\exp(-2t)-1}\right)\\
&= \quad \exp(-2c^2t)\exp\left(\frac{4c^2(1-\exp(-t))}{\exp(-t)+1}\right).
\end{aligned}$$

■

3.2 Upper bound of $|||P_t^{(a,\mathcal{V}_p)}|||$

To get a matching upper bound of $|||\exp(tM)|||$, consider the subspace $\mathcal{Z}$ of $z \in l^2(\mathbb{Z}_+)$ which are finite linear combinaisons of vectors of the previous type, namely that can be written under the form

$$z \quad = \quad \sum_{n\in\mathbb{Z}_+}\sum_{l\in[\![r]\!]}\nu_l\frac{\rho_l^n}{n!}\xi_n, \tag{29}$$

where $r \in \mathbb{N}$ and ν_l, ρ_l are real numbers, for $l \in [\![r]\!]$.

Lemma 23 *The subspace $\mathcal{Z}$ is dense in $l^2(\mathbb{Z}_+)$.*

Proof

Consider $z \in l^2(\mathbb{Z}_+)$ orthogonal to $\mathcal{Z}$, we want to show that $z = 0$. The orthogonality of z to $\mathcal{Z}$ is equivalent to the fact that for any $\rho \in \mathbb{R}$,

$$\left\langle z, \sum_{n\in\mathbb{Z}_+}\frac{\rho^n}{n!}\xi_n\right\rangle \quad = \quad 0$$

This means that the series $\sum_{n\in\mathbb{Z}_+} \frac{\langle z,\xi_n\rangle}{n!}\rho^n$ vanishes for all real values of ρ and it is possible only if

$$\forall\ n \in \mathbb{Z}_+, \qquad \langle z, \xi_n \rangle \quad = \quad 0.$$

But we have seen in Proposition 6 that this implies that $z = 0$.

■

The previous result suggests that any $z \in l^2(\mathbb{Z}_+)$ can be written under the form

$$z \quad = \quad \sum_{n\in\mathbb{Z}_+} \int \frac{\rho^n}{n!}\nu(d\rho)\xi_n, \tag{30}$$

for an appropriate signed measure ν on $\mathbb{R}$. But we won't push the investigation in this direction (see also the last remark of this section), since what is interesting for us is that by density

$$\forall\ t \geqslant 0, \qquad \|\exp(tM)\| \quad = \quad \sup_{z\in\mathcal{Z}\setminus\{0\}} \frac{\|\exp(tM)z\|}{\|z\|}, \tag{31}$$

and that the norm $\|z\|$ can be computed for $z \in \mathcal{Z}$:

Lemma 24 *Let $z \in \mathcal{Z}$ be given by (29).Then we have*

$$\|z\|^2 \quad = \quad \nu' A(\rho)\nu,$$

where ν (respectively ν') is the column (resp. line) vector of coordinates $(\nu_l)_{l\in[\![r]\!]}$ and where $A(\rho)$ is the $r \times r$-matrix given by

$$\forall\ k, l \in [\![r]\!], \qquad A_{k,l}(\rho) \quad \coloneqq \quad \exp\left(\rho_k\rho_l + 2c(\rho_k + \rho_l)\right).$$

Proof

This is an immediate computation: let us denote for $l \in [\![r]\!]$,

$$z_l \quad \coloneqq \quad \sum_{n\in\mathbb{Z}_+} \frac{\rho_l^n}{n!}\xi_n,$$

so that $z = \sum_{l\in[\![r]\!]} \nu_l z_l$ and

$$\begin{aligned} \langle z, z \rangle \quad &= \quad \sum_{l,k\in[\![r]\!]} \nu_l\nu_k \langle z_l, z_k \rangle \\ &= \quad \sum_{l,k\in[\![r]\!]} \nu_l A_{l,k}(\rho)\nu_k, \end{aligned}$$

according to Lemma 21.

■

The advantage of the decomposition (29) is that it well-behaves under the action of the semi-group under consideration:

$$\forall\ t \geqslant 0, \qquad \exp(tM)z \quad = \quad \exp(-c^2t) \sum_{n\in\mathbb{Z}_+} \sum_{l\in[\![r]\!]} \nu_l \frac{(\exp(-t)\rho_l)^n}{n!}\xi_n.$$

The above lemma then implies that

$$\langle \exp(tM)z, \exp(tM)z \rangle \quad = \quad \exp(-2c^2t)\nu' A(\exp(-t)\rho)\nu. \tag{32}$$

To treat the r.h.s., we need the following result.

Lemma 25 *For any $t \geqslant 0$, any $r \in \mathbb{N}$ and any $\nu = (\nu_k)_{k\in[\![r]\!]}, \rho = (\rho_k)_{k\in[\![r]\!]} \in \mathbb{R}^r$, we have*

$$\nu' A(\exp(-t)\rho)\nu \quad \leqslant \quad \exp\left(-4c^2 \frac{1-\exp(-t)}{1+\exp(-t)}\right) \nu' A(\rho)\nu.$$

Proof

Fix the dimension $r \in \mathbb{N}$ and the time $t > 0$ (for $t = 0$ the announced result is trivial), and consider

$$\rho_0 \quad := \quad -\frac{2c}{1+e^{-t}}, \tag{33}$$

which is the maximizer in (28) (we omit the dependence on t in the sequel). Define next the vector $h = (h_k)_{k\in[\![r]\!]} \in \mathbb{R}^r$ by

$$\forall\ k \in [\![r]\!], \qquad h_k \quad := \quad \rho_k - \rho_0,$$

where $\rho = (\rho_k)_{k\in[\![r]\!]} \in \mathbb{R}^r$ is a vector given as in the statement of the lemma. We compute that for any $k, l \in [\![r]\!]$,

$$e^{-2t}\rho_k\rho_l + 2e^{-t}c(\rho_k + \rho_l) \quad = \quad -4c^2 e^{-t}\frac{2+e^{-t}}{(1+e^{-t})^2} + \frac{2e^{-t}c}{1+e^{-t}}(h_k + h_l) + e^{-2t}h_k h_l,$$

and

$$\rho_k\rho_l + 2c(\rho_k + \rho_l) \quad = \quad -4c^2 \frac{1+2e^{-t}}{(1+e^{-t})^2} + \frac{2e^{-t}c}{1+e^{-t}}(h_k + h_l) + h_k h_l.$$

Note that the terms $h_k + h_l$ have the same factor in the two last expressions. This leads us to introduce the vector η whose coordinates are given by

$$\forall\ k \in [\![r]\!], \qquad \eta_k \quad := \quad \exp\left(\frac{2e^{-t}}{1+e^{-t}} h_k\right) \nu_k,$$

so that we can write

$$\begin{aligned}
\nu' A(e^{-t}\rho)\nu \quad &= \quad \exp\left(-4c^2 e^{-t}\frac{2+e^{-t}}{(1+e^{-t})^2}\right) \eta' B(e^{-t}h)\eta \\
\nu' A(\rho)\nu \quad &= \quad \exp\left(-4c^2 \frac{1+2e^{-t}}{(1+e^{-t})^2}\right) \eta' B(h)\eta,
\end{aligned}$$

where $B(h)$ is the $r \times r$-matrix given by

$$\forall\ k, l \in [\![r]\!], \qquad B_{k,l}(h) \quad := \quad \exp(h_k h_l).$$

Since

$$-4c^2 \frac{1+2e^{-t}}{(1+e^{-t})^2} + 4c^2 e^{-t}\frac{2+e^{-t}}{(1+e^{-t})^2} \quad = \quad -4c^2 \frac{1-e^{-t}}{1+e^{-t}},$$

it remains to prove that for any $\eta = (\eta_k)_{k\in[\![r]\!]} \in \mathbb{R}^r$ and any $h = (h_k)_{k\in[\![r]\!]} \in \mathbb{R}^r$,

$$\eta' B(e^{-t}h)\eta \quad \leqslant \quad \eta' B(h)\eta.$$

To get this bound, it is sufficient to expand these expressions:

$$\begin{aligned}
\eta' B(e^{-t}h)\eta &= \sum_{k,l\in[\![r]\!]} \exp(e^{-2t}h_k h_l)\eta_k\eta_l \\
&= \sum_{k,l\in[\![r]\!]}\sum_{n\in\mathbb{Z}_+} \frac{e^{-2nt}}{n!}(h_k h_l)^n \eta_k\eta_l \\
&= \sum_{n\in\mathbb{Z}_+} \frac{e^{-2nt}}{n!}\left(\sum_{k\in[\![r]\!]} h_k^n\eta_k\right)^2 \\
&\leqslant \sum_{n\in\mathbb{Z}_+} \frac{1}{n!}\left(\sum_{k\in[\![r]\!]} h_k^n\eta_k\right)^2 \\
&= \eta' B(h)\eta.
\end{aligned}$$

■

Coming back to (31) and (32), we get that for any $t \geqslant 0$,

$$\|\!| \exp(tM) |\!\|\ \leqslant\ \exp\left(-c^2\left(t-2\frac{1-\exp(-t)}{1+\exp(-t)}\right)\right),$$

and in conjunction with Lemma 22, it follows that

$$\|\!| \exp(tM) |\!\|\ =\ \exp\left(-c^2\left(t-2\frac{1-\exp(-t)}{1+\exp(-t)}\right)\right).$$

Coming back to the notations of the beginning of this section, we have that for any $a > 0$, any $p \in \mathbb{N}$ and any $t \geqslant 0$,

$$\|\!| P_t^{(a,\mathcal{V}_p)} |\!\|\ =\ \exp\left(-ap^2\left(t-2\frac{1-\exp(-t)}{1+\exp(-t)}\right)\right).$$

Injecting this quantity in Lemma 19, Theorem 1 is proved.

3.3 Final remarks

To finish this section, let us make explicit the functions for which the operator norms of the semi-group are reached. It will appear a posteriori that there is a faster way to justify the introduction of such functions of the form presented in Lemma 21.

From the above computations, it follows that if $t > 0$ is such that $\|\!| P_t^{(a)} - \mu_a |\!\| = \exp(-t)$, then any element of $\mathcal{U}_1\backslash\{0\}$ is a maximizing function for the computation of $\|\!| P_t^{(a)} - \mu_a |\!\|$, for instance the mapping $\mathbb{T}\times\mathbb{R} \ni (x,y) \mapsto y$.
This no longer true if $t > 0$ is such that $\|\!| P_t^{(a)} - \mu_a |\!\| > \exp(-t)$, in which case $z_t \coloneqq \sum_{n\in\mathbb{Z}_+} \frac{\rho_t^n}{n!}\xi_n \in \mathcal{V}_1$ is a maximizing function, where $\rho_t \coloneqq -2\sqrt{a}(1+\exp(-t))^{-1}$ is the quantity defined in (33) when $p = 1$ (let $z'_t \coloneqq \sum_{n\in\mathbb{Z}_+} \frac{\rho_t^n}{n!}\xi'_n \in \mathcal{W}_1$, where the ξ'_n are defined in Proposition 17, with $p = 1$, then any non-null linear combination of z_t and z'_t is also maximizing). So let us compute z_t and more generally:

Lemma 26 *For any $p \in \mathbb{N}$ and $\rho \in \mathbb{R}$, consider $z \coloneqq \sum_{n\in\mathbb{Z}_+} \frac{\rho^n}{n!}\xi_n \in \mathcal{V}_p$. Then we have, almost everywhere in $(x,y) \in \mathbb{T}\times\mathbb{R}$,*

$$z(x,y)\ =\ \frac{2^p p!}{\sqrt{(2p)!}}\exp\left(2\sqrt{a}p\rho + \frac{\rho^2}{2}\right)\cos\left(\frac{\rho y}{\sqrt{a}} + p(x+y)\right).$$

Proof

Recall from Proposition 17, that almost everywhere in $(x, y) \in \mathbb{T} \times \mathbb{R}$, $\xi_n(x, y)$ is the real part of

$$\sum_{l \in [\![0,n]\!]} \binom{n}{l} l!(2c)^{n-l} \frac{h_{l,a}(y)}{\sqrt{l!}} i^l \exp(ip(x+y)),$$

where $c = \sqrt{a}p$. We deduce that $z(x, y)$ is a.e. the real part of

$$\begin{aligned}
&\frac{2^p p!}{\sqrt{(2p)!}} \exp(ip(x+y)) \sum_{n \in \mathbb{Z}_+} \rho^n \sum_{l \in [\![0,n]\!]} \frac{(2c)^{n-l}}{(n-l)!} i^l \frac{h_{l,a}(y)}{\sqrt{l!}} \\
&= \frac{2^p p!}{\sqrt{(2p)!}} \exp(ip(x+y)) \sum_{l \in \mathbb{Z}_+} i^l \frac{h_{l,a}(y)}{\sqrt{l!}} \sum_{n \geqslant l} \frac{(2c)^{n-l}}{(n-l)!} \rho^n \\
&= \frac{2^p p!}{\sqrt{(2p)!}} \exp(ip(x+y)) \sum_{l \in \mathbb{Z}_+} (i\rho)^l \frac{h_{l,a}(y)}{\sqrt{l!}} \exp(2c\rho) \\
&= \frac{2^p p!}{\sqrt{(2p)!}} \exp(ip(x+y)) \exp(2c\rho) \exp\left(\frac{i\rho y}{\sqrt{a}} + \frac{\rho^2}{2}\right),
\end{aligned}$$

where (21) was taken into account.

■

Thus, when $\|P_t^{(a)} - \mu_a\| > \exp(-t)$, functions proportional to

$$\mathbb{T} \times \mathbb{R} \ni (x, y) \quad \mapsto \quad \exp\left(-\frac{2iy}{1+\exp(-t)} + i(x+y)\right),$$

are maximizers for the computation of $\|P_t^{(a)} - \mu_a\|$ in the complexified $\mathbb{L}^2(\mu_a)$.
Lemma 26 leads us to consider for any $p \in \mathbb{N}$ and for any $\rho \in \mathbb{R}$ the mapping

$$F_{p,\rho} \;:\; \mathbb{T} \times \mathbb{R} \ni (x, y) \quad \mapsto \quad \exp(i\rho y + ip(x+y)).$$

If $\mathbb{R}_+ \ni t \mapsto \rho_t \in \mathbb{R}$ is a smooth function, define

$$\forall\ t \geqslant 0,\ \forall\ (x, y) \in \mathbb{T} \times \mathbb{R}, \qquad G_t(x, y) \quad \coloneqq \quad F_{p,\rho_t}(x, y),$$

we compute that

$$\partial_t G_t(x, y) + L_a G_t(x, y) \quad = \quad (i(\rho_t' - \rho_t)y - a(p + \rho_t)^2) G_t(x, y).$$

Thus if we choose $\rho_t \coloneqq \exp(t)\rho$ for given $\rho \in \mathbb{R}$ and all $t \geqslant 0$, we get

$$\partial_t P_t^{(a)}[G_t] \quad = \quad -a(p + \exp(t)\rho)^2 P_t^{(a)}[G_t],$$

whose integration leads to

$$\forall\ t \geqslant 0, \qquad P_t^{(a)}[G_t] \quad = \quad \exp(-a[\rho^2(e^{2t} - 1)/2 + 2p\rho(e^t - 1) + p^2 t]) G_0.$$

This formula can be rewritten under the form

$$P_t^{(a)}[F_{p,\rho}] \quad = \quad \exp(-ap^2 t) \exp(-a[\rho^2(1 - e^{-2t})/2 + 2p\rho(1 - e^{-t})]) F_{p,e^{-t}\rho},$$

and via Lemma 26, this corresponds to (27).
From here it is possible to follow our previous arguments (computing instead $\mu_a[F_{p,\widetilde{\rho}} F_{p,\widehat{\rho}}]$ for $\widetilde{\rho}, \widehat{\rho} \in \mathbb{R}$, namely values of the characteristic function associated to μ_a) to get the same proof of

Theorem 1. So the above manipulations of functions of the form $F_{p,\rho}$ are a short way to avoid the spectral decomposition of L_a.
This approach could also be considered in our second model (or for some quadratic symbol operators), but it would be more tricky, because in the end the corresponding maximizing functions for the computation of the operator norms will be linear and not of the form $\mathbb{R}\times\mathbb{R}\ni(x,y)\mapsto\exp(i\alpha x+i\beta y)$, where $\alpha,\beta\in\mathbb{R}$.

Remark 27 After some elementary manipulations, Lemma 26 enables to translate the question asked around (30) into the following one: is there a set $\mathcal{M}$ of complex measures on $\mathbb{R}$ such that the relation

$$\forall\, y\in\mathbb{R},\qquad f(y)\quad=\quad\int_{\mathbb{R}}\exp(iy\rho)\,\nu(d\rho),$$

induces a bijection between functions $f\in\mathbb{L}^2(\mu_a)$ (complexified) and measures $\nu\in\mathcal{M}$? Whatever it is, $\mathcal{M}$ contains all finite linear combinations of Dirac masses. Note that since $\mathbb{L}^2(\mu_a)$ is not included in the space $\mathcal{S}'$ of tempered distributions, the usual Fourier transform in $\mathcal{S}'$ does not give the answer.

□

4 Spectral decomposition of the Gaussian case

We treat here the spectral decomposition of our second model. Despite it is already known (see for instance Risken [19]), we will proceed differently, rather following an approach based on a decomposition of the generator similar to our roadmap used in Section 2. Apart from underlying the analogies and differences between our two models, this will put us in good position to compute the operators norms.

4.1 Decomposition of the generator on stable subspaces

So for fixed $a>0$, we are interested in the operator $\widetilde{L}_a$ defined in (2). Since the coefficients of $\widetilde{L}_a$ are affine and the associated invariant measure $\widetilde{\mu}_a=\gamma_{1/a}\otimes\gamma$ is Gaussian, it is natural to check how $\widetilde{L}_a$ acts on the Hermite polynomials, renormalized to be orthogonal in $\mathbb{L}^2(\widetilde{\mu}_a)$. The definition of the orthogonal polynomials associated to γ were recalled in (5). To simplify notations, we sightly modify those adopted in Section 2 and rather consider

$$\forall\, p\in\mathbb{N},\,\forall\, x\in\mathbb{R},\qquad h_{p,a}(y)\quad\coloneqq\quad h_p(\sqrt{a}x).$$

The family $(h_{p,a}\otimes h_q)_{p,q\in\mathbb{N}}$ is then an orthogonal basis of $\mathbb{L}^2(\widetilde{\mu}_a)$. In analogy with Lemma 5, we begin by

Lemma 28 *For all $p,q\in\mathbb{N}$, we have*

$$\widetilde{L}_a[h_{p,a}\otimes h_q]\quad=\quad\sqrt{a}ph_{p-1,a}\otimes h_{q+1}-\sqrt{a}qh_{p+1,a}\otimes h_{q-1}-qh_{p,a}\otimes h_q.$$

Proof

Taking into account the following classical relations, valid for all $q\in\mathbb{N}$ and $y\in\mathbb{R}$ (with the convention $h_{-1}=0$),

$$\begin{aligned}h_q''(y)-yh_q'(y)&=-qh_q(y)\\ h_q'(y)&=qh_{q-1}(y)\\ h_{q+1}(y)&=yh_q(y)-qh_{q-1}(y),\end{aligned}$$

we compute that for all $p, q \in \mathbb{N}$ and $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
&\widetilde{L}_a[h_{p,a} \otimes h_q](x,y) \\
&= y\sqrt{a}ph_{p-1,a}(x)h_q(y) - axqh_{p,a}(x)h_{q-1}(y) - qh_{p,a}(x)h_q(y) \\
&= \sqrt{a}ph_{p-1,a}(x)(h_{q+1} + qh_{q-1})(y) - \sqrt{a}q(h_{p+1,a} + ph_{p-1,a})(x)h_{q-1}(y) - qh_{p,a}(x)h_q(y) \\
&= (\sqrt{a}ph_{p-1,a} \otimes h_{q+1} - \sqrt{a}qh_{p+1,a} \otimes h_{q-1} - qh_{p,a} \otimes h_q)(x,y).
\end{aligned}$$

■

This formula leads us to introduce, for $n \in \mathbb{N}$, the subspace H_n of $\mathbb{L}^2(\widetilde{\mu}_a)$ generated by $h_{p,a} \otimes h_{n-p}$, for $p \in [\![0,n]\!]$. Indeed, H_n is left invariant by $\widetilde{L}_a$. Let us consider the matrix $\widetilde{M}_n$ associated to the restriction of $\widetilde{L}_a$ to H_n in the orthonormal basis $(h_{p,a} \otimes h_{n-p})_{p\in[\![0,n]\!]}$. It is the tridiagonal matrix given by

$$\widetilde{M}_n = \begin{pmatrix} -n & \sqrt{an} & 0 & \cdots & 0 \\ -\sqrt{an} & -(n-1) & \sqrt{a2(n-1)} & & \vdots \\ 0 & -\sqrt{a2(n-1)} & -(n-2) & \ddots & 0 \\ & & \ddots & \ddots & \sqrt{an} \\ 0 & \cdots & 0 & -\sqrt{an} & 0 \end{pmatrix}. \tag{34}$$

In order to diagonalize this matrix, it is fruitful to decompose it into its diagonal , above-diagonal and below-diagonal parts, i.e. $\widetilde{M}_n = \widetilde{D}_n + \sqrt{a}S_n - \sqrt{a}S_n^*$, with

$$\forall\ p,q \in [\![0,n]\!], \qquad \widetilde{D}_n(p,q) := \begin{cases} -(n-p) & \text{, if } p = q \\ 0 & \text{, otherwise.} \end{cases}$$

$$\forall\ p,q \in [\![0,n]\!], \qquad S_n(p,q) := \begin{cases} \sqrt{(p+1)(n-p)} & \text{, if } q = p+1 \\ 0 & \text{, otherwise.} \end{cases}$$

and where S_n^* stands here for the transposed matrix associated to S_n.
The next point is crucial to understand the spectral structure of $\widetilde{M}_n$:

Lemma 29 *For any $n \in \mathbb{N}$, the commutators of S_n, S_n^* and $\widetilde{D}_n$ are given by*

$$\begin{aligned}
[S_n, S_n^*] &= -2\widetilde{D}_n - nI_n \\
[S_n, \widetilde{D}_n] &= S_n \\
[S_n^*, \widetilde{D}_n] &= -S_n^*,
\end{aligned}$$

where I_n is the $n \times n$ identity matrix.

Proof
The two first relations are just direct computations: for any $p, q \in [\![0,n]\!]$, we have

$$\begin{aligned}
[S_n, S_n^*](p,q) &= S_nS_n^*(p,q) - S_n^*S_n(p,q) \\
&= S_n(p,p+1)S_n(q,q+1)\delta_{q+1=p+1} - S_n(p-1,p)S_n(q-1,q)\delta_{q-1=p-1} \\
&= ((p+1)(n-p) - p(n-p+1))\delta_{q=p} \\
&= (n-2p)\delta_{q=p} \\
&= -2\widetilde{D}_n(p,q) - nI_n(p,q).
\end{aligned}$$

In a similar way, we have for any $p, q \in [\![0, n]\!]$,

$$\begin{array}{rcl}[S_n, \widetilde{D}_n](p,q) & = & S_n\widetilde{D}_n(p,q) - \widetilde{D}_nS_n(p,q) \\ & = & S_n(p,p+1)\widetilde{D}_n(q,q)\delta_{q=p+1} - \widetilde{D}_n(p,p)S_n(q-1,q)\delta_{p=q-1} \\ & = & -((n-p-1)\sqrt{(p+1)(n-p)} - (n-p)\sqrt{(p+1)(n-p)})\delta_{q=p+1} \\ & = & \sqrt{(p+1)(n-p)}\delta_{q=p-1} \\ & = & S_n(p,q).\end{array}$$

The last equality is a consequence of the previous one:

$$\begin{array}{rcl}[S_n^*, \widetilde{D}_n] & = & [\widetilde{D}_n^*, (S_n^*)^*]^* \\ & = & [\widetilde{D}_n, S_n]^* \\ & = & -[S_n, \widetilde{D}_n]^* \\ & = & -S_n^*.\end{array}$$

■

Thus it appears that the vector space $\widetilde{V}_n$ generated by the four matrices I_n, $\widetilde{D}_n$, S_n and S_n^* is a real Lie subalgebra of $\mathfrak{gl}(n+1, \mathbb{R})$, stable by transposition and containing $\widetilde{M}_n$. It is not difficult to check that for $n \in \mathbb{N}\backslash\{0\}$, the four matrices I_n, $\widetilde{D}_n$, S_n and S_n^* are independent, so that $\dim(\widetilde{V}_n) = 4$ (the case $n = 0$ is different: $\widetilde{V}_0 = \mathbb{R}$ and $\dim(\widetilde{V}_0) = 1$). To avoid trivialities, we assume that $n \in \mathbb{N}\backslash\{0\}$ in the discussion that follows. It is possible to reduce the dimension to 3, by considering the next slight modifications. Define

$$\begin{array}{rcl}D_n & \coloneqq & \widetilde{D}_n + \frac{n}{2}I_n \\ M_n & \coloneqq & \widetilde{M}_n + \frac{n}{2}I_n,\end{array}$$

and let V_n be the vector space generated by the three matrices D_n, S_n and S_n^*. We deduce immediately from the above lemma that

$$\begin{array}{rcl}[S_n, S_n^*] & = & -2D_n \\ [S_n, D_n] & = & S_n \\ [S_n^*, D_n] & = & -S_n^*.\end{array}$$

so V_n is still a real Lie subalgebra of $\mathfrak{gl}(n+1, \mathbb{R})$ stable by transposition. We recognize the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra. Indeed, defining

$$e_1 \coloneqq -D_n \qquad e_2 \coloneqq S_n/\sqrt{2} \qquad e_3 \coloneqq S_n^*/\sqrt{2},$$

these elements satisfy the same Lie bracket relations

$$[e_1, e_2] = e_2 \qquad [e_1, e_3] = -e_3 \qquad [e_2, e_3] = e_1,$$

as the elements of usual basis of $\mathfrak{sl}(2, \mathbb{R})$ given by

$$e_1 \coloneqq \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 \coloneqq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e_3 \coloneqq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For $n = 1$, we even have equality between these elements and if we rather see the V_n, for $n \in \mathbb{N}$, as complex vector spaces, then $(V_n)_{n\in\mathbb{N}}$ is the family of all irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ (see for instance Section 4.4 of the book of Hall [8]).

4.2 Spectral analysis of $\tilde{L}_a$

The Lie algebra structure of V_n suggests that to get informations about the spectral decomposition of $M_n = \sqrt{a}S_n - \sqrt{a}S_n^* + D_n \in V_n$, it is interesting to first investigate the spectral decomposition of the adjoint operator at M_n, which is defined by

$$\mathrm{ad}_{M_n} : V_n \ni X \quad \mapsto \quad [M_n, X] \in V_n.$$

This is the object of the next result, where V_n and ad_{M_n} are replaced by their natural complexifications.

Lemma 30 *Let $n \in \mathbb{N}\backslash\{1\}$ be fixed. The kernel of the operator ad_{M_n} is generated by M_n. For $a \neq 1/4$, there are two other eigenvalues, θ and $-\theta$ where*

$$\theta \quad := \quad \begin{cases} \sqrt{1-4a} & , \text{ if } a \in [0, a/4) \\ \sqrt{4a-1}i & , \text{ if } a > 1/4. \end{cases}$$

The corresponding eigenspaces are respectively generated by

$$\begin{aligned} J_+ &= 4\sqrt{a}D_n + (1-\theta)S_n - (1+\theta)S_n^* \\ J_- &= 4\sqrt{a}D_n + (1+\theta)S_n - (1-\theta)S_n^*. \end{aligned} \tag{35}$$

For $a = 1/4$, the operator ad_{M_n} is not diagonalizable and its matrix is equal to the 3×3 Jordan block $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ associated to the eigenvalue 0, in the basis $(M_n, D_n - 2\sqrt{a}S_n^*, -2\sqrt{a}S_n^*)$.

Proof

Due to the fact that $[M_n, M_n] = 0$, we already know that M_n is an eigenvector associated to the eigenvalue 0 of ad_{M_n}. Using the above bracket relations, we compute that the matrix associated to ad_{M_n} in the basis (D_n, S_n, S_n^*) is given by

$$\begin{pmatrix} 0 & -2\sqrt{a} & -2\sqrt{a} \\ \sqrt{a} & -1 & 0 \\ \sqrt{a} & 0 & 1 \end{pmatrix}.$$

It characteristic polynomial is $-X(X^2 - 1 + 4a)$, so for $a \neq 1/4$, ad_{M_n} admits three distinct eigenvalues which are 0, θ and $-\theta$, defined in the above statement. Computing associated eigenvectors, we get the announced results, for $a \neq 1/4$. For $a = 1/4$, since the characteristic polynomial is $-X^3$, it appears that 0 is the only possible eigenvalue. Furthermore it is clear that the above matrix has rank 2 (in fact for any $a \geqslant 0$), so ad_{M_n} is necessarily similar a 3×3 Jordan block associated to the eigenvalue 0. Already knowing that M_n is in the kernel of ad_{M_n}, it is not difficult to complete it into a basis in which the matrix associated to ad_{M_n} has the required form, e.g. the basis given in the lemma.

■

For the remaining of this section, the case $a = 1/4$ will often be excluded from our study. This value is critical for the spectra of the M_n, $n \in \mathbb{N}\backslash\{1\}$, to be real. More precisely, we will see that for $a \in (0, 1/4]$, the spectrum of M_n is real (so a posteriori complexification was not necessary), while for $a \in (1/4, +\infty)$, it does contain non-real eigenvalues. First we present a simple but very useful technical result.

Lemma 31 *For $a \in (0, +\infty)\backslash\{1/4\}$, $\ker(J_+)$, the kernel of J_+, is of dimension 1.*

Proof

From (35) we remark that J_+ is a tridiagonal matrix, whose supdiagonal has only non-vanishing entries (namely the values $\sqrt{a(p+1)(n-p)}(\theta-1)$, for $p \in [\![0, n]\!]$). So if $f = (f_p)_{p\in[\![0,n]\!]}$ is a vector belonging to $\ker(J_+)$ and if $f_0 = 0$, we deduce by iteration that $f = 0$: indeed the equation $J_+(0,0)f_0+J_+(0,1)f_1 = 0$, implies $f_1 = 0$, next the equation $J_+(1,0)f_0+J_+(1,1)f_1+J_+(1,2)f_2 = 0$ enables us to see that $f_2 = 0$, etc., in the end the nullity f_n is a consequence of the last but one equation. It follows that $\ker(J_+)$ is at most of dimension 1, otherwise we could find a non-zero vector in $\ker(J_+)$ whose first coordinate is zero.

To see that $\ker(J_+)$ is not reduced to $\{0\}$, let be given λ an eigenvalue of (the complexification of) M_n and denote by $\varphi \neq 0$ a corresponding eigenvector. Since we have

$$\begin{aligned} M_n J_+\varphi &= J_+M_n\varphi + \theta\varphi \\ &= (\lambda+\theta)\varphi. \end{aligned}$$

we get that either $\lambda + \theta$ is an eigenvalue of M_n or $J_+\varphi = 0$. If the latter condition is not satisfied, we iterate this operation to see that either $\lambda + 2\theta$ is an eigenvalue of M_n or $J_+^2\varphi = 0$. But $\lambda + p\theta$ cannot be an eigenvalue of M_n for all $p \in \mathbb{N}$, so necessarily there exists $p \in \mathbb{N}$ with $J_+^p\varphi \neq 0$ and $J_+^{p+1}\varphi = 0$, i.e. $J_+^p\varphi \in \ker(J_+)\backslash\{0\}$.

■

By extending to Jordan-type subspaces the latter argument, we will prove the following important result.

Proposition 32 *For $a \neq 1/4$, the matrix M_n is diagonalizable and all the eigenvalues have multiplicity 1. More precisely if λ is an eigenvalue of M_n such that $\lambda + \theta$ is not an eigenvalue of M_n, then the spectrum of M_n is the set $\{\lambda - k\theta : k \in [\![0, n]\!]\}$. Furthermore, for $k \in [\![1, n]\!]$, J_+ (respectively J_-) transforms the spectral line associated to $\lambda - k\theta$ (resp. $\lambda - (k-1)\theta$) into the spectral line associated to $\lambda - (k-1)\theta$ (resp. $\lambda - k\theta$).*

Proof

We define that a subspace V of $\mathbb{C}^{n+1}$ is of type (l, d), with $l \in \mathbb{C}$ and $d \in \mathbb{N}$, if there exists a basis $(\varphi_0, \varphi_1, ..., \varphi_d)$ of V such that

$$\begin{aligned} & M_n\varphi_0 = l\varphi_0, \\ \forall\, p \in [\![1, d]\!], \qquad & M_n\varphi_p = l\varphi_p + \varphi_{p-1}. \end{aligned}$$

The Jordan decomposition implies that M_n is diagonalizable if and only if there is no $(l, 1)$-type subspace for any $l \in \mathbb{C}$ (by taking into account that maximal (l, d)-type subspaces lead to Jordan blocks, which contains $(l, 1)$-type subspaces if $d \geqslant 1$). We are to proceed by a contradictory argument to show that M_n is diagonalizable. First consider V a $(l, 1)$-type subspace endowed of a basis (φ_0, φ_1) as above. The relation $M_nJ_+ = J_+M_n + \theta J_+$ implies that

$$\begin{aligned} M_nJ_+\varphi_0 &= (l+\theta)J_+\varphi_0, \\ M_nJ_+\varphi_1 &= (l+\theta)J_+\varphi_1 + J_+\varphi_0. \end{aligned} \tag{36}$$

Thus if $\ker(J_+) \cap V = \{0\}$, we get that $J_+\varphi_0$ and $J_+\varphi_1$ must be independent, so $J_+(V)$ is a $(l+\theta, 1)$-type subspace. In particular $l + \theta$ is an eigenvalue of M_n. Next let λ be as in the statement of the proposition and assume there exists a $(\lambda, 1)$-type subspace V, endowed of a basis (φ_0, φ_1) as above. Necessarily $\ker(J_+) \subset V$, otherwise the above argument would lead to fact that $\lambda + \theta$ is an eigenvalue of M_n. So let $f \in \ker(J_+)\backslash\{0\}$ be given. The relation $M_nJ_+f = J_+M_nf + \theta J_+f$ implies that $J_+ M_n f = 0$, namely $M_n f \in \ker(J_+)$. Lemma 31 then shows that $M_n f$ is proportional to f, i.e. f is an eigenvector of M_n. The only eigenvectors of M_n belonging to V are proportional to φ_0, thus we deduce that $J_+\varphi_0 = 0$. But (36) (with l replaced by λ) implies that either $\lambda + \theta$ is an eigenvalue of M_n, which is forbidden by our choice of λ, either $J_+\varphi_1 = 0$, which is not more possible, because it would lead to $\dim(\ker(J_+)) \geqslant 2$. It follows that a $(\lambda, 1)$-type subspace does not exist.

Nevertheless, a eigenvector $\varphi \neq 0$ associated to λ exists and necessarily $J_+\varphi = 0$. A consequence of this property and of Lemma 31 is that for any $l \neq \lambda$ and any $(l, 1)$-subspace V, we have $\ker(J_+) \cap V = \{0\}$. As it was already shown, we then get that $J_+(V)$ is a $(l + \theta, 1)$-type subspace. If $l + \theta \neq \lambda$, we reiterate this procedure. Necessarily we end up with a integer $p \in \mathbb{N}\backslash\{0\}$ such that $l + p\theta = \lambda$, otherwise we would construct an infinity of eigenvalues. But another contradiction appears, because $J_+^p(V)$ is in fact a $(\lambda, 1)$-type subspace. In conclusion, there is no $(l, 1)$-type subspace: M_n is diagonalizable.

The other assertions of the proposition are proven in a similar way: first $\ker(J_+)$ is necessarily the eigenspace associated to λ, which by consequence is of multiplicity 1. Next any non-zero eigenvector φ associated to an eigenvalue $l \neq \lambda$ of M_n is such that $J_+\varphi$ is a non-zero eigenvector associated to $l + \theta$. Iterating again, we deduce there exists $p \in \mathbb{N}\backslash\{0\}$ such that $l + p\theta = \lambda$ and $J_+^p\varphi$ belongs to the line eigenspace associated to λ. Another application of Lemma 31 shows that the dimension of the eigenspace associated to l was necessarily 1 (otherwise you could find $\varphi \neq 0$ in this eigenspace and $k \in [\![1, p]\!]$ such that $J_+^{k-1}\varphi$ belongs to $\ker(J_+)$ but not to the eigenspace associated to λ, which is not permitted). This is only possible if the spectrum of M_n coincides with the set $\{\lambda - k\theta \,:\, k \in [\![0, n]\!]\}$ and if for $k \in [\![1, n]\!]$, J_+ transforms the spectral line associated to $\lambda - k\theta$ into the spectral line associated to $\lambda - (k-1)\theta$. Rather working with J_- instead of J_+ leads to the corresponding statement for J_-.

■

To end the determination of the spectrum of M_n, we point out another particular feature of this matrix: M_n is skew-centrosymmetric, i.e. $\mathcal{T}(M_n) = -M_n$, where for any $(n+1) \times (n+1)$ matrix $M = (M_{k,l})_{k,l \in [\![0,n]\!]}$, we define

$$\forall\ k, l \in [\![0, n]\!], \qquad (\mathcal{T}(M))_{k,l} \;\coloneqq\; M_{n-k,n-l}.$$

This transformation $\mathcal{T}$ also applies to vectors by

$$\forall\ f = (f_k)_{k \in [\![0,n]\!]}, \qquad \mathcal{T}(f) \;\coloneqq\; (f_{n-k})_{k \in [\![0,n]\!]}$$

and it is easily checked that for any matrix M and vector f,

$$\mathcal{T}(Mf) \;=\; \mathcal{T}(M)\mathcal{T}(f),$$

(for general references about (skew) centrosymmetric matrices, see for instance the papers of Weaver [22] and Lee [16]). An important consequence of the skew-centrosymmetry of M_n is that its spectrum is symmetric with respect to 0. Indeed, if λ is an eigenvalue of M_n and if φ is a corresponding eigenvector, we get, by using that $\mathcal{T}$ is a linear involution, that

$$\begin{aligned} M_n \mathcal{T}(\varphi) &= \mathcal{T}(\mathcal{T}(M_n)\varphi) \\ &= -\mathcal{T}(M_n \varphi) \\ &= -\lambda \mathcal{T}(\varphi). \end{aligned}$$

This shows that $-\lambda$ is also an eigenvalue of M_n, an associated eigenvector being $\mathcal{T}(\varphi)$. In conjunction with Proposition 32, this observation leads to the determination of the spectrum of M_n.

Proposition 33 *For $a \neq 1/4$, the spectrum of M_n is $\{(k - n/2)\theta \,:\, k \in [\![0, n]\!]\}$. For $a = 1/4$, M_n is similar to the Jordan block of size $n+1$ associated to the eigenvalue 0 (in particular M_n is not diagonalizable for $n \geqslant 1$).*

Proof

The first assertion is an immediate consequence of Proposition 32 and of the symmetry of the spectrum of M_n. Note that as $a \neq 1/4$ goes to $1/4$, θ and the eigenvalues of M_n converge to zero. A usual result on perturbation of spectrum (cf. for instance the chapter 2 of book of Kato [14]) then implies that the spectrum of M_n for $a = 1/4$ is reduced to $\{0\}$. But the arguments of the proof Lemma 31 also apply to the tridiagonal matrix M_n to show that the dimension of $\ker(M_n)$ is at most 1. By the Jordan decomposition, it follows that M_n is necessarily similar to the Jordan block of size $n+1$ associated to the eigenvalue 0.

■

In view of this result it is natural to make the convention that $\theta = 0$ when $a = 1/4$. Recalling that for all $n \in \mathbb{N}$, $\widetilde{M}_n = M_n - \frac{n}{2} I_n$ is (the matrix associated to) the restriction of $\widetilde{L}_a$ to H_n and that $\mathbb{L}^2(\widetilde{\mu}_a) = \bigotimes_{n\in\mathbb{N}} H_n$, where the H_n, $n \in \mathbb{N}$ are mutually orthogonal, we get

Corollary 34 *For any $a > 0$, the spectrum of $\widetilde{L}_a$ in $\mathbb{L}^2(\widetilde{\mu}_a)$ is*

$$\left\{-\frac{n}{2} + (k - n/2)\theta \;:\; n \in \mathbb{N},\, k \in [\![0, n]\!]\right\}.$$

For $a \neq 1/4$, $\widetilde{L}_a$ is diagonalizable in $\mathbb{L}^2(\widetilde{\mu}_a)$, but $\widetilde{L}_{1/4}$ is not diagonalizable and it contains Jordan blocks of all dimensions.

For $a \neq 1/4$ and $n \in \mathbb{N}$, we have seen that a family $(\xi_p)_{p\in[\![0,n]\!]}$ of eigenvectors associated to the eigenvalues $((p - n/2)\theta)_{p\in[\![0,n]\!]}$ of M_n is given by

$$\forall\, p \in [\![0, n]\!], \qquad \xi_p \;\;\coloneqq\;\; J_+^p \xi_0,$$

where ξ_0 is a normalized vector generating the kernel of J_-.
Using this information, it is possible to make explicit the eigenvectors of $\widetilde{L}_a$, which are polynomial. But as seen in Section 3 for our first model, to obtain hypocoercive bounds, it is more crucial to compute the scalar products of the eigenvectors than to known them exactly. This is the objective of next section.

5 Norms of hypocoercive Gaussian semi-groups

We are going to prove Theorem 3, by following the approach of Section 3, namely by investigating scalar products of underlying eigenvectors.

Let $a > 0$ be fixed. Since, on one hand the orthogonal decomposition $\mathbb{L}^2(\widetilde{\mu}_a) = \bigotimes_{n\in\mathbb{N}} H_n$, introduced in the previous section, is left stable by all the elements of the semi-group $(\widetilde{P}_t^{(a)})_{t\geqslant 0}$, and on the other hand $\widetilde{\mu}_a$ correspond to the orthogonal projection on H_0, the space containing the constant functions, we have for all $t \geqslant 0$,

$$\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\|^2_{\mathbb{L}^2(\widetilde{\mu}_a)\circlearrowleft} \;=\; \sup_{n\in\mathbb{N}} \|\widetilde{P}_t^{(a)}\|^2_{H_n\circlearrowleft}.$$

By the isometries introduced at the beginning of Section 4, we have for any $n \in \mathbb{N}$ and $t \geqslant 0$, $\|\widetilde{P}_t^{(a)}\|_{H_n\circlearrowleft} = \|\exp(t\widetilde{M}_n)\|$, where $\widetilde{M}_n$ is the $(n+1)\times(n+1)$ matrix defined in (34) and where $\|\cdot\|$ stands for the operator norm with respect to the canonical Hermitian norm on $\mathbb{C}^{n+1}$. We are thus brought back to the finite dimensional setting of Section 4, $n \in \mathbb{N}$ being fixed.

5.1 Identification of eigenvectors when $a \in (0, 1/4)$

For the first part of this section, we restrict ourself to the case $a \in (0, 1/4)$, so that θ is real and belongs to $(0, 1)$. Recall that

$$D_n \coloneqq \begin{pmatrix} -\frac{n}{2} & 0 & 0 & \cdots & 0 \\ & -\frac{n}{2}+1 & 0 & & \vdots \\ 0 & 0 & -\frac{n}{2}+2 & \ddots & 0 \\ & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \frac{n}{2} \end{pmatrix} \quad \text{and} \quad S_n \coloneqq \begin{pmatrix} 0 & \sqrt{n} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2(n-1)} & & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ & & \ddots & \ddots & \sqrt{n} \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

and that

$$\begin{aligned} \widetilde{M}_n &\coloneqq M_n + \frac{n}{2} I_n \\ M_n &\coloneqq D_n + \sqrt{a}(S_n - S_n^*) \\ \theta &\coloneqq \sqrt{1-4a} \\ J_+ &\coloneqq 4\sqrt{a} D_n + (1-\theta) S_n - (1+\theta) S_n^* \\ J_- &\coloneqq 4\sqrt{a} D_n + (1+\theta) S_n - (1-\theta) S_n^*. \end{aligned}$$

Furthermore, ξ_0 is a normalized vector generating the kernel of J_- and for all $p \in [\![1, n]\!]$, $\xi_p \coloneqq J_+^p \xi_0$, so that $(\xi_p)_{p \in [\![0,n]\!]}$ is a family of eigenvectors of M_n associated to the eigenvalues $((p - n/2)\theta)_{p \in [\![0,n]\!]}$. We begin by checking that $\xi_0^2 \coloneqq (\xi_0^2(p))_{p \in [\![0,n]\!]}$ is the binomial distribution of parameter $(1-\theta)/2$.

Lemma 35 *We can take*

$$\forall\, p \in [\![0, n]\!], \qquad \xi_0(p) \;=\; \sqrt{\binom{n}{p} \left(\frac{1-\theta}{2}\right)^p \left(\frac{1+\theta}{2}\right)^{n-p}}. \tag{37}$$

Proof

Let ξ be the vector whose coordinates are given by the r.h.s. of (37). It is sufficient to show that $J_- \xi = 0$. By definition, we have for any $p \in [\![0, n]\!]$ (with the convention $\xi(-1) = \xi(n+1) = 0$),

$$\begin{aligned} D_n \xi(p) &= \left(-\frac{n}{2} + p\right) \xi(p), \\ S_n \xi(p) &= \sqrt{(p+1)(n-p)}\xi(p+1) \\ &= \sqrt{(p+1)(n-p)} \sqrt{\frac{n-p}{p+1}} \sqrt{\frac{1-\theta}{1+\theta}} \xi(p) \\ &= (n-p) \sqrt{\frac{1-\theta}{1+\theta}} \xi(p), \\ S_n^* \xi(p) &= \sqrt{p(n-p+1)}\xi(p-1) \\ &= \sqrt{p(n-p+1)} \sqrt{\frac{p}{n-p+1}} \sqrt{\frac{1+\theta}{1-\theta}} \xi(p) \\ &= p \sqrt{\frac{1-\theta}{1+\theta}} \xi(p). \end{aligned}$$

It follows that

$$\begin{aligned} J_- \xi(p) &= \left(4\sqrt{a}\left(p - \frac{n}{2}\right) + (1+\theta)(n-p)\sqrt{\frac{1-\theta}{1+\theta}} - (1-\theta)p\sqrt{\frac{1+\theta}{1-\theta}}\right) \xi(p) \\ &= \left(2\sqrt{a}\,(2p-n) + (n-p)\sqrt{(1-\theta)(1+\theta)} - p\sqrt{(1+\theta)(1-\theta)}\right) \xi(p) \\ &= 2\sqrt{a}(2p - n + n - p - p)\xi(p) \\ &= 0. \end{aligned}$$

$\blacksquare$

Following the same arguments of Section 3, we are now looking for a family $(Q_p)_{p\in[\![0,n]\!]}$ of polynomials, with Q_p of degree p for $p \in [\![0,n]\!]$, such that

$$\forall\ p \in [\![0,n]\!], \qquad \xi_p \;=\; Q_p(D_n)\xi_0. \tag{38}$$

To do so, we will need two preliminary computations

Lemma 36 *We have*

$$J_+ \;=\; \frac{8a-2}{\sqrt{a}}D_n + \frac{2}{\sqrt{a}}M_n - J_-,$$

and for any $p \in \mathbb{Z}_+$,

$$[J_-, J_+^p]\xi_0 \;=\; 4p(p-1-n)\theta^2 J_+^{p-1}\xi_0.$$

Proof

By definition, we see that

$$\begin{aligned} J_+ + J_- &= 8\sqrt{a}D_n + 2S_n - 2S_n^* \\ &= 8\sqrt{a}D_n + \frac{2}{\sqrt{a}}(M_n - D_n), \end{aligned}$$

and the first equality follows at once.

The second equality is trivial for $p = 0$ since by convention J_+^0 is the $(n+1)\times(n+1)$ identity matrix I_n, so that $[J_-, J_+^0] = 0$. For $p = 1$, we are going to show that

$$[J_-, J_+] \;=\; 8\theta M_n. \tag{39}$$

Let us first remark a priori from the actions of J_- and J_+ on the eigenspaces of M_n, that J_-J_+, J_+J_- and $[J_-, J_+]$ are functions of M_n, at least for $a \neq 1/4$, when all the eigenvalues of M_n are distinct. Indeed, recalling that $[D_n, S_n] = -S_n$, $[D_n, S_n^*] = S_n^*$ and $[M_n, J_-] = -\theta J_-$, we get

$$\begin{aligned} [J_-, J_+] &= \left[J_-, \frac{8a-2}{\sqrt{a}}D_n + \frac{2}{\sqrt{a}}M_n - J_-\right] \\ &= \frac{8a-2}{\sqrt{a}}[J_-, D_n] + \frac{2}{\sqrt{a}}[J_-, M_n] \\ &= \frac{8a-2}{\sqrt{a}}[4\sqrt{a}D_n + (1+\theta)S_n - (1-\theta)S_n^*, D_n] + \frac{2}{\sqrt{a}}\theta J_- \\ &= \frac{8a-2}{\sqrt{a}}(1+\theta)[S_n, D_n] - \frac{8a-2}{\sqrt{a}}(1-\theta)[S_n^*, D_n] + \frac{2}{\sqrt{a}}\theta J_- \\ &= \frac{8a-2}{\sqrt{a}}(1+\theta)S_n + \frac{8a-2}{\sqrt{a}}(1-\theta)S_n^* + \frac{2}{\sqrt{a}}\theta J_- \\ &= -\frac{2\theta^2}{\sqrt{a}}(1+\theta)S_n - \frac{2\theta^2}{\sqrt{a}}(1-\theta)S_n^* + \frac{2}{\sqrt{a}}\theta(4\sqrt{a}D_n + (1+\theta)S_n - (1-\theta)S_n^*) \\ &= \frac{2\theta}{\sqrt{a}}(-\theta((1+\theta)S_n + (1-\theta)S_n^*) + 4\sqrt{a}D_n + (1+\theta)S_n - (1-\theta)S_n^*) \\ &= \frac{2\theta}{\sqrt{a}}(4\sqrt{a}D_n + (1-\theta^2)S_n - (1-\theta^2)S_n^*) \\ &= \frac{2\theta}{\sqrt{a}}(4\sqrt{a}D_n + 4aS_n - 4aS_n^*) \\ &= 8\theta M_n. \end{aligned}$$

Hence, from (39) we deduce that for any $p \in \mathbb{N}$,

$$\begin{aligned}
[J_-, J_+^p] &= [J_-, J_+]J_+^{p-1} + J_+[J_-, J_+]J_+^{p-2} + \cdots + J_+^{p-1}[J_-, J_+] \\
&= 8\theta(M_n J_+^{p-1} + J_+ M_n J_+^{p-2} + \cdots + J_+^{p-1} M_n).
\end{aligned}$$

Applying this formula to the vector ξ_0 and taking into account that $J_+^q \xi_0$ is an eigenvector of M_n associated to the eigenvalue $(q - n/2)\theta$ for all $q \in [\![0, n]\!]$ (and that the relation $M_n J_+^q \xi_0 = (q - n/2)\theta J_+^q \xi_0$ is also true for $q > n$, since then $J_+^q \xi_0 = 0$), we obtain for all $p \in \mathbb{N}$,

$$\begin{aligned}
[J_-, J_+^p]\xi_0 &= 8\theta^2((p-1-n/2) + (p-2-n/2) + \cdots + (-n/2))J_+^{p-1}\xi_0 \\
&= 4p(p-1-n)\theta^2 J_+^{p-1}\xi_0,
\end{aligned}$$

as announced.

■

We can now find $(Q_p(X))_{p \geqslant 0}$ such that (38) is satisfied.

Lemma 37 *Consider the family of polynomials $(Q_p(X))$ defined by the recurrence relation:*

$$Q_0(X) = 1 \quad \text{and} \quad Q_1(X) = -\left(\frac{2}{\sqrt{a}}\theta^2 X + \frac{n}{\sqrt{a}}\theta\right),$$

$$\forall\ p \in [\![1, n-1]\!], \qquad Q_{p+1}(X) = \left(-\frac{2}{\sqrt{a}}\theta^2 X + \frac{2p-n}{\sqrt{a}}\theta\right) Q_p(X) + 4\theta^2 p(n+1-p)Q_{p-1}(X).$$

Then for any $p \in [\![0, n]\!]$, Q_p is of degree p and (38) is fulfilled.

Proof

Of course $\xi_0 = Q_0(D_n)\xi_0$ and if we assume that for some $p \in [\![0, n-1]\!]$, $\xi_q = Q_q(D_n)\xi_0$ for all $q \in [\![0, p]\!]$, then we can write, using the first relation of Lemma 36, and that $J_-\xi_0 = 0$

$$\begin{aligned}
\xi_{p+1} &= J_+(J_+^p \xi_0) \\
&= \left(\frac{8a-2}{\sqrt{a}} D_n + \frac{2}{\sqrt{a}} M_n - J_-\right) J_+^p \xi_0 \\
&= \frac{8a-2}{\sqrt{a}} D_n Q_p(D_n)\xi_0 + \frac{2}{\sqrt{a}}\left(p - \frac{n}{2}\right)\theta J_+^p \xi_0 - [J_-, J_+^p]\xi_0 - J_+^p J_- \xi_0 \\
&= \left(\frac{8a-2}{\sqrt{a}} D_n + \frac{2}{\sqrt{a}}\theta\left(p - \frac{n}{2}\right)\right) Q_p(D_n)\xi_0 - [J_-, J_+^p]\xi_0.
\end{aligned}$$

If $p = 0$, since $[J_-, J_+^0] = 0$, this gives $\xi_1 = Q_1(D_n)\xi_0$ with Q_1 the polynomial described in the lemma, recalling that $\theta^2 = 1 - 4a$.

For $p > 1$, the second relation of Lemma 36 enables to replace $[J_-, J_+^p]\xi_0$ by $4p(p-1-n)\theta^2 J_+^{p-1}\xi_0 = 4p(p-1-n)\theta^2 Q_{p-1}(D_n)\xi_0$, due to our iterative assumption. So we end up with the announced recurrence relation for the family $(Q_p(X))_{p \in [\![0,n]\!]}$, which clearly implies that for any $p \in [\![0, n]\!]$, Q_p is of degree p.

■

Remark 38 Let us define Q_{n+1} by extending the above recurrence:

$$Q_{n+1}(X) = \left(-\frac{2}{\sqrt{a}}\theta^2 X + \frac{n}{\sqrt{a}}\theta\right) Q_n(X) + 4\theta^2 n Q_{n-1}(X).$$

The computations of the above proof show that $Q_{n+1}(D_n)\xi_0 = J_+^{n+1}\xi_0 = 0$. It follows that for any $p \in [\![0, n]\!]$, $Q_{n+1}(D_n)\xi_p = Q_{n+1}(D_n)Q_p(D_n)\xi_0 = Q_p(D_n)Q_{n+1}(D_n)\xi_0 = 0$. Thus $Q_{n+1}(D_n) = 0$, because $(\xi_p)_{p\in[\![0,n]\!]}$ is a basis of the underlying vector space. The operator D_n has $n+1$ distinct eigenvalues (given by the elements of its diagonal), so its minimal polynomial has degree $n+1$ and Q_{n+1} must be proportional to it. Indeed, an immediate analysis of the leading monomials proves that

$$Q_{n+1}(X) \;=\; \left(\frac{-2\theta^2}{\sqrt{a}}\right)^{n+1} \prod_{p\in[\![0,n]\!]} \left(X + \frac{n}{2} - p\right).$$

A similar observation leads to the uniqueness property of the family $(Q_p(X))_{p\in[\![0,n]\!]}$: assume that for some $p \in [\![0,n]\!]$, $\xi_p = \widetilde{Q}_p(D_n)\xi_0$, where $\widetilde{Q}_p$ is a polynomial of degree p. Then necessarily we have $\widetilde{Q}_p = Q_p$. Indeed, $\widehat{Q}_p \coloneqq Q_p - \widetilde{Q}_p$ would be a polynomial of degree less than p such that $\widehat{Q}_p(D_n)\xi_0 = 0$ and by the above arguments it appears that $\widehat{Q}_p$ must be proportional to Q_{n+1} of degree $n+1$. This is only possible if $\widehat{Q}_p = 0$ as wanted.

□

Since it will be more convenient to work with polynomials whose leading term is 1 and to shift the eigenvalues of D_n by $n/2$ (to end up with the set $[\![0,n]\!]$, which is the support of the binomial law ξ_0^2), we define

$$\forall\, p \in [\![0,n]\!], \qquad P_p(X) \;=\; \left(\frac{-2\theta^2}{\sqrt{a}}\right)^{-p} Q_p(X - n/2).$$

In the sequel, we denote

$$\upsilon \;\coloneqq\; \frac{1}{\theta} - 1.$$

It is easy to see that the recurrence relation holds

$$\begin{aligned} P_0(X) \;&=\; 1,\\ \forall\, p \in [\![0,n-1]\!], \qquad P_{p+1}(X) \;&=\; \left(X - p + \left(\frac{n}{2} - p\right)\upsilon\right) P_p(X) + \frac{1}{4} p(n+1-p)(\upsilon^2 + 2\upsilon) P_{p-1}(X). \end{aligned}$$

Note that the term P_{-1} is not necessary to determine P_1.

These modifications also prompt us to exchange the $(\xi_p)_{p\in[\![0,n]\!]}$ for the $(\zeta_p)_{p\in[\![0,n]\!]}$ defined by

$$\begin{aligned} \forall\, p \in [\![0,n]\!], \qquad \zeta_p \;&\coloneqq\; \left(\frac{-2\theta^2}{\sqrt{a}}\right)^{-p} \xi_p\\ &=\; P_p\left(D_n + \frac{n}{2} I_n\right) \xi_0. \end{aligned} \tag{40}$$

The family $(\zeta_p)_{p\in[\![0,n]\!]}$ still consists of a basis of eigenvectors of M_n (associated to the family of eigenvalues $((p - n/2)\theta)_{p\in[\![0,n]\!]}$), its advantage is encapsulated in the next result.

Lemma 39 *For any $p, q \in [\![0,n]\!]$, we have*

$$\langle \zeta_p, \zeta_q \rangle \;=\; \beta_{(1-\theta)/2}[P_p P_q],$$

where $\beta_{(1-\theta)/2}$ is the binomial distribution of parameter $(1-\theta)/2$.

Proof

This is a direct computation: for any $p, q \in [\![0, n]\!]$, we have

$$\begin{aligned}
\langle \zeta_p, \zeta_q \rangle &= \sum_{m \in [\![0,n]\!]} \zeta_p(m)\zeta_q(m) \\
&= \sum_{m \in [\![0,n]\!]} \left(P_p\left(D_n + \frac{n}{2}I_n\right)\xi_0\right)(m) \left(P_q\left(D_n + \frac{n}{2}I_n\right)\xi_0\right)(m) \\
&= \sum_{m \in [\![0,n]\!]} P_p(m)P_q(m)(\xi_0(m))^2 \\
&= \beta_{(1-\theta)/2}[P_pP_q],
\end{aligned}$$

where we have used (40) and (37).

■

The recurrence relation satisfied by $(P_p(X))_{p\in[\![0,n]\!]}$ may lead the reader to think that they are the orthogonal polynomials associated to some law on $[\![0, n]\!]$. This law cannot be $\beta_{(1-\theta)/2}$, because the family $(\zeta_p)_{p\in[\![0,n]\!]}$ is not orthogonal: this is the heart of the subject and the motivation for the computations of this section.

It is time now to provide an explicit formula for the polynomials $P_p(X)$, $p \in [\![0, n]\!]$. In analogy with Section 3 again, it is more convenient to express them in the basis $(\Pi_p)_{p\in[\![0,n]\!]}$, where

$$\forall\ p \in [\![0, n]\!], \qquad \Pi_p(X) \ := \prod_{k\in[\![0,p-1]\!]} (X - k),$$

(slightly abusing notations, the r.h.s. could also be written $X^{(p)}$).

Lemma 40 *For any $p \in [\![0, n]\!]$, we have*

$$P_p(X) = \sum_{k\in[\![0,p]\!]} \frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!} \left(\frac{\upsilon}{2}\right)^{p-k} \Pi_k(X). \tag{41}$$

Proof

After computing the first elements of the family $(P_p(X))_{p\in[\![0,n]\!]}$, one guesses that they will be of the form

$$P_p(X) = \sum_{k\in[\![0,p]\!]} \alpha_{p,k}\upsilon^{p-k}\Pi_k(X),$$

for some coefficients $(\alpha_{p,k})_{p\in[\![0,n]\!],k\in[\![0,p]\!]}$ independent of the parameter υ. Putting such a form in the recurrence relation, it appears that to be conserved for P_{p+1} (assuming it is true for P_p and P_{p-1}), one must have $\alpha_{p,k} = \frac{p(n+1-p)}{2(p-k)}\alpha_{p-1,k}$. Since necessarily $\alpha_{p,p} = 1$, this leads to the announced formula. Once the latter is suspected, it is sufficient to check it by induction: assuming that (41) is true for P_p and P_{p-1} (this is immediate for P_0 and P_1), we compute that

$$\begin{aligned}
&\left(X-p+\left(\frac{n}{2}-p\right)\upsilon\right)P_p(X)+\frac{1}{4}p(n+1-p)(\upsilon^2+2\upsilon)P_{p-1}(X)\\
&= \sum_{k\in\mathbb{Z}_+}\frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!}\left(\frac{\upsilon}{2}\right)^{p-k}\left(X-p+\left(\frac{n}{2}-p\right)\upsilon\right)\Pi_k(X)\\
&\quad+\frac{1}{4}\sum_{k\in\mathbb{Z}_+}\frac{(p-1)^{(p-1-k)}(n-k)^{(p-1-k)}}{(p-1-k)!}p(n+1-p)(\upsilon^2+2\upsilon)\left(\frac{\upsilon}{2}\right)^{p-1-k}\Pi_k(X)\\
&= \sum_{k\in\mathbb{Z}_+}\frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!}\left(\frac{\upsilon}{2}\right)^{p-k}(X-k)\,\Pi_k(X)\\
&\quad+\sum_{k\in\mathbb{Z}_+}\frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!}\left(k-p+\left(\frac{n}{2}-p\right)\upsilon\right)\left(\frac{\upsilon}{2}\right)^{p-k}\Pi_k(X)\\
&\quad+\frac{1}{2}\sum_{k\in\mathbb{Z}_+}\frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!}(p-k)(\upsilon+2)\left(\frac{\upsilon}{2}\right)^{p-k}\Pi_k(X)\\
&= \sum_{k\in\mathbb{Z}_+}\frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!}\left(\frac{\upsilon}{2}\right)^{p-k}\Pi_{k+1}(X)\\
&\quad+\sum_{k\in\mathbb{Z}_+}\frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!}\left(k-p+\left(\frac{n}{2}-p\right)\upsilon+(p-k)\left(1+\frac{\upsilon}{2}\right)\right)\left(\frac{\upsilon}{2}\right)^{p-k}\Pi_k(X)\\
&= \sum_{k\in\mathbb{Z}_+}\frac{p^{(p-k+1)}(n-k+1)^{(p-k+1)}}{(p-k+1)!}\left(\frac{\upsilon}{2}\right)^{p-k+1}\Pi_k(X)\\
&\quad+\sum_{k\in\mathbb{Z}_+}\frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!}(n-p-k)\frac{\upsilon}{2}\left(\frac{\upsilon}{2}\right)^{p-k}\Pi_k(X)\\
&= \sum_{k\in\mathbb{Z}_+}\frac{(p+1)^{(p+1-k)}(n-k)^{(p+1-k)}}{(p+1-k)!}\frac{k(n-k+1)}{(p+1)(n-p)}\left(\frac{\upsilon}{2}\right)^{p+1-k}\Pi_k(X)\\
&\quad+\sum_{k\in\mathbb{Z}_+}\frac{(p+1)^{(p+1-k)}(n-k)^{(p+1-k)}}{(p+1-k)!}\frac{(p+1-k)(n-p-k)}{(p+1)(n-p)}\left(\frac{\upsilon}{2}\right)^{p+1-k}\Pi_k(X)\\
&= \sum_{k\in\mathbb{Z}_+}\frac{(p+1)^{(p+1-k)}(n-k)^{(p+1-k)}}{(p+1-k)!}\left(\frac{\upsilon}{2}\right)^{p+1-k}\Pi_k(X),
\end{aligned}$$

which is the wanted expression for $P_{p+1}(X)$.

■

Remark 41 It is possible to give a compact formula for the r.h.s. of (41): introduce two free variables Z_1 and Z_2 and consider the following interpretations:

$$\begin{aligned}
(n-k)^{(p-k)} &= (-1)^{p-k}\frac{d^{p-k}}{dZ_1^{p-k}}\frac{1}{Z_1^{n-p+1}}\bigg|_{Z_1=1},\\
\Pi_k(X) &= \frac{d^p}{dZ_2^p}Z_2^X\bigg|_{Z_2=1}.
\end{aligned}$$

The r.h.s of (41) can then be seen as

$$\sum_{k\in[\![0,p]\!]}\binom{p}{p-k}\left(\frac{-\upsilon}{2}\right)^{p-k}\frac{d^{p-k}}{dZ_1^{p-k}}\frac{d^p}{dZ_2^p}\frac{Z_2^X}{Z_1^{n-p+1}}\bigg|_{Z_1=1,Z_2=1} = \left(\frac{d}{dZ_2}-\frac{\upsilon}{2}\frac{d}{dZ_1}\right)^p\frac{Z_2^X}{Z_1^{n-p+1}}\bigg|_{Z_1=1,Z_2=1}.$$

Unfortunately, it is not obvious to obtain the wanted recurrence relation from this expression.

□

From the two last lemmas, we can deduce an explicit formula for $\langle \zeta_p, \zeta_q \rangle$, $p, q \in [\![0, n]\!]$, but it is not so easy to handle. It rather suggests to consider certain particular vectors.

5.2 Exact computation of $\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\|_{\mathbb{L}^2(\widetilde{\mu}_a)\circlearrowleft}$, $a \in (0; 1/4)$

To any $\rho \in \mathbb{R}$, associate the vector

$$z(\rho) \quad \coloneqq \quad \sum_{p \in [\![0,n]\!]} \frac{\rho^p}{p!} \zeta_p. \tag{42}$$

Lemma 42 *For any $\widetilde{\rho}, \widehat{\rho} \in \mathbb{R}$, we have*

$$\langle z(\widetilde{\rho}), z(\widehat{\rho}) \rangle \quad = \quad \left(1 + \frac{1-\theta^2}{2\theta}(\widetilde{\rho} + \widehat{\rho}) + \frac{1-\theta^2}{4\theta^2} \widetilde{\rho}\widehat{\rho}\right)^n.$$

Proof

From Lemma 39, we get that

$$\begin{aligned}
\langle z(\widetilde{\rho}), z(\widehat{\rho}) \rangle \quad &= \quad \beta_{(1-\theta)/2}\left[\sum_{p \in [\![0,n]\!]} \frac{\widetilde{\rho}^p}{p!} P_p \sum_{q \in [\![0,n]\!]} \frac{\widehat{\rho}^q}{q!} P_q\right] \\
&= \quad 2^{-n} \sum_{m \in [\![0,n]\!]} \binom{n}{m} (1-\theta)^m (1+\theta)^{n-m} \sum_{p \in [\![0,n]\!]} \frac{\widetilde{\rho}^p}{p!} P_p(m) \sum_{q \in [\![0,n]\!]} \frac{\widehat{\rho}^q}{q!} P_q(m). \qquad (43)
\end{aligned}$$

From (41), we have for any $m \in [\![0, n]\!]$ and $p \in [\![0, n]\!]$,

$$P_p(m) \quad = \quad \sum_{k \in \mathbb{Z}_+} \frac{p^{(p-k)}(n-k)^{(p-k)}}{(p-k)!} \left(\frac{\upsilon}{2}\right)^{p-k} m^{(k)}.$$

Hence, using the relation $p^{(p-k)}/(p!) = 1/(k!)$ and exchanging sums, we obtain

$$\begin{aligned}
\sum_{p \in [\![0,n]\!]} \frac{\widetilde{\rho}^p}{p!} P_p(m) \quad &= \quad \sum_{k \in \mathbb{Z}_+} \frac{1}{k!} \widetilde{\rho}^k m^{(k)} \sum_{p \geqslant k} \widetilde{\rho}^{p-k} \binom{n-k}{p-k} \left(\frac{\upsilon}{2}\right)^{p-k} \\
&= \quad \sum_{k \in \mathbb{Z}_+} \frac{1}{k!} \widetilde{\rho}^k m^{(k)} \left(1 + \frac{\upsilon\widetilde{\rho}}{2}\right)^{n-k} \\
&= \quad \left(1 + \frac{\upsilon\widetilde{\rho}}{2}\right)^n \sum_{k \in \mathbb{Z}_+} \binom{m}{k} \widetilde{\rho}^k \left(1 + \frac{\upsilon\widetilde{\rho}}{2}\right)^{-k} \\
&= \quad \left(1 + \frac{\upsilon\widetilde{\rho}}{2}\right)^n \left(1 + \widetilde{\rho}\left(1 + \frac{\upsilon\widetilde{\rho}}{2}\right)^{-1}\right)^m \\
&= \quad \left(1 + \frac{\upsilon\widetilde{\rho}}{2}\right)^{n-m} \left(1 + \widetilde{\rho}\left(1 + \frac{\upsilon}{2}\right)\right)^m.
\end{aligned}$$

Coming back to (43), it appears that $2^n \langle z(\widetilde{\rho}), z(\widehat{\rho}) \rangle$ is equal to

$$\begin{aligned}
&\sum_{m \in [\![0,n]\!]} \binom{n}{m} (1-\theta)^m \left(1 + \widetilde{\rho}\left(1 + \frac{\upsilon}{2}\right)\right)^m \left(1 + \widehat{\rho}\left(1 + \frac{\upsilon}{2}\right)\right)^m (1+\theta)^{n-m} \left(1 + \frac{\upsilon\widetilde{\rho}}{2}\right)^{n-m} \left(1 + \frac{\upsilon\widehat{\rho}}{2}\right)^{n-m} \\
&\quad = \quad \left((1-\theta)\left(1 + \widetilde{\rho}\left(1 + \frac{\upsilon}{2}\right)\right)\left(1 + \widehat{\rho}\left(1 + \frac{\upsilon}{2}\right)\right) + (1+\theta)\left(1 + \frac{\upsilon\widetilde{\rho}}{2}\right)\left(1 + \frac{\upsilon\widehat{\rho}}{2}\right)\right)^n \\
&\quad = \quad (2 + A(\widetilde{\rho} + \widehat{\rho}) + B\widetilde{\rho}\widehat{\rho})^n,
\end{aligned}$$

where

$$\begin{aligned} A &:= (1-\theta)\left(1+\frac{v}{2}\right)+(1+\theta)\frac{v}{2} \\ &= \frac{1}{2}(1-\theta)\frac{1+\theta}{\theta}+\frac{1}{2}(1+\theta)\frac{1-\theta}{\theta} \\ &= \frac{1-\theta^2}{\theta}, \end{aligned}$$

and

$$\begin{aligned} B &:= (1-\theta)\left(1+\frac{v}{2}\right)^2+(1+\theta)\left(\frac{v}{2}\right)^2 \\ &= \frac{1}{4}(1-\theta)\left(\frac{1+\theta}{\theta}\right)^2+\frac{1}{4}(1+\theta)\left(\frac{1-\theta}{\theta}\right)^2 \\ &= \frac{1}{2}\frac{1-\theta^2}{\theta^2}. \end{aligned}$$

as announced.

■

The main interest of vectors of the form $z(\rho)$, for $\rho \in \mathbb{R}$, is that they well-behave under the action of the semi-group associated to M_n, more precisely:

$$\forall\ t \geqslant 0, \qquad \exp(tM_n)z(\rho) \;=\; \exp\left(-\frac{n\theta t}{2}\right)z(\exp(\theta t)\rho).$$

To take advatange of this property, let us consider a basis of $\mathbb{R}^{n+1}$, of the form $(z(\rho_k))_{k\in[\![0,n]\!]}$: indeed, classical Vandermonde determinants show that such a family will be a basis as soon all the ρ_k are distinct. Since powers play an important role in the kind of formulas that we have obtained so far, it is convenient to chose a basis of the form $(z(\rho^k))_{k\in[\![0,n]\!]}$, where ρ is a real different from -1, 0 and 1. Then any $z \in \mathbb{R}^{n+1}$ can be written under the form

$$z \;=\; \sum_{k\in[\![0,n]\!]} \nu_k z(\rho^k), \tag{44}$$

where the ν_k, for $k \in [\![0,n]\!]$, are the appropriate coordinates.

Lemma 43 *With the notation (44), we have*

$$\langle z,z\rangle \;=\; (1-\theta^2)^n \sum_{p\in[\![0,n]\!]} \binom{n}{p}\gamma^{n-p}\left(\sum_{k\in[\![0,n]\!]}\left(1+\frac{\rho^k}{2\theta}\right)^p \nu_k\right)^2,$$

where $\gamma := \theta^2/(1-\theta^2) > 0$.

It follows that for any given $t \geqslant 0$, the operator norm of $\exp(tM_n)$ is equal to the square root of the largest eigenvalue of the symmetric matrix $\exp(-n\theta t)\, B^{-1/2}A^*BAB^{-1/2}$, where A and B are respectively the triangular and diagonal matrices defined by

$$\begin{aligned} \forall\ k,l \in [\![0,n]\!], \qquad A_{k,l} &:= \binom{k}{l}(1-\exp(\theta t))^{k-l}\exp(\theta l t), \\ \forall\ k \in [\![0,n]\!], \qquad B_{k,k} &:= \binom{n}{k}\gamma^{n-k}. \end{aligned}$$

Proof

From Lemma 42 and by definition of γ, we get that for any $k, l \in [\![0, n]\!]$,

$$\begin{aligned}
\left\langle z(\rho^k), z(\rho^l)\right\rangle &= \left(1 + \frac{1-\theta^2}{2\theta}(\rho^k + \rho^l) + \frac{1-\theta^2}{4\theta^2}\rho^{k+l}\right)^n \\
&= (1-\theta^2)^n \left(\gamma + \left(1 + \frac{\rho^k}{2\theta}\right)\left(1 + \frac{\rho^l}{2\theta}\right)\right)^n \\
&= (1-\theta^2)^n \sum_{p\in[\![0,n]\!]} \binom{n}{p} \gamma^{n-p} \left(1 + \frac{\rho^k}{2\theta}\right)^p \left(1 + \frac{\rho^l}{2\theta}\right)^p.
\end{aligned}$$

Thus, if we using (44) and expand $\langle z, z\rangle$, we get

$$\begin{aligned}
\langle z, z\rangle &= \sum_{k,l\in[\![0,n]\!]} \nu_k\nu_l \left\langle z(\rho^k), z(\rho^l)\right\rangle \\
&= (1-\theta^2)^n \sum_{p\in[\![0,n]\!]} \binom{n}{p} \gamma^{n-p} \sum_{k,l\in[\![0,n]\!]} \nu_k\nu_l \left(1 + \frac{\rho^k}{2\theta}\right)^p \left(1 + \frac{\rho^l}{2\theta}\right)^p \\
&= (1-\theta^2)^n \sum_{p\in[\![0,n]\!]} \binom{n}{p} \gamma^{n-p} \left(\sum_{k\in[\![0,n]\!]} \left(1 + \frac{\rho^k}{2\theta}\right)^p \nu_k\right)^2,
\end{aligned}$$

as announced.

Concerning the second part of the lemma, since for any $t \geqslant 0$,

$$\exp(tM_n)z = \exp\left(-\frac{n\theta t}{2}\right) \sum_{k\in[\![0,n]\!]} \nu_k z(\exp(\theta t)\rho^k),$$

replacing in the above computation the ρ^k by $\exp(\theta t)\rho^k$, for $k \in [\![0, n]\!]$, leads to

$$\begin{aligned}
&\exp(n\theta t)(1-\theta^2)^{-n} \langle \exp(tM_n)z, \exp(tM_n)z\rangle \\
&= \sum_{p\in[\![0,n]\!]} \binom{n}{p} \gamma^{n-p} \left(\sum_{k\in[\![0,n]\!]} \left(1 + \exp(\theta t)\frac{\rho^k}{2\theta}\right)^p \nu_k\right)^2 \\
&= \sum_{p\in[\![0,n]\!]} \binom{n}{p} \gamma^{n-p} \left(\sum_{k\in[\![0,n]\!]} \left(1 - \exp(\theta t) + \exp(\theta t)\left(1 + \frac{\rho^k}{2\theta}\right)\right)^p \nu_k\right)^2 \\
&= \sum_{p\in[\![0,n]\!]} \binom{n}{p} \gamma^{n-p} \left(\sum_{k\in[\![0,n]\!]} \sum_{l\in[\![0,p]\!]} \binom{p}{l} (1-\exp(\theta t))^{p-l} \exp(\theta l t) \left(1 + \frac{\rho^k}{2\theta}\right)^l \nu_k\right)^2 \\
&= \sum_{p\in[\![0,n]\!]} \binom{n}{p} \gamma^{n-p} \left(\sum_{l\in[\![0,p]\!]} \binom{p}{l} (1-\exp(\theta t))^{p-l} \exp(\theta l t)\eta_l\right)^2,
\end{aligned}$$

where we have defined

$$\forall\, l \in [\![0, n]\!], \qquad \eta_l := \sum_{k\in[\![0,n]\!]} \left(1 + \frac{\rho^k}{2\theta}\right)^l \nu_k.$$

Since $\rho \notin \{-1, 1\}$, Vandermonde determinant insures that the linear morphism $\mathbb{R}^{n+1} \ni (\nu_k)_{k\in[\![0,n]\!]} \mapsto \eta := (\eta_l)_{l\in[\![0,n]\!]} \in \mathbb{R}^{n+1}$ is bijective. Using the matrices A, B defined in the lemma, we can write

$$\langle \exp(tM_n)z, \exp(tM_n)z\rangle = (\exp(-\theta t)(1-\theta^2))^n \langle A\eta, BA\eta\rangle,$$

and considering the same expression at time $t = 0$, it appears that

$$\begin{aligned}\sup_{z\in\mathbb{R}^{n+1}\setminus\{0\}} \frac{\langle \exp(tM_n)z, \exp(tM_n)z\rangle}{\langle z, z\rangle} &= \exp(-n\theta t) \sup_{\eta\in\mathbb{R}^{n+1}\setminus\{0\}} \frac{\langle A\eta, BA\eta\rangle}{\langle \eta, B\eta\rangle} \\ &= \exp(-n\theta t) \sup_{\eta\in\mathbb{R}^{n+1}\setminus\{0\}} \frac{\langle AB^{-1/2}\eta, BAB^{-1/2}\eta\rangle}{\langle \eta, \eta\rangle} \\ &= \exp(-n\theta t) \sup_{\eta\in\mathbb{R}^{n+1}\setminus\{0\}} \frac{\langle \eta, B^{-1/2}A^*BAB^{-1/2}\eta\rangle}{\langle \eta, \eta\rangle}.\end{aligned}$$

The variational caracterization of eigenvalues then implies the second part of the above lemma.

■

The matrix B being non degenerate, the largest eigenvalue of $B^{-1/2}A^*BAB^{-1/2}$ is also the largest eigenvalue of $B^{-1}A^*BA$. Next result determines it:

Lemma 44 *The largest eigenvalue of $B^{-1}A^*BA$ is $\Lambda_{\theta,t}^n \exp(2\theta nt)$, where*

$$\Lambda_{\theta,t} \coloneqq e^{-2\theta t} + \frac{1-\theta^2}{2}\left(\frac{1-e^{-\theta t}}{\theta}\right)^2 + \frac{1-e^{-2\theta t}}{2}\left(1 + \frac{1}{\theta}\sqrt{1 + \frac{1-\theta^2}{\theta^2}\left(\frac{e^{\theta t}-1}{e^{\theta t}+1}\right)^2}\right). \quad (45)$$

Proof

Consider J the $(n+1)$-diagonal matrix with $J_{k,k} = (-1)^k$ for $k \in [\![0, n]\!]$ and $\widetilde{A} \coloneqq JAJ$, whose entries are non-negative. Being conjugate, the two matrices $B^{-1}A^*BA$ and $JB^{-1}A^*BAJ = B^{-1}\widetilde{A}^*B\widetilde{A}$ have the same spectrum. The advantage of $B^{-1}\widetilde{A}^*B\widetilde{A}$ is that all its entries are positive, so Perron-Frobenius theorem asserts that if we can find an eigenvector vector of $B^{-1}\widetilde{A}^*B\widetilde{A}$ whose coordinates are positive, then the corresponding eigenvalue is the largest one. In view of the entries of $\widetilde{A}$ and B, it is natural to try vectors $\eta \coloneqq (\eta_l)_{l\in[\![0,n]\!]} \in \mathbb{R}^{n+1}$ whose coordinates are powers, namely of the form

$$\forall\, l \in [\![0, n]\!], \qquad \eta_l = r^l,$$

where $r > 0$ is to be determined so that η is an eigenvector vector of $B^{-1}\widetilde{A}^*B\widetilde{A}$. We compute that for any $k \in [\![0, n]\!]$,

$$\begin{aligned}(\widetilde{A}\eta)_k &= \sum_{l\in[\![0,n]\!]} \binom{k}{l}(e^{\theta t}-1)^{k-l}e^{\theta lt}r^l \\ &= \left(e^{\theta t} - 1 + e^{\theta t}r\right)^k.\end{aligned}$$

To simplify notation, let $s \coloneqq e^{\theta t} - 1 + e^{\theta t}r$. Then for any $k \in [\![0, n]\!]$, we have

$$\begin{aligned}(\widetilde{A}^*B\widetilde{A}\eta)_k &= \sum_{l\in[\![0,n]\!]} \binom{l}{k}(e^{\theta t}-1)^{l-k}e^{\theta kt}\binom{n}{l}\gamma^{n-l}s^l \\ &= \gamma^n\left(\frac{e^{\theta t}}{e^{\theta t}-1}\right)^k \frac{1}{k!}\sum_{l\geqslant k} l^{(k)}\left(\frac{(e^{\theta t}-1)s}{\gamma}\right)^l \frac{n^{(l)}}{l!} \\ &= \gamma^n\left(\frac{e^{\theta t}}{e^{\theta t}-1}\right)^k \frac{1}{k!}\left(\frac{(e^{\theta t}-1)s}{\gamma}\right)^k n^{(k)}\sum_{l\geqslant k}\frac{(n-k)^{(l-k)}}{(l-k)!}\left(\frac{(e^{\theta t}-1)s}{\gamma}\right)^{l-k} \\ &= \gamma^n\left(\frac{e^{\theta t}s}{\gamma}\right)^k \frac{n^{(k)}}{k!}\left(1 + \frac{(e^{\theta t}-1)s}{\gamma}\right)^{n-k}.\end{aligned}$$

Thus in the end, we get for any $k \in [\![0,n]\!]$,

$$(B^{-1}\widetilde{A}^*B\widetilde{A}\eta)_k \;=\; \left(e^{\theta t}s\right)^k\left(1+\frac{(e^{\theta t}-1)s}{\gamma}\right)^{n-k}.$$

It appears that the vector η is an eigenvector for $B^{-1}\widetilde{A}^*B\widetilde{A}$ if and only if we have

$$e^{\theta t}s\left(1+\frac{(e^{\theta t}-1)s}{\gamma}\right)^{-1} \;=\; r,$$

and in this case the corresponding eigenvalue will be

$$\begin{aligned}\Lambda \;&:=\; \left(1+\frac{(e^{\theta t}-1)s}{\gamma}\right)^n\\ &=\; \left(1+\frac{(e^{\theta t}-1)}{\gamma}(e^{\theta t}-1+e^{\theta t}r)\right)^n.\end{aligned} \tag{46}$$

Expanding the above condition, we end up with the second order equation in r:

$$(e^{\theta t}-1)[e^{\theta t}r^2+(e^{\theta t}-1-\gamma[1+e^{\theta t}])r-\gamma e^{\theta t}] \;=\; 0.$$

For $t>0$, this equation admits a positive solution as required, namely

$$\begin{aligned}r \;&=\; \frac{1}{2}e^{-\theta t}\left(1-e^{\theta t}+\gamma[1+e^{\theta t}]+\sqrt{(e^{\theta t}-1-\gamma[1+e^{\theta t}])^2+4\gamma e^{2\theta t}}\right)\\ &=\; \frac{1}{2}e^{-\theta t}\left(1-e^{\theta t}+\gamma[1+e^{\theta t}]+(1+e^{\theta t})\sqrt{(\gamma+1)\left(\gamma+\left(\frac{e^{\theta t}-1}{e^{\theta t}+1}\right)^2\right)}\right).\end{aligned}$$

Inserting this value in (46), we obtain

$$\begin{aligned}\Lambda \;&=\; \left(1+\frac{(e^{\theta t}-1)}{2\gamma}\left(e^{\theta t}-1+(1+e^{\theta t})\left(\gamma+\sqrt{(\gamma+1)\left(\gamma+\left(\frac{e^{\theta t}-1}{e^{\theta t}+1}\right)^2\right)}\right)\right)\right)^n\\ &=\; \left(1+\frac{(e^{\theta t}-1)^2}{2\gamma}+\frac{e^{2\theta t}-1}{2}\left(1+\sqrt{\left(1+\frac{1}{\gamma}\right)\left(1+\frac{1}{\gamma}\left(\frac{e^{\theta t}-1}{e^{\theta t}+1}\right)^2\right)}\right)\right)^n,\end{aligned}$$

and this leads to the assertion of the lemma.

■

Now we can come back to our project of computing the $\mathbb{L}^2$-operator norms of the hypocoercive semi-group associated to the generator $\widetilde{L}_a$, at least for $a \in (0,1/4)$.

Theorem 45 *For any $a \in (0,1/4)$ and any $t \geqslant 0$, we have*

$$\|\widetilde{P}_t^{(a)}-\widetilde{\mu}_a\|_{\mathbb{L}^2(\widetilde{\mu}_a)\circlearrowleft} \;=\; \sqrt{\Lambda_{\theta,t}}\exp\left(-\frac{1-\theta}{2}t\right),$$

where we recall that $\theta := \sqrt{1-4a}$ and that $\Lambda_{\theta,t}$ was defined in (45).

Proof

We have seen at the beginning of this section that for any $a \in (0,1/4)$ and any $t \geqslant 0$,

$$\|\widetilde{P}_t^{(a)}-\widetilde{\mu}_a\|^2_{\mathbb{L}^2(\widetilde{\mu}_a)\circlearrowleft} \;=\; \sup_{n\in\mathbb{N}}\exp(-nt)\|\exp(tM_n)\|^2. \tag{47}$$

According to Lemmas 43 and 44, we have

$$\|\exp(tM_n)\|^2 \quad = \quad (\exp(\theta t)\Lambda_{\theta,t})^n\,,$$

so that

$$\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\|^2_{\mathbb{L}^2(\widetilde{\mu}_a)\circlearrowleft} \quad = \quad \sup_{n\in\mathbb{N}} (\exp(-(1-\theta)t)\Lambda_{\theta,t})^n\,.$$

Since we know a priori that the l.h.s. is less or equal to 1, necessarily the quantity $\exp(-(1-\theta)t)\Lambda_{\theta,t}$ is less or equal to 1 and the above supremum is attained for $n = 1$.

■

We remark that for any fixed time $t \geqslant 0$, as $\theta > 0$ goes to zero, $\Lambda_{\theta,t}$ converges toward

$$\Lambda_{0,t} \quad := \quad 1 + \frac{t^2}{2} + t\sqrt{1+\left(\frac{t}{2}\right)^2}\,.$$

Since on one hand, for any fixed $n \in \mathbb{N}$ and $t \geqslant 0$, the operator $\exp(tM_n)$ is a continuous function of the (hidden) parameter $a > 0$ and on the other hand, (47) is always true, the previous theorem can be extended for the value $a = 1/4$:

Corollary 46 *For any $t \geqslant 0$, we have*

$$\|\exp(tL_{1/4}) - \widetilde{\mu}_{1/4}\|_{\mathbb{L}^2(\widetilde{\mu}_{1/4})\circlearrowleft} \quad = \quad \sqrt{\Lambda_{0,t}}\exp\left(-\frac{1}{2}t\right).$$

5.3 Exact computation of $\|\widetilde{P}_t^{(a)} - \widetilde{\mu}_a\|_{\mathbb{L}^2(\widetilde{\mu}_a)\circlearrowleft}$, $a > 1/4$

For the remaining part of this section, we consider the situation where $a > 1/4$. The parameter θ is now purely imaginary and we choose $\theta = \sqrt{4a-1}i$. Most of the previous arguments can be extended and we will only insist on the main changes.

First (37) is still valid if we rather rewrite it under the form

$$\forall\ p \in [\![0,n]\!], \qquad \xi_0(p) \quad = \quad \sqrt{\binom{n}{p}}\left(\frac{\sqrt{1-\theta}}{\sqrt{2}}\right)^p\left(\frac{\sqrt{1+\theta}}{\sqrt{2}}\right)^{n-p},$$

where the signs of the two complex numbers $\sqrt{1-\theta}$ and $\sqrt{1+\theta}$ are chosen so that their product is equal to $2\sqrt{a}$. Anyway, the important object is $(|\xi_0(p)|^2)_{p\in[\![0,n]\!]}$, which is just $((1+|\theta|^2)/4)^{n/2}\left(\binom{n}{p}\right)_{p\in[\![0,n]\!]}$, since $|1+\theta| = |1-\theta| = \sqrt{1+|\theta|^2}$. Indeed, Lemmas 36, 37 and 40 don't need to be modified, since they only deal with algebraic properties of ξ_0, J_- and J_+. Similarly, we consider the family $(\zeta_p)_{p\in[\![0,n]\!]}$ defined by (40). The next change comes with Lemma 39, which must rather state that for any $p, q \in [\![0,n]\!]$, we have

$$\langle\zeta_p,\zeta_q\rangle \quad = \quad (1+|\theta|^2)^{n/2}\beta_{1/2}[P_p\overline{P_q}], \tag{48}$$

where $\beta_{1/2}$ is the binomial distribution of parameter 1/2 (note that $\langle\cdot,\cdot\rangle$ now stands for the usual Hermitian product on $\mathbb{C}^{n+1}$).

Definition 42 can be extended to any $\rho \in \mathbb{C}$, but Lemma 42 must be replaced by

Lemma 47 *For any $\widetilde{\rho}, \widehat{\rho} \in \mathbb{C}$, we have*

$$\langle z(\widetilde{\rho}), z(\widehat{\rho})\rangle \quad = \quad \left(1+|\theta|^2\right)^{-n/2}\left(\gamma + \left(1+\frac{\widetilde{\rho}}{\delta}\right)\overline{\left(1+\frac{\widehat{\rho}}{\delta}\right)}\right)^n,$$

where

$$\gamma \;:=\; |\theta|^2 \qquad \textit{and} \qquad \delta \;:=\; \frac{2\theta}{1+|\theta|^2}.$$

Proof

The first part of the proof of Lemma 42 and (48) show that

$$\begin{aligned}
&\langle z(\widetilde{\rho}), z(\widehat{\rho})\rangle \\
&= \left(\frac{1+|\theta|^2}{4}\right)^{n/2}\left(\left(1+\widetilde{\rho}\frac{1+\theta}{2\theta}\right)\overline{\left(1+\widehat{\rho}\frac{1+\theta}{2\theta}\right)}+\left(1+\widetilde{\rho}\frac{1-\theta}{2\theta}\right)\overline{\left(1+\widehat{\rho}\frac{1-\theta}{2\theta}\right)}\right)^n \\
&= \left(\frac{1+|\theta|^2}{4}\right)^{n/2}\left(2+\frac{\widetilde{\rho}}{\theta}+\frac{\overline{\widehat{\rho}}}{\overline{\theta}}+\frac{|1+\theta|^2}{2}\frac{\widetilde{\rho}}{\theta}\frac{\overline{\widehat{\rho}}}{\overline{\theta}}\right)^n \\
&= \left(\frac{\sqrt{1+|\theta|^2}}{2}\right)^n\left(2-\frac{2}{1+|\theta|^2}+\left(\sqrt{\frac{2}{1+|\theta|^2}}+\sqrt{\frac{1+|\theta|^2}{2}}\frac{\widetilde{\rho}}{\theta}\right)\left(\sqrt{\frac{2}{1+|\theta|^2}}+\sqrt{\frac{1+|\theta|^2}{2}}\frac{\overline{\widehat{\rho}}}{\overline{\theta}}\right)\right)^n.
\end{aligned}$$

expression which coincides with the announced one.

■

Fix any complex number ρ whose norm is different from 0 and 1. Then $(z(\rho^k))_{k\in[\![0,n]\!]}$ is a basis of $\mathbb{C}^{n+1}$ and any $z\in\mathbb{C}^{n+1}$ can be written under the form (44), where the ν_k, for $k\in[\![0,n]\!]$, are the appropriate complex coordinates.

Using Lemma 47 and the notations introduced there, we obtain as in Lemma 43 that

$$\begin{aligned}
\langle z,z\rangle &= \left(1+|\theta|^2\right)^{-n/2}\sum_{p\in[\![0,n]\!]}\binom{n}{p}\gamma^{n-p}\left|\sum_{k\in[\![0,n]\!]}\left(1+\frac{\rho^k}{\delta}\right)^p\nu_k\right|^2 \\
&= \left(1+|\theta|^2\right)^{-n/2}\sum_{p\in[\![0,n]\!]}\binom{n}{p}\gamma^{n-p}\left|\eta_p\right|^2,
\end{aligned}$$

with

$$\forall\ l\in[\![0,n]\!],\qquad \eta_l \ \coloneqq\ \sum_{k\in[\![0,n]\!]}\left(1+\frac{\rho^k}{\delta}\right)^l\nu_k.$$

As in the proof of Lemma 43, we also compute that for any $t\geqslant 0$,

$$\begin{aligned}
&\langle \exp(tM_n)z, \exp(tM_n)z\rangle \\
&= \left(1+|\theta|^2\right)^{-n/2}|\exp(-n\theta t/2)|^2\sum_{p\in[\![0,n]\!]}\binom{n}{p}\gamma^{n-p}\left|\sum_{l\in[\![0,p]\!]}\binom{p}{l}(1-\exp(\theta t))^{p-l}\exp(\theta l t)\eta_l\right|^2 \\
&= \left(1+|\theta|^2\right)^{-n/2}\sum_{p\in[\![0,n]\!]}\binom{n}{p}\gamma^{n-p}\left|\sum_{l\in[\![0,p]\!]}\binom{p}{l}(1-\exp(\theta t))^{p-l}\exp(\theta l t)\eta_l\right|^2.
\end{aligned}$$

Thus, if the matrices A and B are defined in the same way as in Lemma 43, it appears the $\mathbb{L}^2$-operator norm of $\exp(tM_n)$ is equal to the square root of the largest eigenvalue of the Hermitian matrix $B^{-1/2}A^*BAB^{-1/2}$, where A^* is the Hermitian adjoint matrix associated to A. If $\exp(\theta t)\neq 1$, consider the diagonal matrices $\widetilde{C}$ and $\widehat{C}$ defined by

$$\begin{aligned}
\forall\ k\in[\![0,n]\!],\qquad \widetilde{C}_{k,k} &\coloneqq \frac{(1-e^{\theta t})^k}{|1-e^{\theta t}|^k}, \\
\widehat{C}_{k,k} &\coloneqq \frac{(1-e^{\theta t})^{-k}e^{\theta k t}}{|(1-e^{\theta t})^{-1}|^k}.
\end{aligned}$$

We can write $A = \widetilde{C}\widetilde{A}\widehat{C}$, where the entries of $\widetilde{A}$ are just the absolute values of the entries of A. The interest is that $B^{-1/2}A^*BAB^{-1/2} = \widehat{C}^{-1}B^{-1/2}\widetilde{A}^*B\widetilde{A}B^{-1/2}\widehat{C}$ is conjugate to the symmetric matrix $B^{-1/2}\widetilde{A}^*B\widetilde{A}B^{-1/2}$, whose entries are positive. We can then use Perron-Frobenius theorem to get

Lemma 48 *The largest eigenvalue of $B^{-1/2}A^*BAB^{-1/2}$ is $\Lambda_{\theta,t}^n$, where*

$$\Lambda_{\theta,t} \quad := \quad \begin{cases} 1+\frac{\nu^2}{2|\theta|^2}\left(1+\sqrt{1+4\frac{|\theta|^2}{\nu^2}}\right) & , \text{ if } \nu \neq 0 \\ 1 & , \text{ if } \nu = 0 \end{cases}$$

with $\nu := \left|e^{\theta t}-1\right|$.

Proof

If $\nu = 0$, A is just the identity matrix, so the result is immediate. Assume that $\nu \neq 0$. As in Lemma 44, the wanted largest eigenvalue is also the largest eigenvalue of $B^{-1}\widetilde{A}^*B\widetilde{A}$ and it is sufficient to find a corresponding positive eigenvector. Again we look for a vector $\eta := (\eta_l)_{l\in[\![0,n]\!]} \in \mathbb{R}^{n+1}$ whose coordinates are of the form

$$\forall\, l \in [\![0,n]\!], \qquad \eta_l \quad = \quad r^l,$$

where $r > 0$ is to be determined to insure that η is an eigenvector vector of $B^{-1}\widetilde{A}^*B\widetilde{A}$. Let us denote $s = \nu + r$, so that for any $k \in [\![0,n]\!]$,

$$\begin{aligned}(\widetilde{A}\eta)_k \quad &= \quad \sum_{l\in[\![0,n]\!]}\binom{k}{l}\nu^{k-l}r^l \\ &= \quad s^k.\end{aligned}$$

It follows that for any $k \in [\![0,n]\!]$,

$$\begin{aligned}(\widetilde{A}^*B\widetilde{A}\eta)_k \quad &= \quad \sum_{l\in[\![0,n]\!]}\binom{l}{k}\nu^{l-k}\binom{n}{l}\gamma^{n-l}s^l \\ &= \quad \gamma^n\frac{1}{k!}\sum_{l\geqslant k} l^{(k)}\nu^{l-k}\left(\frac{s}{\gamma}\right)^l\frac{n^{(l)}}{l!} \\ &= \quad \gamma^{n-k}\frac{s^k}{k!}n^{(k)}\sum_{l\geqslant k}\binom{n-k}{l-k}\left(\frac{\nu s}{\gamma}\right)^{l-k} \\ &= \quad \binom{n}{k}\gamma^{n-k}s^k\left(1+\frac{\nu s}{\gamma}\right)^{n-k}.\end{aligned}$$

Finally, it appears that for any $k \in [\![0,n]\!]$,

$$(B^{-1}\widetilde{A}^*B\widetilde{A}\eta)_k \quad = \quad s^k\left(1+\frac{\nu s}{\gamma}\right)^{n-k}.$$

Thus η will be the wanted eigenvector, with $\Lambda := (1+\nu(\nu+r)/\gamma)^n$ as associated eigenvalue, if the following equation has a positive solution r:

$$\nu + r \quad = \quad r\left(1+\frac{\nu(\nu+r)}{\gamma}\right).$$

Since $\nu \neq 0$, this is equivalent to $r^2+\nu r-\gamma = 0$, which admits $r = (-\nu+\sqrt{\nu^2+4\gamma})/2$ as positive solution. Expanding Λ, we end up with the announced result.

■

The arguments of the proof of Theorem 45 enable to conclude the computation of $\mathbb{L}^2$-operator norms of the hypocoercive semi-group associated to the generator $\widetilde{L}_a$, for $a \in (1/4, \infty)$:

Theorem 49 *For any $a > 1/4$ and any $t \geqslant 0$, we have*

$$\| \widetilde{P}_t^{(a)} - \widetilde{\mu}_a \|^2_{\mathbb{L}^2(\widetilde{\mu}_a)\circlearrowleft} \quad = \quad C_a(t) e^{-t},$$

with

$$C_a(t) \quad \coloneqq \quad 1 + \frac{\left| e^{\sqrt{4a-1}it} - 1 \right|}{2(4a-1)} \left(\left| e^{\sqrt{4a-1}it} - 1 \right| + \sqrt{\left| e^{\sqrt{4a-1}it} - 1 \right|^2 + 4(4a-1)} \right).$$

Let us finish this section by noting that for all $a > 0$, the maximizing functions for the computation of the operator norms of the semi-group $(\widetilde{P}_t^{(a)})_{t \geqslant 0}$ belong to H_1, namely are linear mapping (but they are not eigenfunctions of $\widetilde{L}_a$). This justifies the assertion made before Remark 27.

6 Concluding remarks

One common feature of the previous analysis of L_a or $\widetilde{L}_a$, for $a > 0$, is that the underlying $\mathbb{L}^2$ space was decomposed into $\oplus_{p \in \mathcal{P}} V_p$, where the subspaces V_p are orthogonal and left invariant by the generator at hand. In the first model the index set $\mathcal{P}$ is $\mathbb{Z}_+ \sqcup \mathbb{N} \sqcup \mathbb{N}$ and $\mathbb{Z}_+$ in the second model. These decompositions were maximal, in the sense that each of the V_p, $p \in \mathcal{P}$, cannot be non-trivially decomposed further (due to the non-orthogonality of all the eigenvectors belonging to V_p). Inside each of the V_p, $p \in \mathcal{P}$, the restriction of the generator was written under the form $K_p + R_p - R_p^*$, where K_p is self-adjoint in V_p and where the brackets of the operators K_p, R_p and R_p^* have nice forms (especially $[K_p, R_p] = R_p$, which implies that there is a basis consisting of eigenvectors of K_p in which the matrix of R_p has a below-diagonal form, thus among decompositions of the type $K_p + R_p - R_p^*$, R_p is in some sense minimal). Indeed, everything was deduced from the relations satisfied by these brackets. So it is natural to wonder if something is left of these observations for more general models.

First we note that the decompositions of the restriction of generator to the subspace V_p, $p \in \mathcal{P}$, can be lifted into a decomposition $K + R - R^*$ of the initial generator, where $K \coloneqq \oplus_{p \in \mathcal{P}} K_p$ and $R \coloneqq \oplus_{p \in \mathcal{P}} R_p$. More precisely, in the first model we get

$$\begin{aligned} K &= a\partial_y^2 - y\partial_y \\ R &= y\partial_x - a\partial_x\partial_y \\ R^* &= -a\partial_x\partial_y, \end{aligned}$$

with

$$[K, R] = R, \qquad [R, R^*] = aJ,$$

where $J = \partial_x^2$ is a coercive operator on $\mathbb{T}$ (and for any $\alpha, \beta > 0$, $\alpha J + \beta K$ is coervive on $\mathbb{T} \times \mathbb{R}$). Similarly, in the second model we have

$$\begin{aligned} K &= \partial_y^2 - y\partial_y \\ R &= y\partial_x - \partial_x\partial_y \\ R^* &= ax\partial_y - \partial_x\partial_y, \end{aligned}$$

with

$$[K, R] = R, \qquad [R, R^*] = J - aK,$$

where $J \coloneqq \partial_x \partial_x^* = \partial_x^2 - ax\partial_x$ is a coercive Ornstein-Ulhenbeck operator on $\mathbb{R}$.

In the literature about hypocoercivity, the authors have often a predilection for brackets of first order operators (this is maybe due to the importance of Hörmander's condition in hypoellipticity), but it seems that in the considered toy models, the key is given by brackets between second order operators.

More generally, let be given a smooth potential $U : \mathbb{T} \to \mathbb{R}$ and consider on $\mathbb{T} \times \mathbb{R}$ the kinetic operator

$$L \quad \coloneqq \quad y\partial_x - U'(x)\partial_y + \partial_y^2 - y\partial_y.$$

The following remarks can be adapted to the situation of potentials defined on $\mathbb{R}$, under appropriate conditions. The associated invariant probability is $\mu \coloneqq \nu \times \gamma_1$, where ν is the Gibbs measure on $\mathbb{T}$ whose density with respect to the Lebesgue measure λ is proportional to $\exp(-U)$. As above, we can write $L = K + R - R^*$, where

$$\begin{aligned} K &= \partial_y^2 - y\partial_y \\ R &= y\partial_x - \partial_x\partial_y \\ R^* &= U'(x)\partial_y - \partial_x\partial_y \end{aligned}$$

The operator K is still self-adjoint in $\mathbb{L}^2(\mu)$ and R^* is adjoint to R. We furthermore have

$$[K, R] = R, \qquad\qquad [R, R^*] = J - U''K$$

where $J \coloneqq \partial_x^* \partial_x = \partial_x^2 - U'(x)\partial_x$ is the usual coercive Langevin operator associated to U on $\mathbb{T}$ (note that the formulation $L = K + R - R^*$ is different from the one proposed by Villani $L = A^*A + B$ in the first chapter of [21] since our operator K is a second order one).

We are wondering if these properties could not be used to deduce, in a direct manner, hypocoercive bounds for the semi-group $(P_t)_{t\geqslant 0}$ associated to L. So let $f \in \mathbb{L}^2(\mu)$ be given with $\mu[f] = 0$ and denote for $t \geqslant 0$, $F_t \coloneqq \mu[(P_t[f])^2]$. Since we expect behaviors such as (1), (3) and (4) to be valid again, it is natural to look for inequalities satisfied by F_t, F_t', F_t'' and F_t'''. So let us compute formally (a justification would require regularity assumptions on f) these derivatives: using the relation $[K, R] = R$, we get that for all $t \geqslant 0$,

$$\begin{aligned} F_t' &= 2\left\langle Kf_t, f_t\right\rangle, \\ F_t'' &= 4\left\langle K^2 f_t, f_t\right\rangle - 4\left\langle f_t, Rf_t\right\rangle, \\ F_t''' &= 8\left\langle K^3 f_t, f_t\right\rangle - 24\left\langle Kf_t, Rf_t\right\rangle - 12\left\langle f_t, Rf_t\right\rangle + 4\left\langle [R, R^*]f_t, f_t\right\rangle. \end{aligned}$$

where f_t is a short hand for $P_t[f]$ and where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{L}^2(\mu)$. In view of $[R, R^*]$, which brings the missing coercivity through J, hope is rising.

We first tried to find three constants $A, B, C > 0$ such that for regular functions f and for all $t \geqslant 0$,

$$AF_t + BF_t' + CF_t'' + F_t''' \quad \leqslant \quad 0.$$

It is sufficient to prove such a differential inequality with $t = 0$. Interpreting $AF_0 + BF_0' + CF_0'' + F_0'''$ as a quadratic form in f, we would like to find $A, B, C > 0$ so that it is non-positive definite. In fact, we were able to attain this objective in the case $U \equiv 0$, then up to appropriate changes of the constants $A, B, C > 0$ (where would enter the supremum norms of U' and U''), it could be extended to all smooth potential U. That is where we are brought back to the first toy model (with $a = 1$). Unfortunately, in this simple case, we can show that there is no choice of the constants $A, B, C > 0$ so that the quadratic form $AF_0 + BF_0' + CF_0'' + F_0'''$ is non-positive definite. Despite the fact that for any $p \in \mathbb{N}$, it is possible to find "constants" $A_p, B_p, C_p > 0$ such that the restriction to $\mathcal{V}_p$ (and to $\mathcal{W}_p$, with the notations of Section 2) of the quadratic form $A_pF_0 + B_pF_0' + C_pF_0'' + F_0'''$

is non-positive definite (an analogous statement is valid in the Gaussian case). Thus an idea is missing to push further this alternative approach.

Furthermore, these considerations are maybe not without links with the traditional approach, where the $\mathbb{L}^2$ norm is modified by the addition of terms, since among them, $\langle f, Rf\rangle = \langle \partial_x f, \partial_y f\rangle$ plays a major role (see for instance Villani [21]).

Aknowledgments: We are grateful to Francis Nier for pointing out the papers of Hérau [10] and Ottobre, Pavliotis and Pravda-Starov [17] and for related conversations. We also aknowledge fruitful discussions with Dominique Bakry.

References

[1] Dominique Bakry, Patrick Cattiaux, and Arnaud Guillin. Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *J. Funct. Anal.*, 254(3):727–759, 2008.

[2] L. Desvillettes and C. Villani. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. *Comm. Pure Appl. Math.*, 54(1):1–42, 2001.

[3] Persi Diaconis, Laurent Miclo, and Jessica Zuñiga. On the spectral analysis of second-order Markov chains. Unpublished preprint, 2010.

[4] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. preprint available at `http://hal.archives-ouvertes.fr/hal-00482286/PDF/DMS-I-20.pdf`, 2010.

[5] J.-P. Eckmann and M. Hairer. Non-equilibrium statistical mechanics of strongly anharmonic chains of oscillators. *Comm. Math. Phys.*, 212(1):105–164, 2000.

[6] J.-P. Eckmann and M. Hairer. Spectral properties of hypoelliptic operators. *Comm. Math. Phys.*, 235(2):233–253, 2003.

[7] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[8] Brian C. Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003. An elementary introduction.

[9] Bernard Helffer and Francis Nier. *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, volume 1862 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.

[10] Frédéric Hérau. Short and long time behavior of the Fokker-Planck equation in a confining potential and applications. *J. Funct. Anal.*, 244(1):95–118, 2007.

[11] Frédéric Hérau and Francis Nier. Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. *Arch. Ration. Mech. Anal.*, 171(2):151–218, 2004.

[12] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[13] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.

[14] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[15] Otared Kavian. Personal communication.

[16] Anna Lee. Centro-Hermitian and skew-centro-Hermitian matrices. *Linear Algebra Appl.*, 29:205–210, 1980.

[17] M. Ottobre, G. A. Pavliotis, and K. Pravda-Starov. Exponential return to equilibrium for hypoelliptic quadratic systems. *ArXiv e-prints*, June 2011.

[18] L. Rey-Bellet and L. E. Thomas. Fluctuations of the entropy production in anharmonic chains. *Ann. Henri Poincaré*, 3(3):483–502, 2002.

[19] H. Risken. *The Fokker-Planck equation*, volume 18 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, second edition, 1989. Methods of solution and applications.

[20] Gábor Szegő. *Orthogonal polynomials*. American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[21] Cédric Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202(950):iv+141, 2009.

[22] James R. Weaver. Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, and eigenvectors. *Amer. Math. Monthly*, 92(10):711–717, 1985.

[23] Kōsaku Yosida. *Functional analysis*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

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