ON SECOND ORDER DIFFERENTIAL EQUATIONS WITH ASYMPTOTICALLY SMALL DISSIPATION

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ABSTRACT. We investigate the asymptotic properties as $t \to \infty$ of the differential equation

$$(\mathcal{S}) \qquad \ddot{x}(t) + a(t)\dot{x}(t) + \nabla G(x(t)) = 0, \quad t \geq 0,$$

where $x(\cdot)$ is $\mathbb{R}$-valued, the map $a : \mathbb{R}_+ \to \mathbb{R}_+$ is non increasing, and $G : \mathbb{R} \to \mathbb{R}$ is a potential with locally Lipschitz continuous derivative. We identify conditions on the function $a(\cdot)$ that guarantee or exclude the convergence of solutions of this problem to points in $\operatorname{argmin} G$, in the case where G is convex and $\operatorname{argmin} G$ is an interval. The condition $\int_0^\infty e^{-\int_0^t a(s)\,ds} dt < \infty$ is shown to be necessary for convergence of trajectories, and a slightly stronger condition is shown to be sufficient.

1. INTRODUCTION

In this note, we study the differential equation

$$(\mathcal{S}) \qquad \ddot{x}(t) + a(t)\dot{x}(t) + \nabla G(x(t)) = 0, \quad t \geq 0$$

where $x(\cdot)$ is $\mathbb{R}$-valued, the map $G : \mathbb{R} \to \mathbb{R}$ is at least of class $\mathcal{C}^1$, and $a : \mathbb{R}_+ \to \mathbb{R}_+$ is a non increasing function. In a previous paper [3], we studied this differential equation in a finite- or infinite-dimensional Hilbert space $\mathcal{H}$. We are interested in the case where $a(t) \to 0$ as $t \to \infty$. Broadly speaking, convergence of solutions can be expected if $a(t) \to 0$ sufficiently slowly. One of the questions left open in that paper was whether solutions converge to a limit if the property

$$\int_0^\infty e^{-\int_0^t a(s)ds} dt = \infty \tag{1}$$

does *not* hold and if $\operatorname{argmin} G$ consists of more than just one point. In this note, we give a positive answer to this question, in the one dimensional case.

2. PRELIMINARY FACTS

Throughout this paper, we will denote by $G : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^1$ function for which the derivative $g = G'$ is Lipschitz continuous, uniformly on bounded sets. The function $a : \mathbb{R}_+ \to \mathbb{R}_+$ will always be assumed to be continuous and non-increasing. We also define the energy

$$\mathcal{E}(t) = G(x(t)) + \frac{1}{2}|\dot{x}(t)|^2. \tag{2}$$

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Here are some basic results for solutions of $(\mathcal{S})$ from [3].

For any $(x_0,\, x_1) \in \mathbb{R}^2$, the problem $(\mathcal{S})$ has a unique solution $x(\cdot) \in \mathcal{C}^2([0,T),\mathbb{R})$ satisfying $x(0) = x_0$, $\dot{x}(0) = x_1$ on some maximal time interval $[0,T) \subset [0,\infty)$. For every $t \in [0,T)$, the energy identity holds

$$\frac{d}{dt}\mathcal{E}(t) = -a(t)|\dot{x}(t)|^2. \tag{3}$$

If in addition G is bounded from below, then

$$\int_0^T a(t)|\dot{x}(t)|^2 dt < \infty\,, \tag{4}$$

and the solution exists for all $T > 0$. If also $G(\xi) \to \infty$ as $|\xi| \to \infty$ (i.e. if G is *coercive*), then all solutions to $(\mathcal{S})$ remain bounded together with their first and second derivatives for all $t > 0$. The bound depends only on the initial data. If a solution x to $(\mathcal{S})$ converges toward some $\overline{x} \in \mathbb{R}$, then $\lim_{t\to\infty} \dot{x}(t) = \lim_{t\to\infty} \ddot{x}(t) = 0$ and $\nabla G(\overline{x}) = 0$. If $\int_0^\infty a(s)\, ds < \infty$ and if $\inf G > -\infty$, then solutions $x(\cdot)$ of $(\mathcal{S})$ for which $(x(0), \dot{x}(0)) \notin \operatorname{argmin} G \times \{0\}$ cannot converge to a point in $\operatorname{argmin} G$.

For the remainder of this note we shall assume that $\operatorname{argmin} G \neq \emptyset$. Without loss of generality, we may assume that $\min_{\mathbb{R}} G = 0$ and $G(0) = 0$. If for some $\rho \in \mathbb{R}_+$

$$(G) \qquad \forall x \in \mathbb{R}, \qquad G(x) - G(z) \leq \rho\, G'(x)(x - z).$$

then it is possible to show that any solution x to the differential equation $(\mathcal{S})$ satisfies

$$\int_0^\infty a(t)\, \mathcal{E}(t)\, dt < \infty. \tag{5}$$

Since $t \mapsto \mathcal{E}(t)$ is decreasing, this estimate implies that $\mathcal{E}(t) \to \min G = 0$ as $t \to \infty$, provided that $\int_0^\infty a(t)\, dt = \infty$. If now $\operatorname{argmin} G = \{\overline{x}\}$ is a singleton, then trajectories must converge to $\overline{x}$ under fairly weak additional conditions. The reader is referred to [3] for details.

3. CONVEX POTENTIALS WITH NON-UNIQUE MINIMA

In this section, we investigate the convergence of the trajectories of $(\mathcal{S})$ when $\operatorname{argmin} G$ is *not* a singleton. While the previous discussion shows that $\int_0^\infty a(s)ds = \infty$ is a necessary condition for trajectories to converge to a point in $\operatorname{argmin} G$, this condition is clearly not sufficient, as the particular case $G \equiv 0$ shows. In this case, the solution is given by

$$x(t) = x(0) + \dot{x}(0) \int_0^t e^{-\int_0^s a(u)\, du} ds$$

and the solution x converges if and only if (1) does not hold. Therefore it is natural to ask whether for a general potential G, the trajectory x is convergent if this condition does not hold. The potential G is assumed to have all the properties listed in the previous section. A general result of non-convergence of the trajectories under the condition (1) is shown in [3]. There, we assume that G is coercive, $\inf_{\mathbb{R}} G = 0$, $\operatorname{argmin} G = [\alpha, \beta]$ for some $\alpha < \beta$, and that G is non-increasing on $(-\infty, \alpha]$ and non-decreasing on $[\beta, \infty)$. It is also assumed that a satisfies condition (1). Then either a solution satisfies $(x(0), \dot{x}(0)) \in [\alpha, \beta] \times \{0\}$, or else the ω- limit set $\omega(x_0, \dot{x}_0)$ contains $[\alpha, \beta]$ and hence the trajectory x does not converge.

We now ask if the converse assertion is true: do the trajectories x of $(\mathcal{S})$ converge if (1) does not hold? We give a positive answer when the map a satisfies the following stronger condition

$$\int_0^\infty e^{-\theta \int_0^s a(u)\,du} ds < \infty, \tag{6}$$

for some $\theta \in (0,1)$.

Theorem 3.1. *Let $G : \mathbb{R} \to \mathbb{R}$ be a convex function of class $\mathcal{C}^1$ such that G' is Lipschitz continuous on the bounded sets of $\mathbb{R}$. Assume that* $\operatorname{argmin} G = [\alpha, \beta]$ *with $\alpha < \beta$ and that there exists $\delta > 0$ such that*

$$\forall \xi \in (-\infty, \alpha], \quad G'(\xi) \le 2\,\delta\,(\xi - \alpha) \quad \text{and} \quad \forall \xi \in [\beta, \infty), \quad G'(\xi) \ge 2\,\delta\,(\xi - \beta).$$

Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable non increasing map such that $\lim_{t\to\infty} a(t) = 0$ *and such that condition (6) holds for some positive $\theta < 1$. Then, for any solution non constant solution x to the differential equation $(\mathcal{S})$,* $\lim_{t\to\infty} x(t)$ *exists.*

Proof. We may assume without loss of generality that $\alpha = 0, \beta = 1$. The conditions on G imply that it is coercive, hence $\lim_{t\to\infty} \mathcal{E}(t) = 0$ and $|x(t)| \le M$ for some $M > 0$, for all $t \in \mathbb{R}_+$.

Define the set $\mathcal{T} = \{t \ge 0 \,|\, \dot{x}(t) = 0\}$ of sign changes of $\dot{x}$. This set must be discrete, for if it had an accumulation point t^*, then $\dot{x}(t^*) = 0$ and also $\ddot{x}(t^*) = 0$ by Rolle's Theorem. Since then $\dot{x}(t^*) = \ddot{x}(t^*) = G'(x(t^*)) = 0$, x would have to be the constant solution 0, which yields a contradiction.

If $\mathcal{T}$ is a finite set, then $\dot{x}$ does not change sign for sufficiently large t, and the trajectory x has a limit. Let us therefore assume that $\mathcal{T} = \{t_n \,|\, n \in \mathbb{N}\}$, where the t_n are increasing and tend to ∞. We want to show that this is impossible. Observe that at each t_n, $\dot{x}$ must change its sign and $G'(x(t_n)) \ne 0$, since otherwise also $\ddot{x}(t_n) = 0$ and we would again have a stationary solution. Without loss of generality, we can assume that $\dot{x}(0) < 0$, $x(0) < 0$ and therefore $x(t_0) < 0$. Since $G'(x(t_0)) < 0$, equation $(\mathcal{S})$ shows that $\ddot{x}(t_0) > 0$, hence the map $\dot{x}$ is positive on (t_0, t_1), $x(t_1) > 1$, $\dot{x}$ is negative on (t_1, t_2), and so on.

The argument so far shows that $G'(x(t))$ vanishes on a union of infinitely many disjoint closed intervals,

$$\{t \,|\, 0 \le x(t) \le 1\} = \cup_{k\ge 0}[u_{2k}, u_{2k+1}]$$

where $0 < t_0 < u_0$ and $u_{2k-1} < t_k < u_{2k}$ for $k = 1, 2, \ldots$. Let us observe that, for every $k \in \mathbb{N}$,

$$1 = |x(u_{2k+1}) - x(u_{2k})| = \int_{u_{2k}}^{u_{2k+1}} |\dot{x}(t)|\,dt \le |u_{2k+1} - u_{2k}| \max_{t \ge u_{2k}} |\dot{x}(t)|.$$

Since $\lim_{t\to\infty} \dot{x}(t) = 0$, we deduce that $\lim_{k\to\infty} |u_{2k+1} - u_{2k}| = \infty$.

We next observe that for $u_{2k} \le t \le u_{2k+1}$ the function $v = \dot{x}$ satisfies $\dot{v}(t) + a(t)v(t) = 0$ and hence

$$\forall t \in [u_{2k}, u_{2k+1}], \qquad \dot{x}(t) = \dot{x}(u_{2k}) e^{-\int_{u_{2k}}^t a(\tau)d\tau}. \tag{7}$$

Claim 3.1. *There is a constant γ such that $u_{2k+2} - u_{2k+1} \le \gamma$ for all $k \in \mathbb{N}$.*

To show this claim, fix $k \in \mathbb{N}$ and assume that $t \in [u_{2k+1}, u_{2k+2}]$. Assume for now that k is odd and thus $x(t) \leq 0$. Define the quantity $A(t) = \exp\left(\frac{1}{2}\int_0^t a(s)\,ds\right)$ and set $y(t) = A(t)\,x(t)$. Then y is the solution of the differential equation

$$\ddot{y}(t) + A(t)\,G'\left(\frac{y(t)}{A(t)}\right) - \left(\frac{a^2(t)}{4} + \frac{\dot{a}(t)}{2}\right) y(t) = 0, \tag{8}$$

and satisfies $y(u_{2k+1}) = y(u_{2k+2}) = 0$ and $\dot{y}(u_{2k+1}) = A(u_{2k+1})\,\dot{x}(u_{2k+1}) < 0$. Since the map a converges to 0, we can choose k large enough so that $a(t) < 2\sqrt{\delta}$ for every $t \in [u_{2k+1}, u_{2k+2}]$. On the other hand, the assumption on G' shows that, for every $t \in [u_{2k+1}, u_{2k+2}]$,

$$A(t)\,G'\left(\frac{y(t)}{A(t)}\right) \leq 2\,\delta\,y(t).$$

Recalling finally that $\dot{a}(t) \leq 0$ for every $t \geq 0$, we deduce from (8) that

$$\forall t \in [u_{2k+1}, u_{2k+2}], \qquad \ddot{y}(t) + \delta\,y(t) \geq 0.$$

The unique solution z of the differential equation $\ddot{z}(t) + \delta\,z(t) = 0$ with the same initial conditions as y has the first zero larger than u_{2k+1} at $u_{2k+1} + \frac{\pi}{\sqrt{\delta}}$. By a standard comparison argument, we deduce that y vanishes before z does, hence

$$u_{2k+2} \leq u_{2k+1} + \gamma, \quad \gamma = \frac{\pi}{\sqrt{\delta}}.$$

The same argument applies if k is even. This proves the claim.

Claim 3.2. *There is a $k_0 \in \mathbb{N}$ such that for $k \geq k_0$*

$$|\dot{x}(u_{2k+2})| \leq |\dot{x}(u_{2k})|\,e^{-\theta\int_{u_{2k}}^{u_{2k+2}} a(s)\,ds}.$$

where θ is as in (6).

To prove this, pick k_0 so large that for all $k \geq k_0$

$$(1-\theta)(u_{2k+2} - u_{2k}) \geq \gamma\theta.$$

This is possible since $u_{2k+2} - u_{2k} \to \infty$ as $k \to \infty$. Since a is non-increasing, this implies that

$$\begin{aligned}
\theta\int_{u_{2k+1}}^{u_{2k+2}} a(\tau)d\tau &\leq \gamma\theta a(u_{2k+1}) \leq (1-\theta)(u_{2k+1} - u_{2k})a(u_{2k+1}) \\
&\leq (1-\theta)\int_{u_{2k}}^{u_{2k+1}} a(\tau)d\tau
\end{aligned}$$

and hence

$$\theta\int_{u_{2k}}^{u_{2k+2}} a(\tau)d\tau \leq \int_{u_{2k}}^{u_{2k+1}} a(\tau)d\tau.$$

Then for $k \geq k_0$,

$$\begin{aligned}
|\dot{x}(u_{2k+2})| &\leq |\dot{x}(u_{2k+1})| = |\dot{x}(u_{2k})|e^{-\int_{u_{2k}}^{u_{2k+1}} a(s)\,ds} \\
&\leq |\dot{x}(u_{2k})|e^{-\theta\int_{u_{2k}}^{u_{2k+2}} a(s)\,ds}
\end{aligned}$$

proving the claim.

Claim 3.3. *If the set $\mathcal{T}$ is unbounded, there must exist a constant C, depending on $\mathcal{T}$ and on $x(0), \dot{x}(0)$ such that for all $t \geq 0$*

$$|\dot{x}(t)| \leq C\, e^{-\theta \int_0^t a(s)\, ds}. \tag{9}$$

By making sure that C is sufficiently large, we only have to prove the estimate for $t \geq u_{2k_0}$. First assume that $u_{2k} \leq t \leq u_{2k+1}$ for some k. Then from (7)

$$|\dot{x}(t)| \leq |\dot{x}(u_{2k})|\, e^{-\int_{u_{2k}}^t a(s)\, ds} \leq |\dot{x}(u_{2k})|\, e^{-\theta \int_{u_{2k}}^t a(s)\, ds}.$$

Using induction, we deduce from Claim 3.2 that

$$|\dot{x}(t)| \leq |\dot{x}(u_{2k_0})|\, e^{-\theta \int_{u_{2k_0}}^t a(s)\, ds} = C_1\, e^{-\theta \int_0^t a(s)\, ds}$$

with $C_1 = |\dot{x}(u_{2k_0})|\, e^{\theta \int_0^{u_{2k_0}} a(s)\, ds}$. Next consider the case where $u_{2k+1} < t \leq u_{2k+2}$ for some k. Then

$$|\dot{x}(t)| \leq |\dot{x}(u_{2k+1})| \leq C_1\, e^{-\theta \int_0^{u_{2k+1}} a(s)\, ds} \leq C_1 e^{\theta \int_{u_{2k+1}}^{u_{2k+2}} a(\tau) d\tau}\, e^{-\theta \int_0^t a(s)\, ds}.$$

Due to Claim 3.1, $e^{\theta \int_{u_{2k+1}}^{u_{2k+2}} a(\tau) d\tau} \leq C_2$ for all k, for some constant C_2. Estimate (9) now follows for $t \geq u_{2k_0}$ with $C = C_1 C_2$. By enlarging C further, the estimate follows for all $t \geq 0$.

Let us now conclude the proof of the theorem. From assumption (6) and estimate (9), we derive that $\dot{x} \in L^1(0, \infty)$. Hence $\lim_{t\to\infty} x(t)$ exists, contradicting the initial assumption. Therefore $\lim_{t\to\infty} x(t)$ exists after all, and the theorem has been proved. □

Remark 3.1. Note that the map $t \mapsto \frac{c}{t+1}$ with $c > 1$ satisfies condition (6) for every $\theta \in (\frac{1}{c}, 1)$. In fact, if merely $a(t) \geq \frac{c}{t+1}$ for t large enough for some $c > 1$, then condition (6) is satisfied. Consider next the family of maps $a : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$a(t) = \frac{1}{t+1} + \frac{d}{(t+1)\, \ln(t+2)},$$

for some $d > 0$. It is immediate to check that condition (1) holds if and only if $d \in (0, 1]$. Thus non-stationary trajectories of $(\mathcal{S})$ do not converge when $d \in (0, 1]$. But condition (6) is never satisfied, for any $\theta \in (0, 1)$ and $d > 0$, and the convergence of trajectories remains an open question. Thus there remains a "logarithmic" gap between the criteria for existence and non-existence of limits.

We conclude with some remarks on convergence results in dimension $n > 1$. It is possible to extend the non-convergence result given at the beginning of this section to the case where the differential equation is given in a Hilbert space $\mathcal{H}$, see [3]. However, it is not clear how to prove that $\lim_{t\to\infty} x(t)$ exists, in a general Hilbert space $\mathcal{H}$ and for the case where G is convex and $\operatorname{argmin} G$ is not a singleton. Since in this case $|\dot{x}(t)| \leq \sqrt{2\mathcal{E}(t)}$, it appears natural to derive convergence results from suitable estimates for $\mathcal{E}(t)$. In [3], we give conditions that imply $\mathcal{E}(t) \leq D a(t)$ for all t, for some constant $D > 0$. However, since we must also assume that $\int_0^\infty a(s) ds = \infty$, these estimates are not strong enough to guarantee the convergence of trajectories.

One could try to extend the proof of Theorem 3.1. Set $a_1(t) = a(t) \cdot \chi_S(x(t))$, where χ_S is the characteristic function of S, then $\frac{d}{dt}\mathcal{E}(t) \le -2a_1(t)\mathcal{E}(t)$, and hence $\mathcal{E}(t) \le \mathcal{E}(0)e^{-2\int_0^t a_1(s)ds}$. If the function $t \mapsto e^{-\int_0^t a_1(s)ds}$ can be shown to be in $L^1(0,\infty)$, it would follow that $|\dot{x}|$ is integrable, implying the convergence of trajectories. This works in the one-dimensional case since the behavior of trajectories is quite simple. However, if $\dim \mathcal{H} > 1$, it is difficult to satisfy this property, since trajectories corresponding to $(\mathcal{S})$ can be expected to behave like trajectories of a billiard problem in $S = \operatorname{argmin} G$ for large times.

When the map a is constant and positive, it is established in [1, 2] that the trajectories of $(\mathcal{S})$ are weakly convergent if the potential $G : \mathcal{H} \to \mathbb{R}$ is convex and $\operatorname{argmin} G \neq \emptyset$, in an arbitrary Hilbert space $\mathcal{H}$. The key ingredient of the proof is the Opial lemma [4], which allows the authors of these papers to prove convergence even if $|\dot{x}(\cdot)|$ is only in $L^2(0,\infty)$ and not in $L^1(0,\infty)$. However, if e.g. $a(t) = \frac{c}{t+1}$, then Opial's lemma requires that we show $\int_0^\infty (t+1)|\dot{x}(t)|^2\,dt < \infty$, while (4) implies only $\int_0^\infty \frac{1}{t+1}|\dot{x}(t)|^2\,dt < \infty$. Hence there remains a gap if arguments similar to those in [1] or [2] are to be used. It is unclear how this gap can be closed.

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