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Quantum Random Walks and Minors of Hermitian Brownian Motion

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Abstract. Considering quantum random walks, we construct discrete-time approximations of the eigenvalues processes of minors of Hermitian Brownian motion. It has been recently proved by Adler, Nordenstam, and van Moerbeke that the process of eigenvalues of two consecutive minors of a Hermitian Brownian motion is a Markov process; whereas, if one considers more than two consecutive minors, the Markov property fails. We show that there are analog results in the noncommutative counterpart and establish the Markov property of eigenvalues of some particular submatrices of Hermitian Brownian motion.

1 Introduction

Let $(M(t), t \geq 0)$ be a 2×2 Hermitian Brownian motion with null trace, *i.e.*,

$$M(t) = \begin{bmatrix} B_1(t) & B_2(t) + iB_3(t) \\ B_2(t) - iB_3(t) & -B_1(t) \end{bmatrix}, \quad t \geq 0,$$

where (B_1, B_2, B_3) is a standard Brownian motion in $\mathbb{R}^3$. Itô's calculus easily shows that the process

$$\Big(B_1(t), \sqrt{B_1^2(t) + B_2^2(t) + B_3^2(t)} \Big), t \geq 0, \tag{1.1}$$

is a Markovian process on $\mathbb{R}^2$. Let us recall how noncommutative discrete-time approximation of this process can be constructed, following [Bia06]. For this, we consider the set $\mathrm{M}_2(\mathbb{C})$ of 2×2 complex matrices, endowed with the state

$$\mathrm{tr}(M) = \tfrac{1}{2} \mathrm{Tr}(M),\ M \in \mathrm{M}_2(\mathbb{C}),$$

and the Pauli matrices

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy the commutation relations

$$[x, y] = 2iz, \quad [y, z] = 2ix, \quad [z, x] = 2iy.$$

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The matrices x, y, and z define three noncommutative Bernoulli variables. Consider the algebra $\mathrm{M}_2(\mathbb{C})^{\otimes\infty}$, endowed with the infinite product state, still denoted tr, defined by

$$\mathrm{tr}(a_1 \otimes \cdots \otimes a_n \otimes I^{\otimes\infty}) = \mathrm{tr}(a_1)\cdots\mathrm{tr}(a_n), \qquad \text{for } a_1, \ldots, a_n \in \mathrm{M}_2(\mathbb{C}),$$

where I is the identity matrix of $\mathrm{M}_2(\mathbb{C})$. Define, for all $i \in \mathbb{N}^*$, the elements

$$x_i = I^{\otimes(i-1)} \otimes x \otimes I^{\otimes\infty}, \quad y_i = I^{\otimes(i-1)} \otimes y \otimes I^{\otimes\infty}, \quad z_i = I^{\otimes(i-1)} \otimes z \otimes I^{\otimes\infty}$$

as well as the partial sums

$$X_n = \sum_{i=1}^n x_i, \quad Y_n = \sum_{i=1}^n y_i, \quad Z_n = \sum_{i=1}^n z_i, \quad n \geq 1.$$

The processes $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$, and $(Z_n)_{n\geq 1}$ define three classical centered Bernoulli random walks. Considered together, they form a noncommutative Markov process that converges, after a proper renormalization, towards a standard Brownian motion in $\mathbb{R}^3$ (see Biane [Bia90] for more details). Furthermore, the family of noncommutative random variables

$$\left(Z_n, \sqrt{X_n^2 + Y_n^2 + Z_n^2},\ n \geq 1\right), \tag{1.2}$$

forms a discrete-time approximation of the Markov process (1.1). Since the noncommutative process (1.2) is also Markovian (see [Bia06]), there is a quite noticing analogy between what happens in the commutative and noncommutative cases.

In higher dimension, there are several natural ways to generalize the construction of processes (1.1) and (1.2). For some of them, the Markov property fails. For instance for $d \geq 3$, in the commutative case, if $(M(t), t \geq 0)$ is a $d \times d$ Hermitian Brownian motion, the process obtained by considering the eigenvalues of two consecutive minors of $(M(t),\ t \geq 0)$ is Markovian; whereas, the Markovian-ness fails if one considers more than two consecutive minors, as has been recently proved in [ANvM10] and announced in [Def10]. As we shall see in the sequel, this result also has an analogue in a noncommutative framework, where eigenvalues of the minors are replaced by generalized Casimir operators.

In this paper we extend to higher dimensions the construction of the noncommutative process (1.2). For this we need some basic facts about representation theory of Lie algebra recalled in Section 2. In Section 3 we recall the construction of quantum Markov chains. The Markovian aspects are studied more specifically in Section 4 using some results of invariant theory proved by Klink and Ton-That in [KTT92]. In particular we discuss the Markovianity of noncommutative analogues of the processes of eigenvalues of consecutive minors. In the last section, considering the limit of the noncommutative processes previously studied, we discuss the Markovianity of some natural generalizations of the process (1.1).

2 Universal Enveloping Algebras

Standard facts about Lie algebras and universal envelopping algebras can be found for instance in [Hum78]. Let $G = \mathrm{GL}_d(\mathbb{C})$ be the group of $d \times d$ invertible matrices, and $\mathfrak{g} = \mathrm{M}_d(\mathbb{C})$ its Lie algebra, which is the algebra of $d \times d$ complex matrices. Letting e_{ij}, $i, j = 1, \ldots, d$, be the standard basis in $\mathrm{M}_d(\mathbb{C})$, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra generated by e_{ij}, $i, j = 1, \ldots, d$, with no relations among the generators other than the commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj},$$

where $[\,\cdot\,,\,\cdot\,]$ is the usual bracket of $\mathfrak{g}$. By the Poincaré–Birkhoff–Witt theorem, there exists a basis of $\mathcal{U}(\mathfrak{g})$ composed of monomials $e_{i_1 j_1} \cdots e_{i_m j_m}$, where the integers i_k, j_k are taken in a certain order. Hence, writing an element of $\mathcal{U}(\mathfrak{g})$ in this basis, its degree is defined as the degree of its leading term. For $n \in \mathbb{N}$, we denote by $\mathcal{U}_n(\mathfrak{g})$ the set of elements of $\mathcal{U}(\mathfrak{g})$ whose leading term is of degree smaller than n. Recall that a representation of $\mathfrak{g}$ in a finite dimensional vector space V is a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$. Then any representation ρ of $\mathfrak{g}$ extends uniquely to the universal enveloping algebra letting

$$\rho(xy) = \rho(x)\rho(y), \quad x, y \in \mathcal{U}(\mathfrak{g}).$$

Let I be the identity matrix of size $d \times d$. The coproduct on $\mathcal{U}(\mathfrak{g})$ is the algebra homomorphism $\Delta\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ defined on the generators by

$$\begin{aligned}
\Delta(I) &= I \otimes I,\\
\Delta(e_{ij}) &= I \otimes e_{ij} + e_{ij} \otimes I, \ \text{ if } i \neq j, i, j = 1, \ldots, d,\\
\Delta(h_i) &= I \otimes h_i + h_i \otimes I, \ i = 1, \ldots, d-1,
\end{aligned}$$

where $h_i = e_{ii} - e_{i+1i+1}$. This characterizes Δ entirely, letting

$$\Delta(xy) = \Delta(x)\Delta(y), \quad x, y \in \mathcal{U}(\mathfrak{g}),$$

where the product on $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is defined in the usual way: $(a \otimes b)(c \otimes d) = ac \otimes bd$, for $a, b, c, d \in \mathcal{U}(\mathfrak{g})$. The tensor product of two representations $\rho_1\colon \mathfrak{g} \to \mathrm{End}(V_1)$ and $\rho_2\colon \mathfrak{g} \to \mathrm{End}(V_2)$ and its extension to $\mathcal{U}(\mathfrak{g})$

$$\rho_1 \otimes \rho_2\colon \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V_1 \otimes V_2)$$

is given by

$$\rho_1 \otimes \rho_2(x) = (\rho_1 \otimes \rho_2)\Delta(x), \quad x \in \mathcal{U}(\mathfrak{g}),$$

where $(\rho_1 \otimes \rho_2)(x_1 \otimes x_2) = \rho_1(x_1) \otimes \rho_2(x_2)$, for $x_1, x_2 \in \mathcal{U}(\mathfrak{g})$. For a representation ρ of $\mathfrak{g}$, we define recursively the representation $\rho^{\otimes n}$ of $\mathcal{U}(\mathfrak{g})$ by

$$\rho^{\otimes n}(x) := (\rho^{\otimes n-1} \otimes \rho)\Delta(x), \quad x \in \mathcal{U}(\mathfrak{g}).$$

3 Quantum Markov Chains

We first recall some basic facts about noncommutative probability, which can be found in [Mey93] or [Bia08], for example. A noncommutative probability space $(\mathcal{A}, \varphi)$ is composed of a unital *-algebra and a state $\varphi\colon \mathcal{A} \to \mathbb{C}$, that is, a positive linear form in the sense that $\varphi(aa^*) \geq 0$ for all $a \in \mathcal{A}$, and normalized, *i.e.*, $\varphi(1) = 1$. Elements of $\mathcal{A}$ are called noncommutative random variables. Note that classical probability is recovered, at least for bounded random variables, by letting $\mathcal{A} = L^\infty(\Omega, \mathbb{P})$ for some probability space $(\Omega, \mathbb{P})$, and φ being the expectation $\mathbb{E}$. The law of a family $(a_1, \dots, a_n)$ of noncommutative random variables is defined as the collection of *-moments $\varphi(a_{i_1}^{\varepsilon_1} \cdots a_{i_k}^{\varepsilon_k})$, where for all $j = 1, \dots, k$, $i_j \in \{1, \dots, n\}$, $\varepsilon_j \in \{1, *\}$, and $k \geq 1$. Thus, convergence in distribution means convergence of all *-moments.

Recall that a von Neumann algebra is a subalgebra of the algebra of bounded operators on some Hilbert space, closed under the strong topology. Define $\mathcal{W} = \mathrm{M}_d(\mathbb{C})^{\otimes\infty}$, the infinite tensor product in the sense of von Neumann algebras, with respect to the product state $\omega = \mathrm{tr}^{\otimes\infty}$, where $\mathrm{tr} = \frac{1}{d}\mathrm{Tr}$ is the normalized trace on $\mathrm{M}_d(\mathbb{C})$. Hence, $(\mathcal{W}, \omega)$ is a noncommutative probability space. For $a_1, \dots, a_n \in \mathrm{M}_d(\mathbb{C})$, we use the notation $a_1 \otimes \cdots \otimes a_n$ instead of $a_1 \otimes \cdots \otimes a_n \otimes I^{\otimes\infty}$. Let us now recall the construction of quantum Markov chains, as in [Bia06]. First, let us see how classical Markov chains can be translated in the noncommutative formalism. If $(X_n)_{n\geq 1}$ is a classical Markov chain defined on some probability space $(\Omega, \mathbb{P})$ and taking values in a measurable space E, then for each $n \geq 1$, the random variable $X_n\colon \Omega \to E$ gives rise to an algebra homomorphism

$$\begin{aligned} \chi_n\colon L^\infty(E) &\to L^\infty(\Omega) \\ f &\mapsto f(X_n). \end{aligned}$$

Hence, one can think of a noncommutative random variable as an algebra homomorphism. The Markov property of $(X_n)_{n\geq 1}$ writes

$$\mathbb{E}(Y f(X_{n+1})) = \mathbb{E}(Y Q f(X_n)),$$

for all $\sigma(X_1, \dots, X_n)$-measurable random variable Y, and where $Q\colon L^\infty(\Omega) \to L^\infty(\Omega)$ is the transition operator of $(X_n)_{n\geq 1}$. Translating this in the homomorphism formalism, we get

$$\mathbb{E}(\psi \chi_{n+1}(f)) = \mathbb{E}(\psi \chi_n(Qf)),$$

where ψ is in the subalgebra of $L^\infty(\Omega)$ generated by $X_1, \dots, X_n$.

Let us pass to the construction properly speaking of the quantum Markov chain considered here. Let ρ be the standard representation of $\mathfrak{g}$. We consider the morphism

$$\begin{aligned} j_n\colon \mathcal{U}(\mathfrak{g}) &\to \mathcal{W}, \\ x &\mapsto \rho^{\otimes n}(x), \end{aligned}$$

for all $n \geq 1$. Define $P\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ by $P = \mathrm{id} \otimes \eta \circ \Delta$, where $\eta(\,\cdot\,) = \mathrm{tr}(\rho(\,\cdot\,))$. Then P is a unital completely positive map, which is the analogue of a Markov operator in the quantum context. We have that $(j_n)_{n\geq 1}$ is a quantum Markov chain, in the sense that it satisfies the following Markov property.

Proposition 3.1 *For all ξ in the von Neumann algebra generated by $\{j_k(\mathcal{U}(\mathfrak{g})), k \leq n-1\}$, and all $x \in \mathcal{U}(\mathfrak{g})$,*

$$\omega(j_n(x)\xi) = \omega(j_{n-1}(Px)\xi).$$

Proof Let $\xi = a_1 \otimes \cdots \otimes a_{n-1}$, where the a_i's are in $\mathrm{M}_d(\mathbb{C})$. Using Sweedler's notation

$$\Delta(x) = \sum x^1 \otimes x^2,$$

we have on one hand

$$\omega(j_n(x)\xi) = \omega((\rho^{\otimes n-1} \otimes \rho)\Delta(x)\xi) = \sum \omega\left(\rho^{\otimes n-1}(x^1) \otimes \rho(x^2)\xi\right),$$

so

$$\omega(j_n(x)\xi) = \sum \mathrm{tr}(\rho^{\otimes n-1}(x^1)\xi)\,\mathrm{tr}(\rho(x^2)).$$

On the other hand,

$$Px = \sum x^1 \eta(x^2).$$

Thus,

$$j_{n-1}(Px) = \sum \eta(x^2)\rho^{\otimes n-1}(x^1),$$

and

$$\omega(j_{n-1}(Px)\xi) = \sum \mathrm{tr}\left(\rho^{\otimes n-1}(x^1)\xi\right) \mathrm{tr}\left(\rho(x^2)\right),$$

which completes the proof. ■

4 Restriction to a Subalgebra

Recall that the group G acts on $\mathfrak{g}$ via the adjoint action, *i.e.,* the conjugation action, given by

$$\mathrm{Ad}(g)x = gxg^{-1}, \quad g \in G,\ x \in \mathfrak{g}.$$

This action extends uniquely to an action on $\mathcal{U}(\mathfrak{g})$ letting

$$\mathrm{Ad}(g)(xy) = (\mathrm{Ad}(g)x)(\mathrm{Ad}(g)y), \quad g \in G,\ x \in \mathcal{U}(\mathfrak{g}).$$

The group G acts on $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ via the action

$$\mathrm{Ad}(g)(x \otimes y) = (\mathrm{Ad}(g)x) \otimes (\mathrm{Ad}(g)y), \quad g \in G,\ x, y \in \mathcal{U}(\mathfrak{g}).$$

Note that the morphism Δ satisfies

$$\Delta(\mathrm{Ad}(g)x) = \mathrm{Ad}(g)\Delta(x), \quad g \in G,\ x \in \mathcal{U}(\mathfrak{g}).$$

The next proposition shows that the operator P commutes with the adjoint action.

Proposition 4.1 *For all $g \in G$, and all $x \in \mathcal{U}(\mathfrak{g})$, we have*

$$\mathrm{Ad}(g)P(x) = P(\mathrm{Ad}(g)x).$$

Proof Using the notation $\Delta x = \sum x^1 \otimes x^2$ for $x \in \mathcal{U}(\mathfrak{g})$, we have

$$\mathrm{Ad}(g)P(x) = \mathrm{Ad}(g)\Big(\sum x^1\eta(x^2)\Big) = \sum \mathrm{Ad}(g)x^1\eta(x^2)$$

and

$$\begin{aligned} P(\mathrm{Ad}(g)x) &= \mathrm{id}\otimes\eta \circ \Delta(\mathrm{Ad}(g)x) = \mathrm{id}\otimes\eta\,\big(\mathrm{Ad}(g)\Delta x\big) \\ &= \sum \mathrm{Ad}(g)x^1\eta(x^2), \end{aligned}$$

since η is a trace. ■

Definition 4.2 For a subgroup K of G, an element $x \in \mathcal{U}(\mathfrak{g})$ is said to be K-invariant if

$$\mathrm{Ad}(g)x = x,\ \forall g \in K.$$

The set of K-invariant elements of $\mathcal{U}(\mathfrak{g})$ is denoted by $\mathcal{U}(\mathfrak{g})^K$. For $n \in \mathbb{N}$, we denote $\mathcal{U}_n(\mathfrak{g})^K$ the subset of $\mathcal{U}(\mathfrak{g})^K$ of elements whose leading term is of degree smaller than n. Proposition 4.1 implies the following one, which is fundamental for our purpose.

Proposition 4.3 *Let K be a subgroup of G. The subalgebra $\mathcal{U}(\mathfrak{g})^K$ of K-invariant elements of $\mathcal{U}(\mathfrak{g})$ is stable by P,* i.e.,

$$P\,\mathcal{U}(\mathfrak{g})^K \subset \mathcal{U}(\mathfrak{g})^K.$$

Hence, the restriction of $(j_n)_{n\geq 1}$ to $\mathcal{U}(\mathfrak{g})^K$ defines a quantum Markov chain.

Let us focus on some particular invariant sets related to the minor process studied in [ANvM10]. For a fixed integer $p \in \{0, \dots, d-1\}$, we consider the block diagonal subgroup $\mathrm{GL}_{d-p}(\mathbb{C}) \times \mathbb{C}^{*p}$ of G, which consists of elements of the form

$$\begin{pmatrix} k & & & \\ & k_1 & & 0 \\ & & \ddots & \\ 0 & & & k_p \end{pmatrix},$$

with $k \in \mathrm{GL}_{d-p}(\mathbb{C})$, and $k_1, \dots, k_p \in \mathbb{C}^*$. For $l, m \in \mathbb{N}^*$, we denote by $\mathcal{M}_{l,m}$ the set of $l \times m$ matrices with noncommutative entries in $\mathcal{U}(\mathfrak{g})$. We let $\mathcal{M}_l = \mathcal{M}_{l,l}$. The rules to add or multiply matrices of $\mathcal{M}_{l,m}$ are the same as those for the commutative case, replacing the usual addition and multiplication in a commutative algebra by the addition and the multiplication in $\mathcal{U}(\mathfrak{g})$. Moreover, if $M = (m_{ij})_{1\leq i,j\leq l}$ is a matrix in

$\mathcal{M}_l$, then the element of $\mathcal{U}(\mathfrak{g})$ equal to $\sum_{i=1}^{l} m_{ii}$ is denoted $\mathbf{T}r(M)$. We partition the matrix $E = (e_{ij})_{1\le i,j\le d}$ in block matrices in the form

$$E = \begin{bmatrix} E_{11} & \dots & E_{1p+1} \\ \vdots & & \vdots \\ E_{p+11} & \dots & E_{p+1p+1} \end{bmatrix},$$

where $E_{11} \in \mathcal{M}_{d-p,d-p}$, $E_{1i} \in \mathcal{M}_{d-p,1}$, $E_{i1} \in \mathcal{M}_{1,d-p}$, $i \in \{2, \dots, p+1\}$, and $E_{ij} \in \mathcal{U}(\mathfrak{g})$, $i, j \in \{2, \dots, d\}$.

Notation Entries of a matrix will be always denoted by small letters, while capital letters will refer to the partition defined above.

The next theorem, which was proved by Klink and Ton-That, gives the generators of the subalgebra $\mathcal{U}(\mathfrak{g})^{\mathrm{GL}_{d-p}(\mathbb{C})\times\mathbb{C}^{*p}}$, which are called generalized Casimir operators.

Theorem 4.4 ([KTT92]) *The subalgebra* $\mathcal{U}(\mathfrak{g})^{\mathrm{GL}_{d-p}(\mathbb{C})\times\mathbb{C}^{*p}}$ *is finitely generated by the constants and elements*

$$\mathbf{T}r(E_{i_1 i_2}\cdots E_{i_q i_1}), \quad q \in \mathbb{N}^*,\ i_1, \dots, i_q \in \{1, \dots, p+1\}.$$

The two extreme cases of this theorem give the following classical results. Actually for $p = 0$, it implies that the center of $\mathcal{U}(\mathfrak{g})$ is generated by Casimir operators (see [Žel73])

$$\mathbf{T}r(E^k), \quad k \in \mathbb{N}.$$

For $p = d-1$, we recover that the commutant of $\{e_{ii},\ i = 1, \dots, d\}$ in $\mathcal{U}(\mathfrak{g})$ is generated by elements

$$e_{i_1 i_2}\cdots e_{i_q i_1}, \quad q \in \mathbb{N},\ i_1, \dots, i_q \in \{1, \dots, p+1\}.$$

For $a_1, \dots, a_n \in \mathcal{U}(\mathfrak{g})$, we denote by $\langle a_1, \dots, a_n\rangle$, the subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by the constants and elements $a_1, \dots, a_n$. Let us focus on the subalgebra $\mathcal{U}(\mathfrak{g})^{\mathrm{GL}_{d-p}(\mathbb{C})\times\mathbb{C}^{*p}}$ and its generators in the case when $p = 1$ and $p = 2$. First we need the following lemmas.

Lemma 4.5 *If* $A = (a_{ij})_{1\le i,j\le d}, B = (b_{ij})_{1\le i,j\le d} \in \mathcal{M}_d$, *with* $a_{ij} \in \mathcal{U}_n(\mathfrak{g})$, $b_{ij} \in \mathcal{U}_m(\mathfrak{g})$, *for* $i, j = 1, \dots, d$, *then*

$$\mathbf{T}r(AB) - \mathbf{T}r(BA) \in \mathcal{U}_{m+n-1}(\mathfrak{g}).$$

Proof This is a consequence of the commutation relations in $\mathcal{U}(\mathfrak{g})$. ∎

The following lemma claims that the subset of invariants $\mathcal{U}(\mathfrak{g})^{\mathrm{GL}_{d-1}(\mathbb{C})\times\mathbb{C}^*}$ is generated by the Casimir elements associated with the Lie algebra $\mathrm{M}_d(\mathbb{C})$ and those associated with the subalgebra $\{M \in \mathrm{M}_d(\mathbb{C}) : m_{id} = m_{di} = 0, i = 1, \dots, d\} \simeq \mathrm{M}_{d-1}(\mathbb{C})$.

Lemma 4.6 *The subalgebra $\mathcal{U}(\mathfrak{g})^{\mathrm{GL}_{d-1}(\mathbb{C})\times\mathbb{C}^*}$ is generated by*

$$\mathbf{T}r(E_{11}^{k-1}),\ \mathbf{T}r(E^k),\quad k=1,\dots,d.$$

Proof For $q\in\mathbb{N}^*$, let $\mathcal{T}_q$ be the subalgebra

$$\left\langle \mathbf{T}r(E_{i_1i_2}\cdots E_{i_ki_1}),\ k\in\{1,\dots q\},\ i_1,\dots,i_k=1,2\right\rangle.$$

It is sufficient to prove that for every $q\in\mathbb{N}^*$,

$$\mathcal{T}_q=\left\langle \mathbf{T}r(E_{11}^k),\mathbf{T}r(E^k),\ k\in\{1,\dots,q\}\right\rangle. \tag{4.1}$$

For every $q\in\mathbb{N}^*$ the inclusion

$$\left\langle \mathbf{T}r(E_{11}^k),\mathbf{T}r(E^k),\ k\in\{1,\dots,q\}\right\rangle\subset\mathcal{T}_q$$

follows from the fact that

$$\mathbf{T}r(E^k)=\sum\mathbf{T}r(E_{i_1i_2}\cdots E_{i_ki_1}),$$

where the sum runs over all sequences $i_1,\dots,i_k$ of integers in $\{1,2\}$. Let us prove the reverse inclusion by induction on q. It is clearly true for $q=1$. For $q=2$, let us write

$$\mathbf{T}r(E^2)=\mathbf{T}r(E_{21}E_{12})+\mathbf{T}r(E_{12}E_{21})+\mathbf{T}r(E_{11}^2)+\mathbf{T}r(E_{22}^2).$$

Thus the inclusion

$$\mathcal{T}_2\subset\left\langle \mathbf{T}r(E_{11}^k),\mathbf{T}r(E^k),\ k=1,2\right\rangle$$

follows from Lemma 4.5, which implies that

$$\mathbf{T}r(E_{21}E_{12})-\mathbf{T}r(E_{12}E_{21})\in\mathcal{U}_1^{\mathrm{GL}_{d-1}\times\mathbb{C}^*}(\mathfrak{g})\subset\langle 1,E_{11},E_{22}\rangle.$$

The case $q=3$ is proved in a similar way. Suppose that (4.1) is true for $q-1$, for a fixed $q\geq 4$. Let $i_1,i_2,\dots,i_q$, be a sequence of integers in $\{1,2\}$. If the sequence $i_1,i_2,\dots,i_q$, contains no successive integers equal to 1 then $E_{i_1i_2}\cdots E_{i_qi_1}$, contains only factors equal to $E_{21}E_{12}$, $E_{12}E_{21}$, or E_{22}. By lemma 4.5 and inclusion

$$\mathcal{U}_{q-1}(\mathfrak{g})^{\mathrm{GL}_{d-1}(\mathbb{C})\times\mathbb{C}^*}\subset\mathcal{T}_{q-1}, \tag{4.2}$$

we can suppose that $E_{i_1i_2}\cdots E_{i_qi_1}$, contains only factors equal to $E_{21}E_{12}$, or E_{22}, which belong to the subalgebra

$$\left\langle \mathbf{T}r(E_{11}^k),\mathbf{T}r(E^k),k=1,2\right\rangle.$$

If $E_{i_1i_2}\cdots E_{i_qi_1}$ contains factors equal to E_{11} but strictly less than $q-2$, then it contains at least one factor equal to $E_{21}E_{11}E_{12}$, $E_{11}E_{12}E_{21}$ or $E_{12}E_{21}E_{11}$. Thanks to Lemma 4.5, and inclusion (4.2) we can suppose that $i_1=i_3=2$ and $i_2=1$. Thus

$$\mathbf{T}r(E_{i_1i_2}\cdots E_{i_qi_1})=(E_{21}E_{11}E_{12})\mathbf{T}r(E_{2i_4}\cdots E_{i_q2}).$$

Then the induction hypothesis implies

$$\mathbf{T}r(E_{i_1 i_2}\cdots E_{i_q i_1}) \in \left\langle \mathbf{T}r(E_{11}^k), \mathbf{T}r(E^k), k\in\{1,\dots,q-1\}\right\rangle.$$

If $E_{i_1 i_2}\cdots E_{i_q i_1} = E_{11}^d$, then

$$\mathbf{T}r(E_{i_1 i_2}\cdots E_{i_q i_1}) = \mathbf{T}r(E_{11}^q) \in \left\langle \mathbf{T}r(E_{11}^k), \mathbf{T}r(E^k), k\in\{1,\dots,q\}\right\rangle.$$

If $E_{i_1 i_2}\cdots E_{i_q i_1}$ contains $q-2$ factors equal to E_{11}, then

$$\mathbf{T}r(E_{i_1 i_2}\cdots E_{i_q i_1}) \in \left\{ \mathbf{T}r(E_{11}^{q-2}E_{12}E_{21}), \mathbf{T}r(E_{21}E_{11}^{q-2}E_{12}), \mathbf{T}r(E_{12}E_{21}E_{11}^{q-2})\right\}.$$

We write

$$\mathbf{T}r(E^q) = \mathbf{T}r(E_{11}^q) + \mathbf{T}r(E_{11}^{q-2}E_{12}E_{21}) + \mathbf{T}r(E_{21}E_{11}^{q-2}E_{12}) + \mathbf{T}r(E_{12}E_{21}E_{11}^{q-2}) \\ + \sum \mathbf{T}r(E_{i_1 i_2}\cdots E_{i_q i_1}),$$

where the sum runs over all sequences $i_1,\dots,i_q$ of integers in $\{1,2\}$ containing strictly less than $q-1$ integers equal to 1. The previous cases, Lemma 4.5, and inclusion (4.2) imply that

$$\mathbf{T}r(E^q) - 3\mathbf{T}r(E_{i_1 i_2}\cdots E_{i_q i_1}) \in \left\langle \mathbf{T}r(E_{11}^k), \mathbf{T}r(E^k), k\in\{1,\dots,q\}\right\rangle.$$

Since it is known (see [Žel73]) that

$$\left\langle \mathbf{T}r(E^k), k\in\{1,\dots,d\}\right\rangle = \left\langle \mathbf{T}r(E^k), k\geq 1\right\rangle,$$

and

$$\left\langle \mathbf{T}r(E_{11}^k), k\in\{1,\dots,d-1\}\right\rangle = \left\langle \mathbf{T}r(E_{11}^k), k\geq 1\right\rangle,$$

the proposition follows. ■

Remark 4.7 When $p\geq 2$, the subalgebra $\mathcal{U}(\mathfrak{g})^{\mathrm{GL}_{d-p}(\mathbb{C})\times\mathbb{C}^{*p}}$ is not generated by the Casimir elements associated with the Lie algebras

$$\{M\in \mathrm{M}_d(\mathbb{C}) : m_{ij} = m_{ji} = 0, i = 1,\dots,d,\, j = d-k+1,\dots,d\} \simeq \mathrm{M}_{d-k}(\mathbb{C}),$$

$k\in\{0,\dots,p\}$. For instance,

$$\mathbf{T}r(E_{13}E_{31}) \notin \left\langle \mathbf{T}r(E_{11}^k), \mathbf{T}r\left(\begin{bmatrix} E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix}^k\right), \mathbf{T}r(E^k),\ k\in\mathbb{N}\right\rangle$$

and thus

$$\left\langle \mathbf{T}r(E_{11}^k), \mathbf{T}r\left(\begin{bmatrix} E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix}^k\right), \mathbf{T}r(E^k),\ k\in\mathbb{N}\right\rangle \subsetneq \mathcal{U}(\mathfrak{g})^{\mathrm{GL}_{d-2}(\mathbb{C})\times\mathbb{C}^{*2}}.$$

The following theorem is a quantum analogue of [ANvM10, Theorem 2.2].

Theorem 4.8 *The restriction of the j_n's to the subalgebra*

$$\left\langle \mathbf{T}r(E_{11}^{k-1}), \mathbf{T}r(E^k),\ k \in \{1, \ldots, d\} \right\rangle$$

defines a quantum Markov process.

Proof This follows immediately from Proposition 4.3 and Lemma 4.6. ■

Note that the subalgebra

$$\left\langle \mathbf{T}r(E_{11}^{k-1}), \mathbf{T}r(E^k),\ k \in \{1, \ldots, d\} \right\rangle$$

is commutative. Thus, as in [Bia06], which focuses on the $d = 2$ case, the quantum Markov process in Theorem 4.8 is a noncommutative process, with a commutative Markovian operator. Taking $d = 2$, the Markovianity of the process (1.2) follows. The following theorem is an analogue of [ANvM10, Theorem 2.4] in a noncommutative context. The non-Markovianity comes from Remark 4.7.

Theorem 4.9 *The restriction of the j_n's to the subalgebra*

$$\left\langle \mathbf{T}r(E_{11}^{k}), \mathbf{T}r\left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}^k \right), \mathbf{T}r(E^k),\ k \in \mathbb{N} \right\rangle$$

does not define a quantum Markov process.

Proof We have to prove that the subalgebra

$$\mathcal{B} := \left\langle \mathbf{T}r(E_{11}^{k}), \mathbf{T}r\left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}^k \right), \mathbf{T}r(E^k),\ k \in \mathbb{N} \right\rangle$$

is not stable by the operator P. Indeed, the partition of E for $p = 2$ writes

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}.$$

Let us consider the element

$$a = \mathbf{T}r(E_{32} \circ E_{22} \circ E_{23} + E_{23} \circ E_{31} \circ E_{12} + E_{32} \circ E_{21} \circ E_{13} + E_{31} \circ E_{11} \circ E_{13}),$$

where $x \circ y \circ z$ means $xyz + yzx + zxy$. It is easy to see that a is in $\mathcal{B}$. The element a

writes

$$a = e_{dd-1} \circ e_{d-1d-1} \circ e_{d-1d} + \sum_{i=1}^{d-2} (e_{d-1d} \circ e_{di} \circ e_{id-1} + e_{dd-1} \circ e_{d-1i} \circ e_{id})$$
$$+ \sum_{i,j=1}^{d-2} e_{di} \circ e_{ij} \circ e_{jd}$$
$$= \sum_{i=1}^{d-2} (e_{d-1d} \circ e_{di} \circ e_{id-1} + e_{dd-1} \circ e_{d-1i} \circ e_{id}) + \sum_{\substack{i,j=1 \\ i \neq j}}^{d-2} e_{di} \circ e_{ij} \circ e_{jd}$$
$$+ \sum_{i=1}^{d-1} e_{di} \circ (e_{ii} - e_{dd}) \circ e_{id} + \sum_{i=1}^{d-1} e_{di} \circ e_{dd} \circ e_{id}$$

Since the sums $\sum_{i=1}^{d-1} e_{di} \circ e_{dd} \circ e_{id}$ and $\sum_{i=1}^{d-2} (e_{ii} - e_{dd})$ are in $\mathcal{B}$, the element b defined by

$$b = \sum_{i=1}^{d-2} (e_{d-1d} \circ e_{di} \circ e_{id-1} + e_{dd-1} \circ e_{d-1i} \circ e_{id}) + \sum_{\substack{i,j=1 \\ i \neq j}}^{d-2} e_{di} \circ e_{ij} \circ e_{jd}$$
$$+ \sum_{i=1}^{d-1} e_{di} \circ (e_{ii} - e_{dd}) \circ e_{id}$$

is also in $\mathcal{B}$, as the element $c = b \sum_{i=1}^{d-2} (e_{ii} - e_{dd})$. One can prove by straightforward calculation that

$$Pc - \frac{3}{d} \sum_{i=1}^{d-2} e_{id} e_{di}$$

is in $\mathcal{B}$. Since the element $\sum_{i=1}^{d-2} e_{id} e_{di}$, which is equal to $\mathbf{T}r(E_{13}E_{31})$ is not in $\mathcal{B}$, the theorem follows. ■

Let us choose an integer m large enough such that the subalgebras

$$\left\langle \mathbf{T}r(E_{i_1 i_2} \cdots E_{i_q i_1}), \quad q \in \mathbb{N}^*, \ i_1, \dots, i_q \in \{1, \dots, p+1\} \right\rangle$$

and

$$\left\langle \mathbf{T}r(E_{i_1 i_2} \cdots E_{i_q i_1}), \quad q = 1, \dots, m, \ i_1, \dots, i_q \in \{1, \dots, p+1\} \right\rangle$$

are equal. In the framework of this paper, the natural process that "contains" the one of Theorem 4.9 and remains Markovian is given in the following theorem, taking $p = 2$.

Theorem 4.10 *The restriction of the j_n's to the subalgebra*

$$\left\langle \mathbf{Tr}(E_{i_1 i_2} \cdots E_{i_q i_1}), \quad q \in \{1, \ldots, m\},\ i_1, \ldots, i_q \in \{1, \ldots, p+1\} \right\rangle,$$

defines a quantum Markov process.

Proof Theorem 4.10 follows from Theorem 4.4 and Proposition 4.3. ■

5 Random Matrices

Let H_d and H_d^0 be respectively the set of $d \times d$ complex Hermitian matrices and the set of $d \times d$ complex Hermitian matrices with null trace, both endowed with the scalar product given by

$$\langle M, N \rangle = \mathrm{Tr}(MN), \quad M, N \in \mathrm{H}_d \text{ (resp. } \mathrm{H}_d^0).$$

Recall that a standard Gaussian variable on a real Euclidean space E with finite dimension n is a random variable with density

$$x \mapsto (2\pi)^{-n/2} e^{-\langle x,x \rangle /2}.$$

A Brownian motion $(X_t)_{t \geq 0}$ on E is a Levy process on E such that for all $t \geq 0$, the random variable $(1/\sqrt{t})X_t$ is a standard Gaussian variable on E. Thus our choice of the Euclidean structure above defines a notion of standard Brownian motion on H_d and H_d^0.

For $k, l \in \mathbb{N}^*$, we denote by $\mathrm{M}_{k,l}(\mathbb{C})$ the set of $k \times l$ complex matrices and let $\mathrm{M}_k(\mathbb{C}) = \mathrm{M}_{k,k}(\mathbb{C})$. As in the noncommutative case, we partition a matrix $M \in \mathrm{M}_d(\mathbb{C})$ in block matrices in the form

$$M = \begin{bmatrix} M_{11} & \ldots & M_{1p+1} \\ \vdots & & \vdots \\ M_{p+11} & \ldots & M_{p+1p+1} \end{bmatrix},$$

where $M_{11} \in \mathrm{M}_{d-p,d-p}(\mathbb{C})$, $M_{1i} \in \mathrm{M}_{d-p,1}(\mathbb{C})$, $M_{i1} \in \mathrm{M}_{1,d-p}(\mathbb{C})$, $i \in \{2, \ldots, p+1\}$, and $M_{ij} \in \mathbb{C}$, $i, j \in \{2, \ldots, d\}$.

Define the elements $(x_{ij})_{1 \leq i,j \leq d}$ of $\mathcal{U}(\mathfrak{g})$ by

$$x_{ij} = e_{ij}, \quad \text{for } 1 \leq i \neq j \leq d \quad \text{and } x_{ii} = e_{ii} - \tfrac{1}{d} I, \text{ for } 1 \leq i \leq d.$$

Note all the x_{ij}'s are traceless elements of $\mathfrak{g}$. Let $\nu = \frac{d}{d-1}\,\mathrm{tr}(\rho(x_{ii})\rho(x_{ii}))$, which does not depend on i. Then we have the following theorem, due to Biane.

Theorem 5.1 ([Bia95]) *The law of the family of random variables on $(\mathcal{W}, \omega)$*

$$\left(\frac{1}{\sqrt{n\nu}} j_{\lfloor nt \rfloor}(x_{ij}) \right)_{t \in \mathbb{R}_+, 1 \leq i,j \leq d}$$

converges as n goes to infinity towards the law of $(m_{jk}(t))_{t \in \mathbb{R}_+, 1 \leq i,j \leq d}$, where $(M(t) = (m_{ij}(t))_{1 \leq i,j \leq d}, t \geq 0)$ is a standard Brownian motion on H_d^0.

By this theorem, we see that the law of the noncommutative process

$$\left(\frac{1}{\sqrt{nv}} j_{\lfloor nt \rfloor}\right)_{t \geq 0} \tag{5.1}$$

restricted to the subalgebra of Theorem 4.8 converges, as n goes to infinity, towards the law of $(\mathrm{Tr}(M_{11}(t)^{k-1}), \mathrm{Tr}(M(t)^k), k \geq 1, t \geq 0)$. We will see that this process, which is in bijection with the process of eigenvalues of two consecutive minors of $(M(t), t \geq 0)$, is Markovian. More generally, if K is a subgroup of G, the law of the noncommutative process (5.1) restricted to the subalgebra $\mathcal{U}(\mathfrak{g})^K$ converges, as n goes to infinity, to a commutative process that remains Markovian. The fact that the limit process is a Markov process will follow by Itô's calculus and invariant theory in a commutative framework. Even if the limit process of Theorem 5.1 is a Brownian motion on H_d^0, we consider in this section Brownian motion on H_d, which is more common in the framework of random matrices. Let us recall how they are related. If $(B_t)_{t \geq 0}$ is a standard Brownian motion on H_d, then the process $(B_t^0)_{t \geq 0}$ defined by

$$B_t^0 = B_t - \frac{1}{d} \mathrm{Tr}(B_t) I, t \geq 0,$$

is a standard Brownian motion on H_d^0, independent of the process

$$(\mathrm{Tr}(B_t), t \geq 0).$$

Since we have

$$B_t = B_t^0 + \tfrac{1}{d} \mathrm{Tr}(B_t) I, \quad t \geq 0,$$

Markovian aspects of the minors can be indifferently studied for Brownian motion on H_d or H_d^0. In the sequel, a function $f \colon \mathrm{M}_d(\mathbb{C}) \to \mathbb{C}$ is seen as a function from $\mathbb{C}^{d^2}$ to $\mathbb{C}$.

Definition 5.2 Let K be a subgroup of G. A function f from $\mathrm{M}_d(\mathbb{C})$ to $\mathbb{C}$ is said to be K-invariant if

$$\forall k \in K \quad \forall M \in \mathrm{M}_d(\mathbb{C}), \quad f(kMk^{-1}) = f(M).$$

Let $\mathcal{P}(\mathfrak{g})$ denote the algebra of all complex-valued polynomial functions on $\mathrm{M}_d(\mathbb{C})$, *i.e.,* $\mathcal{P}(\mathfrak{g})$ is the set of all polynomials in coordinates of a matrix of $\mathrm{M}_d(\mathbb{C})$. For any subgroup K of G, the set of K-invariant elements of $\mathcal{P}(\mathfrak{g})$ is denoted $\mathcal{P}(\mathfrak{g})^K$. The following theorem, which is a commutative version of Theorem 4.4, was proved in [KTT92].

Theorem 5.3 ([KTT92]) *It exists $m \in \mathbb{N}$, such that the subalgebra $\mathcal{P}(\mathfrak{g})^{\mathrm{GL}_{d-p}(\mathbb{C}) \times \mathbb{C}^{*p}}$ is generated by the constants and polynomials*

$$M \in \mathrm{M}_d(\mathbb{C}) \mapsto \mathrm{Tr}(M_{i_1 i_2} \cdots M_{i_q i_1}), \quad q \in \{1, \ldots, m\},\ i_1, \ldots, i_q \in \{1, \ldots, p+1\}.$$

Let us recall the following property of Brownian motion and invariant functions. In what follows we denote by $\langle \cdot , \cdot \rangle$ the usual quadratic covariation, and by d and d^2 the usual first and second order differentials.

Proposition 5.4 *Let $g \in \mathrm{GL}_d(\mathbb{C})$, and f and h be twice differentiable functions from $\mathrm{M}_d(\mathbb{C})$ to $\mathbb{C}$ such that*

$$\forall M \in \mathrm{M}_d(\mathbb{C}) \quad f(gMg^{-1}) = f(M) \text{ and } h(gMg^{-1}) = h(M). \tag{5.2}$$

If B is a standard Brownian motion on H_d, then

$$\left\langle \mathrm{d}f(gMg^{-1})(dB), \mathrm{d}h(gMg^{-1})(dB)\right\rangle = \left\langle \mathrm{d}f(M)(dB), \mathrm{d}h(M)(dB)\right\rangle$$

and

$$\left\langle \mathrm{d}^2 f(gMg^{-1})(dB), dB\right\rangle = \left\langle \mathrm{d}^2 f(M)(dB), dB\right\rangle.$$

Proof Since B is a standard Brownian motion on H_d,

$$\left\langle (gdBg^{-1})_{ij}, (gdBg^{-1})_{kl}\right\rangle = \langle dB_{ij}, dB_{kl}\rangle, \quad i, j, k, l \in \{1, \dots, d\}.$$

Thus

$$\left\langle \mathrm{d}f(gMg^{-1})(dB), \mathrm{d}h(gMg^{-1})(dB)\right\rangle = \\ \left\langle \mathrm{d}f(gMg^{-1})(gdBg^{-1}), \mathrm{d}h(gMg^{-1})(gdBg^{-1})\right\rangle,$$

and

$$\left\langle \mathrm{d}^2 f(gMg^{-1})(dB), dB\right\rangle = \left\langle \mathrm{d}^2 f(gMg^{-1})(gdBg^{-1}), gdBg^{-1}\right\rangle.$$

Property (5.2) implies

$$\left\langle \mathrm{d}f(gMg^{-1})(gdBg^{-1}), \mathrm{d}h(g^{-1}Mg)(gdBg^{-1})\right\rangle = \left\langle \mathrm{d}f(M)(dB), \mathrm{d}h(M)(dB)\right\rangle$$
$$\left\langle \mathrm{d}^2 f(gMg^{-1})(gdBg^{-1}), gdBg^{-1}\right\rangle = \left\langle \mathrm{d}^2 f(M)(dB), dB\right\rangle. \qquad \blacksquare$$

The previous proposition implies the following one.

Proposition 5.5 *Let K be a subgroup of $\mathrm{GL}_d(\mathbb{C})$. If f and h are elements in $\mathcal{P}(\mathfrak{g})^K$, then the functions*

$$M \in \mathrm{M}_d(\mathbb{C}) \mapsto \left\langle \mathrm{d}f(M)(dB), \mathrm{d}h(M)(dB)\right\rangle,$$

and

$$M \in \mathrm{M}_d(\mathbb{C}) \mapsto \left\langle \mathrm{d}^2 f(M)(dB), dB\right\rangle$$

are also K-invariant polynomial functions.

For a twice continuously differentiable function $f\colon \mathrm{M}_d(\mathbb{C}) \to \mathbb{C}$, multidimensional Itô's formula writes

$$df(B) = \mathrm{d}f(B)(dB) + \tfrac{1}{2}\langle \mathrm{d}^2 f(B)(dB), dB\rangle.$$

Thus Proposition 5.5 leads to the next proposition in which the integer m is the one introduced in Theorem 5.3.

Proposition 5.6 *If $(B(t), t \geq 0)$ is a standard Brownian motion on H_d, the processes*

$$\left(\mathrm{Tr}(B_{i_1 i_2}(t) \cdots B_{i_q i_1}(t)), t \geq 0\right),$$

$q \in \{1, \ldots, m\}$, $i_1, \ldots, i_q \in \{1, \ldots, p+1\}$, form a Markov process on $\mathbb{R}^r$, with $r = \sum_{k=1}^m (p+1)^k$.

Proof For p and q two integers in $\{1, \ldots, m\}$ and two sequences $i_1, \ldots, i_p$ and $j_1, \ldots, j_q$ of integers of $\{1, \ldots, p+1\}$, let us consider the functions f, g and h from $\mathrm{M}_d(\mathbb{C})$ to $\mathbb{C}$ defined by

$$f(M) = \mathrm{Tr}(M_{i_1 i_2} \cdots M_{i_p i_1}), \quad g(M) = \mathrm{Tr}(M_{i_1 i_p} \cdots M_{i_2 i_1}), \quad M \in \mathrm{M}_d(\mathbb{C}),$$

and

$$h(M) = \mathrm{Tr}(M_{j_1 j_2} \cdots M_{j_q j_1}), \quad M \in \mathrm{M}_d(\mathbb{C}).$$

Since $\overline{f(M)} = g(M)$, when $M \in \mathrm{H}_d$, we have

$$\left\langle \overline{\mathrm{d}f(B)(dB)}, \mathrm{d}h(B)(dB)\right\rangle = \left\langle \mathrm{d}g(B)(dB), \mathrm{d}h(B)(dB)\right\rangle.$$

Proposition (5.5) implies that

$$\left\langle \mathrm{d}f(B)(dB), \mathrm{d}h(B)(dB)\right\rangle, \quad \left\langle \overline{\mathrm{d}f(B)(dB)}, \mathrm{d}h(B)(dB)\right\rangle, \quad \left\langle \mathrm{d}^2 f(B)(dB), dB\right\rangle$$

are polynomial functions in the processes $\mathrm{Tr}(B_{i_1 i_2} \cdots B_{i_q i_1})$, $q \in \{1, \ldots, m\}$, $i_1, \ldots, i_q \in \{1, \ldots, p+1\}$. Thus the proposition follows from the usual properties of diffusions (see [Øks03] for example). ■

Let us give a formulation of the last proposition in term of eigenvalues of some particular submatrices of Brownian motion on H_d. In the following lemma a polynomial function $M \in \mathrm{M}_d(\mathbb{C}) \mapsto f(M)$, is simply denoted $f(M)$.

Lemma 5.7 *For any positive integer q, and any sequence of integers $i_1, \ldots, i_q$ in $\{1, \ldots, p+1\}$, the polynomial function $\mathrm{Tr}(M_{i_1 i_2} \cdots M_{i_q i_1})$ is equal to a finite product of factors of the form*

$$\mathrm{Tr}(M_{11}^n), \mathrm{Tr}(M_{1i} M_{i1} M_{11}^m), \mathrm{Tr}(M_{1i} M_{ij} M_{j1} M_{11}^n),$$
$$M_{ii}, M_{ij} M_{ji}, (M_{ij} M_{ji})^{-1}, M_{ij} M_{jk} M_{ki},$$

where $n \in \mathbb{N}$, and i, j, k, are distinct integers in $\{2, \ldots, p+1\}$.

Proof The lemma, which is is clearly true for $q = 1, 2, 3$, is proved by induction on $q \in \mathbb{N}^*$. Suppose such a decomposition exists up to $q - 1$, for a fixed integer q greater than 4. Let us consider a sequence of integers $i_1, \dots, i_q$ in $\{1, \dots, p + 1\}$. If all, or none, of the integers of the sequence are equal to 1, then the decomposition exists. If two successive integers exist, say i_1 and i_2, such that $i_1 = 1$, $i_2 \neq 1$, then integers $k \leq q - 1$ and $p \leq q$ exist such that $i_p \neq 1$, and

$$\mathrm{Tr}(M_{i_1 i_2} \cdots M_{i_q i_1}) = M_{i_2 i_3} \cdots M_{i_p 1} M_{11}^k M_{1 i_2}.$$

If $i_p = i_2$, then

$$\mathrm{Tr}(M_{i_1 i_2} \cdots M_{i_q i_1}) = (M_{i_2 i_3} \cdots M_{i_{p-1} i_2})(M_{i_2 1} M_{11}^k M_{1 i_2}).$$

If $i_p \neq i_2$, then

$$\mathrm{Tr}(M_{i_1 i_2} \cdots M_{i_q i_1})(M_{i_2 i_p} M_{i_p i_2}) = (M_{i_2 i_3} \cdots M_{i_{p-1} i_p} M_{i_p i_2})(M_{i_p 1} M_{11}^k M_{1 i_2} M_{i_2 i_p}).$$

Induction hypothesis implies that the above polynomials can be written as a product of factors given in the lemma. ■

Proposition 5.8 *If B is a Brownian motion on H_d, then the processes*

$$\mathrm{Tr}(B_{11}^n), \mathrm{Tr}(B_{1i} B_{i1} B_{11}^m), \mathrm{Tr}(B_{1i} B_{ij} B_{j1} B_{11}^n),$$
$$B_{ii}, B_{ij} B_{ji}, B_{ij} B_{jk} B_{ki},$$

where $n \in \mathbb{N}$, and i, j, k, are distinct integers in $\{2, \dots, p + 1\}$, taken together, form a Markov process.

Proof Lemma 5.7 implies that there is a bijection between the Markov process of Proposition 5.6 and the process of Proposition 5.8, which is consequently Markovian too. ■

The following theorem is an immediate consequence of the previous proposition.

Theorem 5.9 *Let p be a positive integer and B be a Brownian motion on H_d. Then the processes of the eigenvalues of the matrices,*

$$B_{11}, \begin{pmatrix} B_{11} & B_{1i} \\ B_{i1} & B_{ii} \end{pmatrix}, \quad \begin{pmatrix} B_{11} & B_{1i} & 0 \\ 0 & 0 & B_{ij} \\ B_{j1} & 0 & 0 \end{pmatrix},$$

and the complex processes, $B_{ij} B_{ji}$, $B_{ij} B_{jk} B_{ki}$, where i, j, k, are distinct integers in $\{2, \dots, p + 1\}$, taken together, form a Markov process.

Taking $p = 1$ in Theorem 5.9 we obtain the following corollary, which has been already proved in [ANvM10].

Corollary 5.10 *If $(\Lambda^{(d)}(t), t \geq 0)$ is the process of eigenvalues of a standard Brownian motion on H_d and $(\Lambda^{(d-1)}(t), t \geq 0)$ is the process of eigenvalues of its principal minor of order $d - 1$, then the process $\left(\Lambda^{(d)}(t), \Lambda^{(d-1)}(t), t \geq 0\right)$ is Markovian.*

References

[ANvM10] M. Adler, E. Nordenstam, and P. van Moerbeke, *Consecutive minors for Dyson's Brownian motions.* arxiv:1007.0220.

[Bia90] P. Biane, *Marches de Bernoulli quantiques.* In: Séminaire de Probabilités, XXIV, 1988/89, Lecture Notes in Math., 1426, Springer, Berlin, 1990, pp. 329–344.

[Bia95] ______, *Permutation model for semi-circular systems and quantum random walks.* Pacific J. Math. **171**(1995), no. 2, 373–387.

[Bia06] ______, *Le théorème de Pitman, le groupe quantique* $\mathrm{SU}_q(2)$*, et une question de P. A. Meyer.* In: In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX, Lecture Notes in Math., 1874, Springer, Berlin, 2006, pp. 61–75.

[Bia08] ______, *Introduction to random walks on noncommutative spaces.* In: Quantum potential theory, Lecture Notes in Math., 1954, Springer, Berlin, 2008, pp. 61–116.

[Def10] M. Defosseux, *Orbit measures, random matrix theory and interlaced determinantal processes.* Ann. Inst. Henri Poincaré Probab. Stat. **46**(2010), no. 1, 209–249. http://dx.doi.org/10.1214/09-AIHP314

[Hum78] J. E. Humphreys, *Introduction to Lie algebras and representation theory.* Second Printing, revised, Graduate Texts in Mathematics, 9, Springer-Verlag, New York-Berlin, 1978.

[KTT92] W. H. Klink and T. Ton-That, *Invariant theory of the block diagonal subgroups of* $\mathrm{GL}(n,\mathbf{C})$ *and generalized Casimir operators.* J. Algebra **145**(1992), no. 1, 187–203. http://dx.doi.org/10.1016/0021-8693(92)90184-N

[Mey93] P.-A. Meyer, *Quantum probability for probabilists.* Lecture Notes in Mathematics, 1538, Springer-Verlag, Berlin, 1993.

[Øks03] B. Øksendal, *Stochastic differential equations. An introduction with applications.* Universitext, Springer-Verlag, Berlin, 2003.

[Žel73] D. P. Želobenko, *Compact Lie groups and their representations.* Translations of Mathematical Monographs, 40, American Mathematical Society, Providence, RI, 1973.

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