Topological Quantum Field Theory, Reciprocity and the Weil representation

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A brief introduction

The ingredients of this book are three: the abelian topological quantum field theory (TQFT), the reciprocity between discriminant quadratic functions and the Weil representation.

A TQFT in dimension 3 is roughly speaking a functor from the category of oriented 3-cobordisms between surfaces to the category of finite dimensional unitary linear operators. Hence to a 3-cobordism M such that $\partial M = -\Sigma_- \coprod \Sigma_+$, a TQFT τ assigns a unitary operator $\tau(M) : \mathcal{T}(\Sigma_-) \to \mathcal{T}(\Sigma_+)$. Introduced in 1989 by E. Witten [**66**], TQFTs turned out to be a fascinating framework to produce topological invariants of 3-manifolds [**53**], [**37**], [**1**] at the crossroads of representation theory, physics and combinatorics.

Abelian TQFT is algebraically simpler than its nonabelian versions. It has been studied from several combinatorial viewpoint. The common procedure is formally the same as in the nonabelian case: it consists in using a surgery presentation of a 3-cobordism M and by a combinatorial count on the presentation, one defines the operator τ.

Our goal in this book is to unify the Abelian TQFTs. Our Abelian TQFT depends on a fixed quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$ on a finite Abelian group G. We describe it thoroughly from classical topological invariants of 3-manifolds, especially from the linking pairing of 3-manifolds and its various refinements. In particular, our description is intrinsic in the sense that it is independent of the surgery presentation. We also establish that the invariants produced by this machinery classify linking pairings with prescribed elements.

The main tool to achieve our goal is a **reciprocity** identity between discriminant quadratic functions. A linking pairing is a symmetric bilinear pairing on a finite abelian group. Any such pairing can be lifted to a bilinear lattice $\tilde{\lambda}$. The map $\tilde{\lambda} \mapsto \lambda$ is known as the discriminant map and plays a fundamental role in this book. The properties of discriminant pairings and their quadratic refinements have been thoroughly studied and applied (see for instance [**36**], [**64**], [**46**], [**18**]). A key observation here is that the discriminant preserves orthogonal sums but not tensor products. The reciprocity measures how far the discriminant is from preserving tensor products, in the framework of the Witt group of quadratic functions.

Reciprocity is also used in this book to classify pointed linking pairings and pointed quadratic functions over abelian groups. See 3, Theorems 3.1 and 3.2. In particular, it answers a question raised independently by C. Gille and myself in [**8**] regarding the classification of pointed linking pairings (called isometries by C. Gille) by the invariant τ of links in closed oriented 3-manifolds. See 3.1.

In turn, many of these results rely on the tensor product of linking pairings and the construction of an order 2 characteristic element (§2.9). Of the realization of this element by a certain G-valued 1-cycle θ depends the non-vanishing of the invariant $\tau(M, \theta)$ and the explicit construction of its extension to a full TQFT.

The last ingredient is the **Weil representation**. This is a projective representation introduced in the sixties by A. Weil (and also by Shale and Segal) of the symplectic group $\mathrm{Sp}(A)$. This representation arises when one modifies the (irreducible) Heisenberg representation by a symplectomorphism. By the Stone-Weyl theorem, the original and the modified Heisenberg representation are equivalent and related by a new projective representation, which is the Weil representation. The Weil representation appears in the context of 3-TQFTs as follows. A fundamental feature of 3-TQFTs is a projective representation of the mapping class group of surfaces (when one restricts the TQFT functor to cobordisms that are cylinders over a surface). The Abelian TQFT representation factors through the symplectic group $\mathrm{Sp}(H, \bullet)$ of the surface Σ, where $H = H_1(\Sigma)$ is the 1-homology of the surface and $\bullet$ is the symplectic intersection pairing on H. It turns out that the Abelian TQFT representation is the Weil representation. This fact has been known for some time (see for instance Funar, Manoliu), especially in connection with the computation of the gluing formula (the composition of TQFT operators is only projectively well defined) which involves the Leray-Maslov index. See [**44**] and [**40**, p. 205] and others. It also appears in the context of conformal field theory, see for instance [**21**, §3]. Here we give a new description of the Weil representation based on computations of linking pairings (or intersection pairings on surfaces) in cylinders over surfaces. This gives a topological interpretation of the Weil representation which does not rely on the combinatorics of the TQFT. The topological interpretation consists, roughly speaking, in viewing the Weil representation as an action on $\mathbb{C}[G \otimes \Lambda]$, where Λ is a Lagrangian of H: the action counts a certain weighted sum of G-valued cycles (which are identified to a basis of $\mathbb{C}[G \otimes \Lambda]$). The weights are topological invariants (linking numbers) and are identified using the reciprocity.

TBC

Part 1

Algebra

CHAPTER 1

Linking pairings and finite quadratic functions

1. ε-symmetric bilinear pairings

Let S, T and U be three sets. A *pairing* between S and T with values in U is a map $p : S \times T \to U$. The *left* and *right adjoint map* associated to p are respectively the maps $S \to U^T,\ s \mapsto p(s, -)$ and $T \to U^S,\ t \mapsto p(-, t)$. A pairing p is *left* (resp. *right*) *nondegenerate* if the left (resp. *right*) adjoint map is injective. A pairing is *left* (resp. *right*) *nonsingular* if its left (resp. *right*) adjoint map is bijective. A pairing $p : S \times S \to U$ is said *symmetric* if $p(x, y) = p(y, x)$ for all $x, y \in S$. Assume that U is an abelian group. The pairing $p : S \times S \to U$ is said *antisymmetric* (resp. *symplectic*) if $p(x, y) = -p(y, x)$ for all $x, y \in S$ (resp. if $p(x, x) = 0$ for all $x \in S$). Let $\varepsilon \in \{\pm 1\}$. A pairing is an *$\varepsilon$-symmetric* pairing if it is either symmetric ($\varepsilon = 1$) or antisymmetric ($\varepsilon = -1$).

LEMMA 1.1. *A symplectic pairing $p : S \times S \to U$ is antisymmetric.*

PROOF. $0 = p(x + y, x + y) = p(x, x) + p(x, y) + p(y, x) + p(y, y) = p(x, y) + p(y, x)$. □

Conversely if $p : S \times S \to U$ is antisymmetric, then $2p(x, x) = 0$ for all $x \in S$. In particular, if U has no 2-torsion, then p is symplectic.

Clearly a ε-symmetric pairing is nondegenerate (resp. nonsingular) if and only if one of its adjoint maps is injective (resp. bijective). If p is symmetric then the left adjoint map coincides with the right adjoint map and we denote it $\widehat{p} : S \to U^S$. If p is antisymmetric then the left adjoint map, still denoted $\widehat{p}$, is the opposite to the right adjoint map.

Let $p : S \times S \to U$ be an ε-pairing. The *orthogonal* $V^\perp$ of a subset $V \subseteq S$ is defined as the set

$$V^\perp = \{s \in S \mid p(s, v) = 0 \text{ for all } v \in V\}.$$

For any subset V, $V \subseteq (V^\perp)^\perp$. If $V \subseteq W \subset S$ then $W^\perp \subseteq V^\perp$. Two subsets V, W of S are *orthogonal* if $p(v, w) = 0$ for all $v \in V$ and $w \in W$. Equivalently $V \subseteq W^\perp$.

Suppose that S, T and U are abelian groups. A pairing $p : S \times T \to U$ is *bilinear* if $p(s + s', t) = p(s, t) + p(s', t)$ and $p(s, t + t') = p(s, t) + p(s, t')$ for all $s, s' \in S$ and $t, t' \in T$. It follows that the left (resp. right) adjoint map is a homomorphism $S \to \mathrm{Hom}(T, U)$ (resp. $T \to \mathrm{Hom}(S, U)$).

Two bilinear pairings $p : S \times S \to U$ and $q : T \times T \to V$ are *isomorphic* if there exists an isomorphism $\varphi : S \to T$ such that $q(\varphi(s), \varphi(s')) = p(s, s')$ for all $s, s' \in S$. We write: $\varphi^* q = p$.

If $p : S \times S \to U$ is an ε-symmetric bilinear pairing, then $V^\perp$ is a subgroup of S for any subset $V \subseteq S$. It is also the orthogonal of the subgroup generated by V. For any subgroups V, W of S,

$$V^\perp \cap W^\perp = (V + W)^\perp. \tag{1.1}$$

Since $S^\perp = \operatorname{Ker} \widehat{p}$, an ε-symmetric bilinear pairing $p : S \times S \to U$ is nondegenerate if and only if $S^\perp = 0$. A subgroup V of S is said *isotropic* if $V \subseteq V^\perp$. A subgroup V of S is a *Lagrangian* if $V = V^\perp$. Any isotropic subgroup V induces a quotient ε-symmetric bilinear pairing $\bar{p}$ on $V^\perp/V$.

The *opposite* of a bilinear pairing $p : S \times T \to U$ is the bilinear pairing $-p : S \times T \to U$ defined by $(-p)(s, t) = -p(s, t)$.

Let $p : S \times S \to U$ and $p' : S' \times S' \to U$ be two bilinear pairings, both symmetric (resp. both antisymmetric). The *orthogonal sum* of p and p' is the symmetric (resp. antisymmetric) bilinear pairing $p \oplus p' : (S \oplus S') \times (S \oplus S') \to U$ defined by

$$(p \oplus p')(x + x', y + y') = p(x, y) + p'(x', y'), \quad x, y \in S,\ x', y' \in S'.$$

Clearly $S = S \oplus 0$ and $S' = 0 \oplus S$ are mutually orthogonal in $S \oplus S'$, i.e. $S^\perp = S'$ and $S'^\perp = S$. If the pairings on S and S' are implicitly understood, then we denote the orthogonal sum of the pairings (S, p) and (S', p') by $S \oplus S'$.

Conversely if $p'' : S'' \times S'' \to U$ is an ε-symmetric pairing such that there exist subgroups S and S' such that $S^\perp = S'$ and $S'^\perp = S$, then p'' splits as an orthogonal sum

$$p'' = p|_{S \times S} \oplus p|_{S' \times S'}.$$

Lemma 1.2. *Let $p : S \times S \to U$ be a nonsingular ε-symmetric pairing. Let V be a subgroup of S. The following statements are equivalent:*

(1) *$p|_{V \times V} : V \times V \to U$ is nonsingular;*
(2) *$S = V \oplus V^\perp$ and $p|_{V^\perp \times V^\perp} : V^\perp \times V^\perp \to U$ is nonsingular.*

A subgroup V satisfying one of the properties stated in Lemma 1.2 is an *orthogonal summand* of S.

Proof. See [**64**, Lemma (1)]. □

Let $p : S \times S \to U$ and $q : T \times T \to V$ be two bilinear pairings. The respective adjoint maps $\widehat{p} : S \to \operatorname{Hom}(S, U)$ and $\widehat{q} : T \to \operatorname{Hom}(T, V)$ induce a homomorphism

$$S \otimes T \xrightarrow{\widehat{p} \otimes \widehat{q}} \operatorname{Hom}(S, U) \otimes \operatorname{Hom}(T, V) \xrightarrow{\otimes} \operatorname{Hom}(S \otimes T, U \otimes V). \tag{1.2}$$

Definition 1.1. The *tensor product* of p and q is the bilinear pairing

$$p \otimes q : (S \otimes T) \times (S \otimes T) \to U \otimes V$$

whose left adjoint map is the homomorphism above.

Alternatively, the tensor product of p and q can be regarded as the bilinear pairing induced by the multilinear map $p \times q : (S \times T) \times (S \times T) \to U \otimes V$ defined by $(p \times q)(s, t; s', t') = p(s, s') \otimes q(t, t')$, $s, s' \in S$, $t, t' \in T$.

If p and q are both symmetric or both antisymmetric, then $p \otimes q$ is symmetric. If p is symmetric (resp. antisymmetric) and q is antisymmetric (resp. symmetric), then $p \otimes q$ is antisymmetric.

The tensor product of two bilinear pairings take value in the tensor product $U \otimes V$ of the groups where the respective pairings take their values. Here are two examples.

Example 1.1. The tensor product of an antisymmetric bilinear pairing $p : S \times S \to \mathbb{Z}$ on a free abelian group S and a symmetric bilinear pairing $q : T \times T \to \mathbb{Q}/\mathbb{Z}$ on a torsion group T is an antisymmetric bilinear pairing $p \otimes q : (S \otimes T) \times (S \otimes T) \to \mathbb{Q}/\mathbb{Z}$. The tensor product is induced by pointwise product $\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$:

$$(p \otimes q)(x \otimes y, x' \otimes y') = p(x, x') \cdot q(y, y').$$

Example 1.2. Let r, s be positive integers and let t be their greatest common divisor. There is a canonical isomorphism

$$\mathbb{Z}/r\mathbb{Z} \otimes \mathbb{Z}/s\mathbb{Z} \simeq \mathbb{Z}/t\mathbb{Z},\ (1 \bmod r,\ 1 \bmod s) \mapsto 1 \bmod t.$$

The tensor product of two symmetric bilinear pairings $p : S \times S \to \mathbb{Z}/r\mathbb{Z}$ and $q : T \times T \to \mathbb{Z}/s\mathbb{Z}$ on torsion groups S and T respectively is a symmetric bilinear pairing $p \otimes q : (S \otimes T) \times (S \otimes T) \to \mathbb{Z}/t\mathbb{Z}$. In the particular case $r = s$, the tensor product is induced by pointwise product.

Remark 1.1. It is sometimes convenient to simplify the notation and write S for an ε-symmetric bilinear pairing $\lambda : S \times S \to U$ when the underlying pairing λ is implicitly understood. In this case, we write $-S$ for the opposite pairing, $S \oplus T$ for orthogonal sum, etc.

2. ε-linking pairings

Let G be a finite abelian group. The *dual group* G^* of G is $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Let $\varepsilon \in \{\pm 1\}$.

Definition 2.1. An *ε-linking pairing* is an ε-symmetric bilinear pairing on a finite abelian group.

The definition forces the value group to be a finite subgroup of $\mathbb{Q}/\mathbb{Z}$. So a linking (resp. finite symplectic) pairing can be defined as a symmetric (resp. symplectic) bilinear pairing $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$. Alternatively, λ can be defined via its left adjoint map as a homomorphism $\widehat{\lambda} : G \to G^*$.

It is sometimes convenient to take a smaller subgroup of values rather than the whole group $\mathbb{Q}/\mathbb{Z}$. For any integer n, the cyclic group $\mathbb{Z}/n\mathbb{Z}$ canonically embeds in $\mathbb{Q}/\mathbb{Z}$ by the map

$$j_n : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z},\ (1 \bmod n) \mapsto \frac{1}{n} \bmod 1.$$

For a finite group G, let $e_G \in \mathbb{N}^\times$ be the *period* of G, that is the smallest positive integer n such that $n\,x = 0$ for all $x \in G$.

LEMMA 2.1. *Any ε-linking pairing $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ factors through an ε-linking pairing $\lambda' : G \times G \to \mathbb{Z}/e_G\mathbb{Z}$:*

$$\begin{tikzcd} G \times G \arrow[rr, "\lambda"] \arrow[dr, dotted] & & \mathbb{Q}/\mathbb{Z} \\ & \mathbb{Z}/e_G\mathbb{Z} \arrow[ur] & \end{tikzcd}$$

LEMMA 2.2. *An ε-linking pairing is nonsingular if and only if it is nondegenerate.*

PROOF. It suffices to see that nondegenerate implies nonsingular. The adjoint map $G \to G^*$ is injective. Since G is finite, the dual group G^* is also finite with $|G^*| = |G|$. Hence the adjoint map is bijective. □

LEMMA 2.3. *Let $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate ε-linking pairing. For any subgroup H of G,*

$$|G| = |H| \cdot |H^\perp| \quad \text{and } (H^\perp)^\perp = H. \tag{2.1}$$

PROOF. There is a short exact sequence

$$0 \longrightarrow H^\perp \longrightarrow G \xrightarrow{\widehat{\lambda}|_H} H^* \longrightarrow 0,$$

where by definition $\widehat{\lambda}|_H(h) = \lambda(h, -) \in H^*$ for all $h \in H$. Hence $G/H^\perp \simeq H^*$. Therefore

$$|G| = |H| \cdot |H^\perp|.$$

This equality is true for any subgroup H of G. Applying this equality to the subgroup $H^\perp$, we obtain $|G| = |H^\perp|\cdot|(H^\perp)^\perp|$. It follows that $|(H^\perp)^\perp| = |H|$. Since $H \subseteq (H^\perp)^\perp$, the equality follows. □

We define a tensor product for ε-linking pairings. The general definition 1.1 does not apply here since $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = 0$. Instead we resort to the ideas of Lemma 2.1 and of Example 1.2.

Set $\hat{G} = \mathrm{Hom}(G, \mathbb{Z}/e_G\mathbb{Z})$. Let $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ and $\lambda' : G' \times G' \to \mathbb{Q}/\mathbb{Z}$ be two ε-linking pairings given by their left adjoint maps. They induce a homomorphism

$$\Phi :\; G \otimes G' \xrightarrow{\widehat{\lambda}\otimes\widehat{\lambda}'} \hat{G} \otimes \hat{G}' \xrightarrow{\otimes} \widehat{G \otimes G'}. \tag{2.2}$$

DEFINITION 2.2. The *tensor product* $\lambda \otimes \lambda'$ of two ε-linking pairings $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ and $\lambda' : G' \times G' \to \mathbb{Q}/\mathbb{Z}$ is the bilinear pairing defined by

$$(\lambda \otimes \lambda')(x, y) = \Phi(x)(y), \quad x, y \in G \otimes G'.$$

The tensor product is symmetric if both pairings are symmetric or both pairings are antisymmetric. The tensor product is antisymmetric (resp. symplectic) if one of the pairings is symmetric and the other one is antisymmetric (symplectic).

The natural map $\hat{G}\otimes\hat{G}' \to \widehat{G\otimes G'}$, $f\otimes f' \mapsto f(-)\otimes f'(-)$ is an isomorphism. As a consequence of this and functoriality, we record

LEMMA 2.4. *If λ and λ' are nondegenerate, then $\lambda\otimes\lambda'$ is nonsingular.*

3. Decomposition of ε-linking pairings

Let $\lambda : A\times A \to \mathbb{Q}/\mathbb{Z}$ be an ε-linking pairing on a finite abelian group A. *We assume throughout this section that λ is nondegenerate*. Recall that this is equivalent to $A^\perp = 0$.

LEMMA 3.1. *For any subgroup $B\subseteq A$, $|B|\cdot|B^\perp| = |A|$ and $(B^\perp)^\perp = B$.*

PROOF. By definition, $B^\perp$ is the kernel of the surjective map

$$A \mapsto \mathrm{Hom}(B,\mathbb{Q}/\mathbb{Z}),\ a\mapsto \widehat{\lambda}(a)|_B.$$

Thus $|A/B^\perp| = |\mathrm{Hom}(B,\mathbb{Q}/\mathbb{Z})| = |B|$, the first equality follows. Applying the first equality to B and $B^\perp$ respectively yields $|B||B^\perp| = |A| = |B^\perp||B^{\perp\perp}|$, so $|B| = |B^{\perp\perp}|$. Since $B\subseteq B^{\perp\perp}$, the second equality follows. □

LEMMA 3.2. *There exists a canonical orthogonal splitting*

$$(A,\lambda) = \bigoplus_{p \text{ prime}} (A_p,\lambda_p)$$

where $A_p = \{x\in A \mid p^N x = 0 \text{ for some } N\in\mathbb{N}\}$. In particular, each orthogonal summand A_p is a p-group.

DEFINITION 3.1. Each linking pairing (A_p,λ_p) is *the p-component* of (A,λ).

PROOF. Let A_p denote the subgroup of all elements in A of order a power of p. Clearly, $A = \oplus_{p \text{ prime}} A_p$. We claim that $\lambda(A_p,A_q) = 0$ for any two distinct primes p,q. Let $x\in A_p$ and $y\in A_q$. By definition, we have $0 = \lambda(p^k x, y) = p^k\ \lambda(x,y)$ for some integer k. Similarly, we have $0 = \lambda(x, q^l y) = q^l\ \lambda(x,y)$. Thus $\lambda(x,y)$ is annihilated in $\mathbb{Q}/\mathbb{Z}$ by both p^k and q^l which are coprime. It follows that $\lambda(x,y) = 0$. ■ □

LEMMA 3.3. *Let $x\in A$ and let B the subgroup generated by x. Let n be the order of B. The following assertions are equivalent:*

1. *$\lambda(x,x)$ has order n in $\mathbb{Q}/\mathbb{Z}$;*
2. *$\lambda|_{B\times B}$ is nonsingular;*
3. *$A = B\oplus B^\perp$ and $\lambda|_{B^\perp\times B^\perp}$ is nonsingular.*

PROOF. The equivalence (2) $\iff$ (3) follows from Lemma 1.2. Let us prove (1) $\iff$ (2). Suppose that $\lambda(x,x)$ has order n. Let $y = k\ x\in B$. The equation $0 = \lambda(x,y) = k\ \lambda(x,x)$ implies that k is a multiple of n hence $y = 0$. Thus $\lambda|_{B\times B}$ is nondegenerate, hence nonsingular. Conversely, let m be the order of $\lambda(x,x)$. Since $n\lambda(x,x) = \lambda(nx,x) = 0$, m divides n. Now $0 = \lambda(x,mx)$ so $0 = \lambda(kx,mx) = 0$ for all $k\geqslant 0$. Thus $mx\in \mathrm{Ker}\ \widehat{\lambda}$. Since $\lambda|_{B\times B}$ is nonsingular, $mx = 0$. This implies that n divides m. Therefore $m = n$. □

COROLLARY 3.1. *Let $x \in A$. Suppose that λ is antisymmetric. The following assertions are equivalent:*

(1) *x generates a nontrivial orthogonal summand;*
(2) *x generates an orthogonal summand of order 2;*
(3) *$\lambda(x,x)$ has order 2 in $\mathbb{Q}/\mathbb{Z}$.*

PROOF. (2) $\Longrightarrow$ (1) is clear and (2) $\Longleftrightarrow$ (3) follows from Lemma 3.3. Suppose (1) holds. By Lemma 3.3, x generates a nontrivial subgroup B of the same order as the order of $\lambda(x,x)$ in $\mathbb{Q}/\mathbb{Z}$. Since $2\lambda(x,x) = 0$, B has order 2. $\square$

3.1. Symmetric linking pairings. The paragraph is devoted to symmetric linking pairings.

PROPOSITION 3.1 (Symmetric linkings). *Let (A,λ) be a nondegenerate symmetric linking pairing on a finite p-group. There exists an orthogonal splitting $(A,\lambda) = \oplus_k (A_k,\lambda_k)$ where each (A_k,λ_k) is a nondegenerate bilinear pairing such that A_k is*

(*i*) *either a cyclic p-group,*
(*ii*) *or a direct sum of two copies of a cyclic group of order 2^n. In this case, λ_k is represented by a matrix of the form $\begin{bmatrix} 0 & 2^{-n} \\ 2^{-n} & 0 \end{bmatrix}$ or $\begin{bmatrix} 2^{1-n} & 2^{-n} \\ 2^{-n} & 2^{1-n} \end{bmatrix}$.*

Furthermore, if p is odd, only the case (i) *may occur.*

PROOF. The proof goes by induction on $|A|$. If $|A| = p$ then A is cyclic and the assertion holds. Let now p^n denote the period of A. We distinguish two cases:

p odd: we claim that there exists $x \in A$ such that $\lambda(x,x)$ has order exactly p^n in $\mathbb{Q}/\mathbb{Z}$. Otherwise, the order of $\lambda(x,x)$ divides p^{n-1} for all x; then the order of $2\,\lambda(x,y) = \lambda(x+y,x+y) - \lambda(x,x) - \lambda(y,y)$ also divides p^{n-1} (for all x,y); thus $p^{n-1}A \subset A^{\perp}$, contradicting nondegeneracy. So pick up $x \in A$ so that the order of $\lambda(x,x)$ is p^n. The cyclic subgroup B generated by x has order p^n. By Lemma 3.3, B is an orthogonal summand of A: $A = B \oplus B^{\perp}$. We apply the induction hypothesis to $B^{\perp} \subset A$.

p even: if an element x exists such that $\lambda(x,x)$ has order 2^n, the argument above applies. Consider the case when no such elements exists in A. Then nondegeneracy of λ ensures that there exist $x,y \in A$, both of order 2^n, such that $\lambda(x,y)$ has order exactly 2^n. So there exist even integers r and s such that $\lambda(x,x) = \frac{r}{2^n} \pmod 1$ and $\lambda(y,y) = \frac{s}{2^n} \pmod 1$. Let B denote the subgroup generated by x and y. Let $a\,x + b\,y \in B \cap B^{\perp}$. We have

$$0 = \lambda(ax+by,x) = a\lambda(x,x) + b\lambda(x,y) + \frac{ar}{2^n} = \frac{ar}{2^n} + \frac{b}{2^n} \bmod 1.$$

It follows that $ar + b = 0 \bmod 2^n$. Similarly the equality $\lambda(ax + by, y) = 0$ leads to $a + bs = 0 \bmod 2^n$. We deduce that $a = b = 0 \bmod 2^n$. Therefore, B is the direct sum of the cyclic groups generated by x and y and $B \cap B^\perp = 0$. We conclude by again applying induction to $B^\perp$.

The statement about the matrix representatives of λ_k is a consequence of Lemma 3.4 below. ■

□

Denote by $\mathrm{Sym}_2(\mathbb{Z}/2^n\mathbb{Z})$ the algebra of two by two matrices with coefficients in $\mathbb{Z}/2^n\mathbb{Z}$ and by $\mathrm{GL}_2(\mathbb{Z}/2^n\mathbb{Z})$ the group of two by two matrices with coefficients in $\mathbb{Z}/2^n\mathbb{Z}$ that are invertible over $\mathbb{Z}/2^n\mathbb{Z}$. For $1 \leqslant k \leqslant n$, define an equivalence relation $\underset{\mathbb{Z}/2^k\mathbb{Z}}{\sim}$ in $\mathrm{Sym}_2(\mathbb{Z}/2^n\mathbb{Z})$ by $A \underset{\mathbb{Z}/2^k\mathbb{Z}}{\sim} B$ if there exists $C \in \mathrm{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ such that ${}^tCAC = M \bmod 2^k$.

Lemma 3.4. *Let $n \geqslant 1$. We have*

$$\begin{bmatrix} 2r & u \\ u & 2s \end{bmatrix} \underset{\mathbb{Z}/2^n}{\sim} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for any $r, s, u \in \mathbb{Z}$ with u odd.

Proof. Note that for all $n \geqslant 1$,

$$\begin{bmatrix} 2r & u \\ u & 2s \end{bmatrix} \underset{\mathbb{Z}/2^n\mathbb{Z}}{\sim} \begin{bmatrix} 2r & -u \\ -u & 2s \end{bmatrix} \tag{3.1}$$

and

$$\begin{bmatrix} 2r & u \\ u & 2s \end{bmatrix} \underset{\mathbb{Z}/2^n\mathbb{Z}}{\sim} \begin{bmatrix} 2s & u \\ u & 2r \end{bmatrix} \tag{3.2}$$

We proceed inductively on n. For $n = 1$, the result is trivial.
For $n = 2$:

- If $2r \neq 2s \bmod 4$, then by (3.2), we may assume that $2r = 0 \bmod 4$ and $2s = 2 \bmod 4$. Then
$$\begin{bmatrix} 0 & u \\ u & 2 \end{bmatrix} \underset{\mathbb{Z}/4\mathbb{Z}}{\sim} \begin{bmatrix} 0 & u \\ u & 2+2u \end{bmatrix} = \begin{bmatrix} 0 & u \\ u & 0 \end{bmatrix} \underset{\mathbb{Z}/4\mathbb{Z}}{\sim} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
(the last relation is either an equality or follows from (3.1.)
- If $2r = 2s \bmod 4$, then applying (3.1) if necessary, we have
$$\begin{bmatrix} 2r & u \\ u & 2r \end{bmatrix} \underset{\mathbb{Z}/4\mathbb{Z}}{\sim} \begin{bmatrix} 2r & 1 \\ 1 & 2s \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now the result follows by repeated applications of the lemma 3.5. ■

□

Lemma 3.5 (A version of Hansel's lemma). *Let $n \geqslant 2$ and $A, B \in \mathrm{Sym}_2(\mathbb{Z}/2^n\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Z}/2^n\mathbb{Z})$. Suppose that $A \underset{\mathbb{Z}/2^k\mathbb{Z}}{\sim} B$ for some $2 \leqslant k \leqslant n-1$. Then $A \underset{\mathbb{Z}/2^{k+1}\mathbb{Z}}{\sim} B$.*

Proof. There is $M_k \in \mathrm{GL}_2(\mathbb{Z}/2^k)$ such that ${}^tM_kAM_k = B \bmod 2^k$. We expect a solution M_{k+1} to the equation

$$ {}^tM_{k+1}AM_{k+1} = B \bmod 2^{k+1}. \tag{3.3}$$

We look for a solution of the form $M_{k+1} = M_k + X_{k+1}$ where X_{k+1} is a matrix with coefficients in $\mathbb{Z}/2^{k+1}$ such that $X_{k+1} = 0 \bmod 2^k$. Plugging this expression in (3.3) and expanding, we find that a necessary condition is that

$$ {}^tM_kAX_{k+1} + {}^tX_{k+1}AM_k = B - {}^tM_kAM_k \mod 2^{k+1}. \tag{3.4}$$

This equation is of the form $UX + {}^t(UX) = H$, with $U = {}^tM_kA$ and $H = B - {}^tM_kAM_k$. A formal solution is $X = \frac{1}{2}U^{-1}H$. Note that U is invertible over $\mathbb{Z}/2^k\mathbb{Z}$, hence over $\mathbb{Z}/2^{k+1}\mathbb{Z}$. Further, $H = B - {}^tM_kAM_k$ is $0 \bmod 2^k$ by hypothesis. Thus, $X_{k+1} = \frac{1}{2}U^{-1}H$ is a solution of (3.4) and $M_{k+1} = M_k + X_{k+1}$ is a solution of (3.3). Since M_{k+1} is invertible mod 2^k, it is also invertible mod 2^{k+1}, which concludes the proof. ■ □

The proof of Lemma 3.5 contains more than the statement of Proposition 3.1. Denote by $\underset{\mathbb{Z}_2}{\sim}$ the same equivalence relation but defined over the 2-adic integers $\mathbb{Z}_2$.

Corollary 3.2. *Any symmetric matrix M with coefficients in $\mathbb{Z}_2$ is equivalent (for $\underset{\mathbb{Z}_2}{\sim}$) to a block-diagonal matrix with each block of one of the three following types:*

$$[a] \ \ (a \in \mathbb{Z}_2), \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Remark 3.1. Although Proposition 3.1 is fundamental and will be used systematically, in practice one needs to know how to deal with linking pairings that are not canonically split.

Example 3.1. Let p, q, r be three pairwise coprime integers. Consider the cyclic linking pairing $(\frac{r}{pq})$ defined on $\mathbb{Z}/pq$ sending $(1,1)$ to $r/pq \bmod 1$. By Proposition 3.1, this linking pairing must be isomorphic to an orthogonal sum of two cyclic pairings on $\mathbb{Z}/p$ and $\mathbb{Z}/q$ respectively. Let $(\alpha, \beta) \in \mathbb{Z}^2$ be a Bezout pair for (p, q) so that $\alpha p + \beta q = 1$. There is an isomorphism

$$\mathbb{Z}/p \times \mathbb{Z}/q \to \mathbb{Z}/pq, \ (u, v) \mapsto u\alpha q + v\beta p \bmod pq$$

whose inverse is

$$\mathbb{Z}/pq \to \mathbb{Z}/p \times \mathbb{Z}/q, \ x \mapsto (x \bmod p, \ y \bmod q).$$

Using the fact that $\alpha p + \beta q = 1$, we see that

$$(u\alpha q + v\beta p)^2 = u^2\alpha q + v^2\beta p \bmod pq.$$

It follows that

$$(\frac{r}{pq}) \simeq (\frac{r\alpha}{p}) \oplus (\frac{r\beta}{q}). \tag{3.5}$$

Applying (3.5) to $\left(\frac{1}{21}\right)$ for instance, we obtain

$$\left(\frac{1}{21}\right) \simeq \left(\frac{1}{7}\right) \oplus \left(\frac{-2}{3}\right) = \left(\frac{1}{7}\right) \oplus \left(\frac{1}{3}\right).$$

More generally, let $p_1, \dots, p_n$ be pairwise coprime integers. For each $1 \leqslant i \leqslant n$, set $M_i = \prod_{j \neq i} p_j$ and let μ_i be an integer such that $\mu_i M_i = 1 \bmod p_i$. The map

$$\prod_{i=1}^{n} \mathbb{Z}/p_i \to \mathbb{Z}/\prod_{i=1}^{n} p_i,\ (u_1, \dots, u_n) \mapsto \sum_{i=1}^{n} u_i M_i \mu_i$$

is an isomorphism whose inverse is

$$\mathbb{Z}/\prod_{i=1}^{n} p_i \to \prod_{i=1}^{n} \mathbb{Z}/p_i,\ x \mapsto (x \bmod p_1, \dots, x \bmod p_n).$$

Using the fact that $\mu_i M_i = 1 \bmod p_i$, we see that

$$\left(\sum_{i=1}^{n} u_i M_i \mu_i\right)^2 = \sum_{i=1}^{n} u_i^2 M_i \mu_i \mod \prod_{i=1}^{n} p_i.$$

It follows that

$$\left(\frac{1}{\prod_{i=1}^{n} p_i}\right) \simeq \bigoplus_{i=1}^{n} \left(\frac{\mu_i}{p_i}\right). \tag{3.6}$$

For instance, applying (3.6) to $\left(\frac{1}{861}\right)$ $(861 = 3 \times 7 \times 41)$ yields

$$\left(\frac{1}{861}\right) \simeq \left(\frac{1}{3}\right) \oplus \left(\frac{1}{7}\right) \oplus \left(\frac{1}{41}\right).$$

DEFINITION 3.2. Let $\lambda : A_2 \times A_2 \to \mathbb{Q}/\mathbb{Z}$ be an ε-linking pairing on a 2-group A_2 of period 2^k. The *summand evaluation map* is the map $\epsilon : A_2 \to \mathbb{Z}/2\mathbb{Z}$ defined by $\epsilon(x) = 1$ if x generates an orthogonal summand of order 2^k in A and $\epsilon_k(x) = 0$ otherwise.

LEMMA 3.6. *The summand evaluation map is a homomorphism.*

PROOF. Assume that λ is symmetric. Let $x, y \in A$. The order of $x + y$ in A is the l.c.m. of the orders of x and y. Assume that $\epsilon(x) = \epsilon(y) = 1$. By Lemma 3.3, $\lambda(x, x) = \frac{a}{2^k} \bmod 1$ and $\lambda(y, y) = \frac{b}{2^k} \bmod 1$ where a and b are odd integers. Then

$$\begin{aligned} \lambda(x+y, x+y) &= \lambda(x, x) + 2\lambda(x, y) + \lambda(y, y) \\ &= \frac{a}{2^k} + \frac{2c}{2^k} + \frac{b}{2^k} \\ &= \frac{a + 2c + b}{2^k} \ (\text{mod } 1). \end{aligned}$$

Since $a + 2c + b$ is even, $\lambda(x + y, x + y)$ has order strictly less than 2^k.

By Lemma 3.3 again, we conclude that $x+y$ does not generate an orthogonal subgroup of order 2^k. Hence $\epsilon(x + y) = 0$.

Assume that $\epsilon(x) = 1$ and $\epsilon(y) = 0$. By Lemma 3.3, $\lambda(x, x) = \frac{a}{2^k} \bmod 1$ and $\lambda(y, y) = \frac{b}{2^l} \bmod 1$, where a is an odd integer, $b \in \mathbb{Z}$ and l is a natural number distinct from k. Since k is maximal, $k > l$ then $x + y$ has order 2^k, hence generates a subgroup of maximal order 2^k. Furthermore,

$$\begin{aligned}\lambda(x+y,x+y) &= \lambda(x,x) + 2\lambda(x,y) + \lambda(y,y)\\ &= \frac{a}{2^k} + \frac{2c}{2^l} + \frac{b}{2^l}\\ &= \frac{a + 2^{k-l+1}c + 2^{k-l}b}{2^k} \pmod 1.\end{aligned}$$

Since $a + 2^{k-l+1}c + 2^{k-l}$ is odd, $\lambda(x+y,x+y)$ has order exactly 2^k. By Lemma 3.3 again, we conclude that $\epsilon(x+y) = 1$. The case when λ is antisymmetric is similar ($c = 0$ above). □

Corollary 3.3. *If λ is antisymmetric, the summand evaluation map is nontrivial only on groups of period* 2.

Proof. Apply Cor. 3.1. □

By Lemma **??**, the summand evaluation map extends to a map $A \times A \to \mathbb{Z}/2\mathbb{Z}$ for any ε-linking pairing.

Lemma 3.7. *The summand evaluation map is an invariant of isomorphism classes of ε-linking pairings.*

The precise meaning of the Lemma is the following. If $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ and $\lambda' : G' \times G' \to \mathbb{Q}/\mathbb{Z}$ are two ε-linking pairings related by an isomorphism $\varphi : G \to G'$ such that $\lambda' \circ \varphi^{\otimes 2} = \lambda$, then the respective summand evaluation maps are related by $\epsilon_{\lambda'} \circ \varphi = \epsilon_\lambda$.

Proof. □

3.2. Antisymmetric linking pairings. This paragraph is devoted to the decomposition of antisymmetric linking pairings. According to Lemma 1.1 and the remark thereafter, the only difference between symplectic and antisymmetric linking pairings occurs on 2-groups. We begin with three examples of antisymmetric linking pairings: the first one is symplectic, the last two are antisymmetric nonsymplectic.

Example 3.2 (symplectic linking pairing). Let p be a prime number and k a positive number. Let A and B be two copies of a cyclic group of order p^k. Choose a generator $x \in A$ and a generator $y \in B$. There is a uniquely defined symplectic linking pairing h defined on $A \oplus B$ by

$$h_{p,k}(x,y) = \frac{1}{p^k} \bmod 1, \quad h(x,x) = h(y,y) = 0.$$

Example 3.3. The assignment

$$(1 \bmod 2, 1 \bmod 2) \mapsto \frac{1}{2} \bmod 1$$

determines an antisymmetric linking pairing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. It is both symmetric and antisymmetric, but not symplectic. We denote it by $\left(\frac{1}{2}\right)$.

EXAMPLE 3.4 (a noncyclic antisymmetric nonsymplectic linking pairing). Let p be a prime number and k a positive number. Let A and B be two copies of a cyclic group of order 2^k. Choose a generator $x \in A$ and a generator $y \in B$. There is a uniquely defined antisymmetric linking pairing e_k^0 defined on $A \oplus B$ by

$$e_k(x,x) = 0,\ e_k(x,y) = \frac{1}{2^k} \bmod 1,\ e_k(y,y) = \frac{1}{2} \bmod 1.$$

With respect to the system of generators (x, y) for $A \oplus B$, e_k is represented by the matrix $\begin{bmatrix} 0 & 1/2^k \\ -1/2^k & 1/2 \end{bmatrix}$.

It turns out that any symplectic linking pairing occurs as a finite orthogonal sum of pairings of the type of Example 3.2.

PROPOSITION 3.2 (Symplectic linking pairings). *Let (A, λ) be a nondegenerate symplectic linking pairing on a finite p-group. There exists an orthogonal splitting $(A, \lambda) = \oplus_k (A_k, \lambda_k)$ where each (A_k, λ_k) is a nondegenerate bilinear pairing such that A_k is the direct sum of two copies of cyclic p-groups and λ_k is represented by a matrix of the form $\begin{bmatrix} 0 & p^{-n} \\ -p^{-n} & 0 \end{bmatrix}$ for some positive natural number n.*

PROOF. We proceed by induction on the exponent p^n of A. Let $x \in A$ have maximal order p^n. Let G be the subgroup generated by x. Since λ is nondegenerate, there exists $y \in A$ such that $\lambda(x,y) = \frac{1}{p^n} \bmod 1$. In particular, y has also order p^n. Since λ is symplectic, the subgroup H generated by y does not intersect nontrivially G. Thus G and H form a direct sum B in A. We claim that $\lambda|_{B\times B}$ is nondegenerate. Indeed, let $z = a'\,x + b'\,y \in B$ such that $\lambda(a\,x + b\,y, z) = 0$ for all $a, b \in \mathbb{Z}$. We find that

$$0 = \lambda(a\,x + b\,y, z) = ab'\lambda(x,y) + ba'\lambda(y,x) = \frac{ab' - a'b}{p^n} \bmod 1.$$

Hence $ab' - a'b = 0 \bmod p^n$ for all $a, b \in \mathbb{Z}$. This implies that $a' = b' = 0 \bmod p^n$, thus $z = 0$. We conclude by Lemma 1.2 that $A = B \oplus B^\perp$. The proof is now completed by applying the induction to $B^\perp$. $\square$

The following corollary will be used in the theory of the Weil representation.

COROLLARY 3.4. *Given a symplectic linking pairing $\lambda : A \times A \to \mathbb{Q}/\mathbb{Z}$, there exists a bilinear pairing $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ such that*

$$\lambda(x,y) = \beta(x,y) - \beta(y,x), \quad \text{for any } x, y \in A. \tag{3.7}$$

PROOF. Decompose $(A, \lambda) = \oplus_k (A_k, \lambda_k)$ where each component is a symplectic linking pairing of the form described in Prop. 3.2. It suffices to construct a form β satisfying (3.7) on $A = A_k$. Let x, y be two generators of A_k. Define

$$\begin{array}{rcll} \beta(x,y) & = & \lambda(x,y) & \\ \beta(x,x) & = & \beta(y,y) & = 0 \\ \beta(y,x) & = & 0 & \end{array}$$

and extend β to a bilinear map on A_k. $\square$

4. Quadratic functions

We keep notations from the previous section.

Definition 4.1. Let G be an abelian group. A *quadratic function* on G is a map $q : G \to \mathbb{Q}/\mathbb{Z}$ such that

$$b_q : (x, y) \mapsto b_q(x, y) = q(x + y) - q(x) - q(y)$$

is $\mathbb{Z}$-bilinear on G. The (symmetric) linking pairing b_q is called the linking pairing *associated* to q; the map q is said to be a quadratic function *over* b_q.

Given a linking pairing $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$, a *quadratic refinement* (or *enhancement*) of λ is a quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$ such that $b_q = \lambda$. The quadratic function q is said to be *nondegenerate* if the associated bilinear pairing b_q is nondegenerate. The set $\mathrm{Quad}(\lambda)$ of quadratic function over a nondegenerate linking pairing $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ is freely and transitively acted on by G via the formula

$$q \cdot x = q + \lambda(x, -) = q + \widehat{\lambda}(x) \in \mathrm{Quad}(\lambda), \quad q \in \mathrm{Quad}(\lambda), \quad x \in G. \tag{4.1}$$

A quadratic function q on G is *homogeneous* if $q(n\,x) = n^2\,q(x)$ for all $x \in G$. Two quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ are *isomorphic* if there exists a group isomorphism $\phi : G \to G'$ such that $q'(\phi(x)) = q(x)$ for all $x \in G$. If two quadratic functions are isomorphic, then their associated linking pairings are isomorphic.

The set $\mathrm{Quad}(G)$ of all quadratic functions (including degenerate quadratic functions) defined on G is an additive group for the operation defined by

$$(q + q')(x) = q(x) + q'(x), \ x \in G.$$

The map $q \mapsto b_q$ defines a projection onto the additive group $\mathrm{Link}(G)$ of all linking pairings defined on G. Note that $\mathrm{Quad}(G)$ contains as a subgroup the group $\mathrm{Quad}^0(G)$ of all homogeneous quadratic functions. These groups fit into the following diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{Quad}(G) \arrow[r] & \mathrm{Link}(G) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(G, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \arrow[r] \arrow[u, hook] & \mathrm{Quad}^0(G) \arrow[r] \arrow[u, hook] & \mathrm{Link}(G) \arrow[r] \arrow[u, equal] & 0. \end{tikzcd} \tag{4.2}$$

We shall use repeatedly the following basic result.

Proposition 4.1. *The following assertions are equivalent:*

(1) *G has odd order.*
(2) *Multiplication by 2 in G is an automorphism.*
(3) *The second row of* (4.2) *is split.*
(4) *The first row of* (4.2) *is split.*

Proof. (1) $\Longrightarrow$ (2): Since the finite homomorphism $G \to G, x \mapsto 2x$ has trivial kernel, it must be an automorphism.

(2) $\Longrightarrow$ (3): The map $s : \mathrm{Link}(G) \to \mathrm{Quad}^0(G), \lambda \mapsto s(\lambda)$ defined by

$$s(\lambda)(x) = \frac{1}{2}\lambda(x,x),\ x \in G$$

is a section.

(3) $\Longrightarrow$ (4): Any section $\mathrm{Link}(G) \to \mathrm{Quad}^0(G)$ composed with the inclusion $\mathrm{Quad}^0(G) \subset \mathrm{Quad}(G)$ is a section.

(4) $\Longrightarrow$ (1): Assume that G has even order. We show that there exists no section for the first row of (4.2). Let $x \in G$ of order 2^k with k maximal (i.e., the 2-valuation of the order of x is maximal among those of all elements of G). Then x generates a direct summand $\langle x\rangle$ of G, say $G = \langle x\rangle \oplus B$. Let $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ be the (degenerate) linking pairing defined by

$$\lambda(mx, ny) = \frac{mn}{2^k} \bmod 1, \quad \text{for } m, n \in \mathbb{Z} \ \text{ and } \lambda(x, B) = 0.$$

Since λ is as the orthogonal sum of a cyclic linking pairing and a trivial linking pairing, the decomposition $G = \langle x\rangle \oplus B$ is an orthogonal decomposition. Suppose by contradiction that there does exist a section $s : \mathrm{Link}(G) \to \mathrm{Quad}(G)$ splitting the first row (4.2). Then there exists $h \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ such that

$$s(\lambda) = q + h$$

where

$$q(m\ x + b) = \frac{m^2}{2^{k+1}} \bmod 1, \quad \text{for any } m \in \mathbb{Z},\ b \in B.$$

But

$$0 = s(0) = s(2^k\ \lambda) = 2^k\ s(\lambda) = 2^k q + 2^k h.$$

Hence $2^k h = -2^k q \neq 0$ and $2^{k+1} h = -2^{k+1} q = 0$. It follows that h has order 2^{k+1} exactly. This contradicts that the 2-valuation of the order of x is maximal. $\square$

5. Decomposition of finite quadratic functions

The decomposition results for finite quadratic functions on a finite abelian group parallel those for linking pairings.

LEMMA 5.1. *Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate quadratic function. For any $x \in G$ and $n \in \mathbb{Z}$,*

$$q(n\ x) = n\ q(x) + \frac{n(n-1)}{2}\ b_q(x,x). \tag{5.1}$$

If x has odd (resp. even) order n in G, then the order of $q(x)$ divides n (resp. divides $2n$) in $\mathbb{Q}/\mathbb{Z}$.

CHAPTER 2

Reciprocity

1. Witt groups

As the reciprocity formula is best understood as an identity in an appropriate Witt group, we include in this section some material about Witt groups. Throughout this section, we deal with nondegenerate ε-symmetric bilinear pairings.

1.1. The Lagrangian category. The notion of Lagrangian was introduced in Chap. 1, §1.

LEMMA 1.1. *Let A, B, C be three nonsingular ε-symmetric bilinear pairings on finitely generated abelian groups such that the map $V \mapsto V^{\perp}$ is involutive on subgroups. Let Λ be a Lagrangian in $-A \oplus B$ and let Λ' be a Lagrangian in $-B \oplus C$. The subset*
(1.1)
$\Lambda' \circ \Lambda = \{(a, c) \in -A \oplus C \mid \textit{there is } b \in B \text{ such that } (a, b) \in \Lambda \text{ and } (b, c) \in \Lambda'\}$

is a Lagrangian in $-A \oplus C$.

DEFINITION 1.1. Given any pair of ε-symmetric bilinear pairings A, the *diagonal Lagrangian* is defined as $\mathrm{Diag}(A) = \{(a, a) \mid a \in A\} \subseteq -A \oplus A$.

PROOF. Denote by a dot a ε-symmetric bilinear pairing. Let (a, c) and (a', c') be two elements in $\Lambda' \circ \Lambda$. There exist $b, b' \in B$ such that $(a, b) \in \Lambda$ and $(b, c) \in \Lambda$, such that $(a', b') \in \Lambda'$ and $(b', c') \in \Lambda'$. Then

$$\begin{aligned}(a, c) \cdot (a', c') = -a \cdot a' + c \cdot c' &= -a \cdot a' + b \cdot b' - b \cdot b' + c \cdot c' \\ &= (a, b) \cdot (a', b') + (b, c) \cdot (b', c') \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Hence $\Lambda' \circ \Lambda \subseteq (\Lambda' \circ \Lambda)^{\perp}$. Let us prove the converse. We consider the orthogonal sum $-A \oplus B \oplus -B \oplus C$. In this group lies the isotropic subgroup $H = 0 \oplus \mathrm{Diag}(B) \oplus 0$ with orthogonal $H^{\perp} = A \oplus \mathrm{Diag}(B) \oplus C$. The canonical projection $-A \oplus B \oplus B \oplus C \to (-A \oplus B \oplus B \oplus C)/H$ restricts to a projection $p : H^{\perp} \to H^{\perp}/H$ which preserves orthogonality. Consider the subgroup

$$G = (\Lambda \oplus \Lambda' \ + \ H) \cap H^{\perp} \subseteq H^{\perp}.$$

We have $p(G) = \Lambda' \circ \Lambda$. Since p preserves orthogonality, $p(G^\perp) = (\Lambda' \circ \Lambda)^\perp$. We compute

$$\begin{aligned} G^\perp &= ((\Lambda \oplus \Lambda' + H) \cap H^\perp)^\perp \\ &= (\Lambda \oplus \Lambda')^\perp \cap H^\perp) + H^{\perp\perp} \\ &= \Lambda \oplus \Lambda' \cap H^\perp + H \\ &= \Lambda \oplus \Lambda' \cap H^\perp \\ &= G. \end{aligned}$$

□

EXAMPLE 1.1 (Lagrangian associated to an isomorphism). Let $\varphi : A \to B$ be an isomorphism of nondegenerate ε-symmetric bilinear pairings. If L is a Lagrangian of A then the graph

$$\mathrm{Graph}(\varphi) = \{(a, \varphi(a)) \mid a \in A\}$$

is a Lagrangian in $-A \oplus B$. The case when $\varphi = \mathrm{id}_A$ gives the diagonal Lagrangian.

The composition of Lagrangians is associative. The diagonal Lagrangian plays the rôle of the identity.

DEFINITION 1.2. The *category of Lagrangians* $\mathrm{Lag}^\varepsilon(U)$ over U is defined as follows. An object is a nondegenerate ε-symmetric bilinear pairing on a finitely generated abelian group A with values in U. A morphism between two objects A and B is a triple (Λ, A, B) where Λ is a Lagrangian in $-A \oplus B$. The composition of two morphisms (Λ, A, B) and (Λ', B, C) is the morphism $(\Lambda' \circ \Lambda, A, C)$ defined by Eq. (1.1).

For more details, see for instance [**30**], [**61**, IV, §3]. We now fix such a category of Lagrangians.

DEFINITION 1.3. A Lagrangian Λ in $-A \oplus B$ is *decomposable* if $\Lambda = (\Lambda \cap A) \oplus (\Lambda \cap B)$.

A general Lagrangian may not be decomposable: for instance, the diagonal Lagrangian $\mathrm{Diag}(A)$ is not decomposable. The orthogonal sum of two Lagrangians is always decomposable.

DEFINITION 1.4. Two Lagrangians L and L' are *transverse* in A if $L + L' = A$.

REMARK 1.1. If L and L' are two transverse Lagrangians in A then $L \cap L' = 0$.

PROOF. Recall that the underlying ε-symmetric bilinear pairing is nondegenerate. We have

$$0 = A^\perp = (L + L')^\perp = L^\perp \cap L'^\perp = L \cap L'.$$

□

LEMMA 1.2. *Let A, B be two objects in* $\mathrm{Lag}^{\varepsilon}(U)$. *Let L, L' be a Lagrangian in $-A \oplus B$. If L is transverse to L' and if L' is decomposable, then L is decomposable.*

PROOF. The inclusion $(L \cap A) + (L \cap B) \subseteq \Lambda$ always holds. To see the other inclusion, let $z \in L$. Since $L + L' = -A \oplus B$, there exists $w \in L'$ and $(a, b) \in A \times B$ such that $z + w = a + b$. Since L' is decomposable, there exists a decomposition $w = w' + w''$ with $w' \in A, w'' \in B$. Thus $z = (a - w') + (b - w'')$ is a decomposition for z. □

The two cases we have in mind are $U = \mathbb{Z}$ (the category of Lagrangians in ε-lattices) and $U = \mathbb{Q}/\mathbb{Z}$ (the category of Lagrangians in ε-linking pairings).

DEFINITION 1.5. A Lagrangian category is *involutive* if for any object A and any subgroup $V \subseteq A$, $V^{\perp\perp} = V$.

If a Lagrangian category $\mathrm{Lag}^{\varepsilon}(U)$ is involutive then for any ε-symmetric bilinear pairing S and for any subgroups V, W of S,

$$(V \cap W)^{\perp} = V^{\perp} + W^{\perp}.$$

(This identity is the direct result of taking the orthogonal of the identity (1.1).)

LEMMA 1.3. *The Lagrangian categories* $\mathrm{Lag}^{\varepsilon}(\mathbb{Z})$ *and* $\mathrm{Lag}^{\varepsilon}(\mathbb{Q}/\mathbb{Z})$ *are involutive.*

In the rest of this paragraph, we now restrict ourselves to involutive Lagrangians categories.

LEMMA 1.4. *Two Lagrangians L and L' are transverse if and only if $L \cap L' = 0$.*

PROOF. We have $L + L' = L^{\perp} + L'^{\perp} = (L \cap L')^{\perp}$. We use the involutive property to conclude. □

LEMMA 1.5. *Given a Lagrangian L in A, there exists a Lagrangian L' transverse to L.*

PROOF. Consider the set of all isotropic subgroups that intersect trivially with L. This set is not empty since it contains the trivial subgroup. Choose a maximal element L' with respect to the inclusion. First, $A = (L \cap L')^{\perp} = L^{\perp} + L'^{\perp} = L + L'^{\perp}$. Second, we claim that L' is Lagrangian. Otherwise there is $x \in L'^{\perp}$ such that $x \notin L'$. Then $x + L'$ is isotropic and intersects trivially with L, contradicting maximality. □

Given a pair of transverse Lagrangians L, L' in A, the ε-symmetric bilinear pairing on A induces a bilinear pairing $L \times A/L' \to U$.

PROPOSITION 1.1. *Let A be a nonsingular ε-symmetric bilinear pairing. The group $\mathrm{O}(A)$ of automorphisms of A acts transitively on pairs of transverse Lagrangians.*

The following observation is a preparation for a suitable refinement of $\mathrm{Lag}^{\varepsilon}(U)$.

LEMMA 1.6. *For an object $A \in \mathrm{Lag}^{\varepsilon}(U)$, we denote by L_A a Lagrangian in A. Let $(A, L_A), (B, L_B), (C, L_C)$ be three pairs where A, B, C are objects in $\mathrm{Lag}^{\varepsilon}(U)$. Let Λ be a Lagrangian in $-A \oplus B$ and let Λ' be a Lagrangian in $-B \oplus C$. If Λ is transverse to $L_A \oplus L_B$ and if Λ' is transverse to $L_B \oplus L_C$, then $\Lambda' \circ \Lambda$ is transverse to $L_A \oplus L_C$.*

In short, the composition of transverse Lagrangians is transverse.

PROOF. We have to prove that $\Lambda_{N\circ M} \cap L_C = \Lambda_{N\circ M} \cap L_A = 0$. We prove that $\Lambda_{N\circ M} \circ L_C$, the other case is similar. Let $x \in \Lambda_{N\circ M} \cap L_C$. Write $x = (a, c)$ as an element in $-A\oplus C$. Since $(a, c) \in L_C \subseteq C$, $a = 0$ and $c \in L_C$. Since $x = (0, c) \in \Lambda_{N\circ M}$, there exists $b \in B$ such that $(b, c) \in \Lambda_N$. Since Λ_N is transverse to the decomposable Lagrangian $L_B \oplus L_C$, by Lemma 1.2, Λ_N itself is decomposable. Hence $c \in \Lambda_N \cap C$. So finally $c \in \Lambda_N \cap L_C = 0$. $\square$

DEFINITION 1.6. The *category of transverse Lagrangians* $\mathrm{Lag}^{\varepsilon}_{\mathrm{trans}}(U)$ over U is defined as follows. An object in $\mathrm{Lag}^{\varepsilon}_{\mathrm{trans}}(U)$ is a pair (A, L_A) where A is an object in $\mathrm{Lag}^{\varepsilon}(U)$ and L_A is a Lagrangian in A. A morphism between two objects A and B is a Lagrangian Λ in $-A \oplus B$ such that Λ is transverse to $L_A \oplus L_B$.

There is a faithful forgetful functor $\mathrm{Lag}^{\varepsilon}_{\mathrm{trans}}(U) \to \mathrm{Lag}^{\varepsilon}(U)$ that "forgets" the extra Lagrangians and the transversality property.

1.2. Witt groups of ε-symmetric pairings.

2. The discriminant construction

2.1. Lattices. A *lattice* is a finitely generated free abelian group.

DEFINITION 2.1. An *ε-symmetric bilinear lattice* is an ε- symmetric bilinear form $f : V \times V \to \mathbb{Z}$ on a lattice V.

REMARK 2.1. We will use the short term *ε-lattice*, or even *lattice*, if the ε-symmetric bilinear pairing is implicit.

A lattice V generates over $\mathbb{Q}$ a vector space $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$. An ε-lattice (V, f) extends to an ε-symmetric bilinear form $f_{\mathbb{Q}} : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}$. The form $f_{\mathbb{Q}}$ is nonsingular if and only if f is nondegenerate. An ε-lattice (V, f) is said *unimodular* if f is nonsingular. The dual lattice is defined as

$$V^{\sharp} = \{x \in V_{\mathbb{Q}} \mid f_{\mathbb{Q}}(x, V) \subseteq \mathbb{Z}\}.$$

A subgroup of a lattice V is finitely generated and free abelian and is called a *sublattice* of V. More generally, given a sublattice $S \subseteq V$, the dual lattice is defined as

$$S^{\sharp} = \{x \in V_{\mathbb{Q}} \mid f_{\mathbb{Q}}(x, S) \subseteq \mathbb{Z}\}.$$

The map $\widehat{f_{\mathbb{Q}}} : x \mapsto f_{\mathbb{Q}}(x, -)$ restricts to a map between $S^{\sharp}$ and $S^* = \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Z})$. This map is an isomorphism if f is nondegenerate. It follows that $S^{\sharp\sharp} = S$ for any sublattice S of V if and only if f is nondegenerate. One observes that if $S, T \subseteq V$ are sublattices, then

$$S \subseteq T \;\Rightarrow\; T^\sharp \subseteq S^\sharp, \quad (S+T)^\sharp = S^\sharp \cap T^\sharp. \tag{2.1}$$

A sublattice $S \subseteq V$ is *primitive* if the quotient group V/S is a lattice. Let (V, f) be a bilinear lattice.

EXAMPLE 2.1. The *annihilator* Ker $\widehat{f} \subseteq V$ is a primitive sublattice of V. This is equivalent to

LEMMA 2.1. *The quotient* $\bar{V} = V/\mathrm{Ker}\ \widehat{f}$ *is a lattice.*

PROOF. Clearly $\bar{V}$ is finitely generated. Let $[x] = x + \mathrm{Ker}\ \widehat{f} \in \bar{V}$ such that $n\,[x] = 0$. Then $n\,x \in \mathrm{Ker}\ \widehat{f}$. Thus

$$0 = f(n\,x, V) = n\,f(x, V).$$

Since $f(x, V) \subseteq \mathbb{Z}$ and $\mathbb{Z}$ has no torsion, it follows that $f(x, V) = 0$. Hence $x \in \mathrm{Ker}\ \widehat{f}$ and $[x] = 0$. □

For a given sublattice $S \subseteq V$, there is smallest primitive sublattice $\widetilde{S} \subseteq V$ containing S. This lattice is called the *primitive hull* of S. The primitive hull of S has the same rank as S. The following observation is useful.

REMARK 2.2. An isomorphism $\varphi : S \to S'$ between sublattices of V and V' does not necessarily extend to an isomorphism $\widetilde{\varphi} : \widetilde{S} \to \widetilde{S}'$ between their respective primitive hulls. For instance, take $V = V' = \mathbb{Z} \oplus \mathbb{Z}$, $S = \mathbb{Z} \oplus 0$ and $S' = 2\mathbb{Z} \oplus 0$. Clearly the map $x \mapsto 2x$ defines an isomorphism between S and S'. However, this map does not extend to an isomorphism between $\widetilde{S} = S = \mathbb{Z}$ and $\widetilde{S}' = \mathbb{Z}$.

LEMMA 2.2. *An isomorphism* $\varphi : S \to S'$ *between primitive sublattices of* V *extends to an automorphism of* V*.*

PROOF. Since V/S is free, the short exact sequence

$$0 \to S \to V \to V/S \to 0$$

splits. Choose a section $s : V/S \to V$ so that the map

$$\psi : S \oplus V/S \to V,\ (x, y) \mapsto (x, s(y))$$

is an isomorphism. Similarly there is a section $s' : V/S' \to V$ such that $\psi' : (x, y) \mapsto (x, s'(y))$ is an isomorphism from $S' \oplus V/S'$ onto V.

Since S and S' are isomorphic primitive sublattices, there is an isomorphism $g : V/S \simeq V/S'$ of lattices.

Then $\psi' \circ (f \oplus g) \circ \psi^{-1}$ is an automorphism of V extending φ. □

COROLLARY 2.1. *An isomorphism* $\varphi : S \to S'$ *between sublattices of* V *extends to an automorphism of* V *if and only if it extends to an isomorphism* $\widetilde{S} \to \widetilde{S}'$ *between their primitive hull.*

PROOF. In one direction, use Lemma 2.2. For the converse, let $\widetilde{\varphi} : V \to V$ be the automorphism extending φ. We have to show that $\widetilde{\varphi}(\widetilde{S}) = \widetilde{S}'$. An element y lies in $\widetilde{S}$ if and only if there is some $n \in \mathbb{Z}$ such that $ny \in S$. Let $y \in \widetilde{S}$ such that $x = ny \in S$. We have $\varphi(x) = \widetilde{\varphi}(ny) = n\varphi(y) \in S'$. Thus $\varphi(y) \in \widetilde{S}'$. It follows that $\widetilde{\varphi}(\widetilde{S}) \subseteq \widetilde{S}'$. The reverse inclusion is proved similarly using φ^{-1}. $\square$

Let G be a finitely generated abelian group. The quotient group $FG = G/\mathrm{Tors}\, G$ is a lattice. Let $S \subset FG$ be a lattice. A partial section $s : S \to G$ (that is, a map $s : S \to FG$ such that $p \circ s|_S = \mathrm{id}_S$) does not necessarily extend to a full section $FG \to G$.

LEMMA 2.3. *If S is primitive, then any partial section $s : S \to G$ extends to a section $s : S \to G$.*

PROOF. Since V/S is free, the short exact sequence $0 \to S \to V \to V/S \to 0$ gives rise to an exact sequence

$$0 \longrightarrow \mathrm{Hom}(V/S, G) \longrightarrow \mathrm{Hom}(V, G) \longrightarrow \mathrm{Hom}(S, G) \longrightarrow 0.$$

$\square$

2.2. Discriminant ε-linking pairings. An ε-lattice (V, f) is *unimodular* if f is nonsingular. It follows from the previous paragraph that (V, f) is unimodular if and only if $V^\sharp = V$. We are interested in studying the failure of f to be unimodular. A natural invariant is provided by the following

DEFINITION 2.2. To an ε-lattice (V, f), one associates an ε-linking pairing, called the *discriminant pairing*, $\lambda_f : G_f \times G_f \to \mathbb{Q}/\mathbb{Z}$ by the formula:

$$G_f = V^\sharp/V, \quad \lambda_f([x], [y]) = f_{\mathbb{Q}}(x, y) \bmod 1. \tag{2.2}$$

The discriminant pairing (G_f, λ_f) is symmetric (resp. antisymmetric, resp. symplectic) if and only if (V, f) is symmetric (resp. antisymmetric, resp. symplectic). The discriminant construction arises from a particular class of free resolutions of length 1.

LEMMA 2.4. *(G_f, λ_f) is nonsingular if and only if (V, f) is nondegenerate.*

A basic result asserts that almost any nondegenerate ε-linking pairing can be produced by this construction [**64**, Theorem (6)]:

THEOREM 2.1. *The assignment $(V, f) \mapsto (G_f, \lambda_f)$ is surjective onto the monoid of nondegenerate symmetric (resp. symplectic) linking pairings on finite abelian groups.*

As an example, any unimodular lattice ($V^\sharp = V$) yields the trivial linking pairing. Clearly the discriminant construction preserves (orthogonal) sum. It follows from these two observations that the discriminant pairing is unaffected by adding orthogonal summands of unimodular lattices. A converse is known since the work of Puppe. To state it in our setting, it is convenient to introduce some definitions about maps between lattices.

A *bilinear lattice map* between two bilinear lattices (V, f) and (W, g) is a map $\alpha : V \to W$ such that $g(\alpha(x), \alpha(y)) = f(x, y)$ for all $x, y \in V$. This is also denoted $\alpha^* g = f$ in the sequel. If α is injective, then we say that α is an *embedding of bilinear lattices*. If α is bijective, then α is an *isomorphism of bilinear lattices*. Two bilinear lattices (V, f) and (W, g) are *stably equivalent* if there exist unimodular bilinear lattices (U, h) and (U', h') such that $(V, f) \oplus (U, h)$ and $(W, g) \oplus (U', h')$ are isomorphic bilinear lattices. Any bilinear lattice map α extends in a unique fashion to a map $\alpha_{\mathbb{Q}} : V_{\mathbb{Q}} \to W_{\mathbb{Q}}$ and thus restricts to a map $V^\sharp \to W^\sharp$ and therefore induces a map $[\alpha] : G_f = V^\sharp/V \to W^\sharp/W = G_g$. It follows that a stable equivalence induces an isomorphism on the induced discriminant linking pairings. The converse is also true:

THEOREM 2.2. *Two nondegenerate linking pairings are isomorphic if and only if they lift to stably isomorphic bilinear lattices.*

For a proof, see e.g., [**14**].

Our goal consists in recovering the product of two linking pairings from the discriminant of their lattices.

Let (V, f) and (W, g) be nondegenerate bilinear lattices. Set $Z = V \otimes W$ and define a (symmetric nondegenerate) bilinear pairing $f \otimes g : Z \times Z \to \mathbb{Z}$ by

$$(f \otimes g)(x \otimes y, x' \otimes y') = f(x, x')\, g(y, y') \quad \text{for } x, x' \in V,\ y, y' \in W.$$

LEMMA 2.5. *There is a natural isomorphism $V^\sharp \otimes W^\sharp \to Z^\sharp$.*

There are also natural inclusion maps $V^\sharp \otimes W \to (V \otimes W)^\sharp$ and $V \otimes W^\sharp \to (V \otimes W)^\sharp$ (where the dual lattice of the target space refers to the bilinear pairing $f \otimes g$) which we shall use freely without further notice. In particular, we verify directly the fact that

$$(f_{\mathbb{Q}} \otimes g_{\mathbb{Q}})(V^\sharp \otimes W, V \otimes W^\sharp) \subseteq \mathbb{Z}. \tag{2.3}$$

More precisely:

LEMMA 2.6.

$$(V^\sharp \otimes W)^\sharp = V \otimes W^\sharp \quad \text{and} \quad (V \otimes W^\sharp)^\sharp = V^\sharp \otimes W. \tag{2.4}$$

The inclusion $V \otimes W^\sharp \subseteq (V^\sharp \otimes W)^\sharp$ is just the equality (2.3). The lemma asserts that this is an equality.

PROOF. It suffices to prove the first equality since one deduces the second one by using the fact that $Z^{\sharp\sharp} = Z$ (since $f \otimes g$ is nondegenerate). The desired equality will result from the following commutative diagram:

$$\begin{tikzcd}
V \otimes W^\sharp \arrow[r, hook] \arrow[d, "\widehat{f_{\mathbb{Q}}} \otimes \widehat{g_{\mathbb{Q}}}"'] & (V^\sharp \otimes W)^\sharp \arrow[d, "\widehat{f_{\mathbb{Q}} \otimes g_{\mathbb{Q}}}"] \\
\operatorname{Hom}_{\mathbb{Z}}(V^\sharp, \mathbb{Z}) \otimes \operatorname{Hom}_{\mathbb{Z}}(W, \mathbb{Z}) \arrow[r, equal] & \operatorname{Hom}_{\mathbb{Z}}(V^\sharp \otimes W, \mathbb{Z}).
\end{tikzcd}$$

The top horizontal arrow is the natural inclusion. The vertical arrows are the tensor product of adjoint maps and the adjoint map of the tensor product of pairings respectively (and they can be identified once $V_{\mathbb{Q}} \otimes W_{\mathbb{Q}}$ is identified to $(V \otimes W)_{\mathbb{Q}}$). We claim that the vertical arrows are bijective maps. Since the map adjoint to $f_{\mathbb{Q}} \otimes g_{\mathbb{Q}}$ is bijective, it is sufficient to check that

$$(\widehat{f}_{\mathbb{Q}} \otimes \widehat{g}_{\mathbb{Q}})(V \otimes W^{\sharp}) = \mathrm{Hom}_{\mathbb{Z}}(V^{\sharp}, \mathbb{Z}) \otimes \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z})$$

and

$$\widehat{f_{\mathbb{Q}} \otimes g_{\mathbb{Q}}}((V^{\sharp} \otimes W)^{\sharp}) = \mathrm{Hom}_{\mathbb{Z}}(V^{\sharp} \otimes W, \mathbb{Z}).$$

Both identities follow from the nondegeneracy of $f_{\mathbb{Q}}$ and $g_{\mathbb{Q}}$. $\square$

We now consider the linking pairing

$$\lambda_{f\otimes g} : G_{f\otimes g} \times G_{f\otimes g} \to \mathbb{Q}/\mathbb{Z}.$$

The natural inclusion map $V^{\sharp} \otimes W \to (V \otimes W)^{\sharp}$ induces a homomorphism $j_f : G_f \otimes W \to G_{f\otimes g}$ by

$$j_f\,(x\,(\mathrm{mod}\ V) \otimes y) = x \otimes y\ (\mathrm{mod}\ Z)$$

where $x \in V^{\sharp}, y \in W$. Similarly, define a homomorphism $j_g : V \otimes G_g \to G_{f\otimes g}$ by

$$j_g\,(x \otimes y\,(\mathrm{mod}\ W)) = x \otimes y\ (\mathrm{mod}\ Z)$$

where $x \in V, y \in W^{\sharp}$.

LEMMA 2.7. *We have*

$$\lambda_{f\otimes g} \circ j_f^{\otimes 2} = \lambda_f \otimes g$$

and

$$\lambda_{f\otimes g} \circ j_g^{\otimes 2} = f \otimes \lambda_g.$$

The following observation is a consequence of f, g being nondegenerate.

LEMMA 2.8. *The maps j_f and j_g are injective.*

Set $A = j_f(G_f \otimes W) \subseteq G_{f\otimes g}$ and $B = j_g(V \otimes G_g) \subseteq G_{f\otimes g}$.

LEMMA 2.9. *The subgroups A and B are mutually orthogonal in $G_{f\otimes g}$: $A^{\perp} = B$.*

PROOF. Consequence of definitions and (2.4). $\square$

Let $H = A \cap A^{\perp}$. We record the following consequence:

COROLLARY 2.2. $\lambda_{f\otimes g}|_{H\times H} = 0$.

Assume that f and g are both nondegenerate. We now describe H in more details.

LEMMA 2.10. *There are exact sequences*

$$0 \longrightarrow V \otimes G_g \xrightarrow{j_g} G_{f\otimes g} \longrightarrow G_f \otimes W^{\sharp} \longrightarrow 0$$

and

$$0 \longrightarrow G_f \otimes W \xrightarrow{j_f} G_{f\otimes g} \longrightarrow V^{\sharp} \otimes G_g \longrightarrow 0.$$

PROOF. Let us identify Coker j_f:

$$\operatorname{Coker} j_f = \frac{G_{f\otimes g}}{j_f(V\otimes G_g)} = \frac{\frac{Z^\sharp}{Z}}{\frac{V^\sharp\otimes W}{Z}} \simeq \frac{Z^\sharp}{V^\sharp\otimes W} \simeq \frac{V^\sharp\otimes W^\sharp}{V^\sharp\otimes W} \simeq V^\sharp\otimes\frac{W^\sharp}{W} = V^\sharp\otimes G_g.$$

The identification of Coker j_g is similar. □

LEMMA 2.11. *There is a natural isomorphism $H^* \simeq G_f\otimes G_g$ and a short exact sequence*

$$0 \longrightarrow H^\perp \xrightarrow{j} G_{f\otimes g} \longrightarrow G_f\otimes G_g \longrightarrow 0.$$

PROOF. On the one hand, there is a short exact sequence

$$0 \to H^\perp \to G_{f\otimes g} \to H^* \to 0.$$

On the other hand, $H^\perp = (A\cap A^\perp)^\perp = A + A^\perp$. Thus

$$H^* = G_{f\otimes g}/(A+A^\perp).$$

There remains to see that the latter group is isomorphic to $G_f\otimes G_g$. There is a natural epimorphism

$$G_{f\otimes g} \to G_f\otimes G_g.$$

Indeed, this map can be defined in two ways

$$p_2\circ p_1 = q_2\circ q_1$$

as the following commutative diagram with exact rows and columns indicates:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & V\otimes G_g \arrow[r, equal] \arrow[d] & V\otimes G_g \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_f\otimes W \arrow[r] \arrow[d, equal] & G_{f\otimes g} \arrow[r, "p_1"] \arrow[d, "q_1"] & V^\sharp\otimes G_g \arrow[r] \arrow[d, "p_2"] & 0 \\
0 \arrow[r] & G_f\otimes W \arrow[r] & G_f\otimes W^\sharp \arrow[r, "q_2"] \arrow[d] & G_f\otimes G_g \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

It follows that the kernel of the epimorphism is

$$\operatorname{Ker}(p_2\circ p_1) = p_1^{-1}(j_g(V\otimes G_g)) = j_f(G_f\otimes W)+j_g(V\otimes G_g) = q_1^{-1}(j_f(G_f\otimes W)) = \operatorname{Ker}(q_2\circ q_1).$$

Thus the quotient map

$$\psi : G_{f\otimes g}/(A+A^\perp) \xrightarrow{\simeq} G_f\otimes G_g \tag{2.5}$$

is an isomorphism. □

Remark. The isomorphism between $G_f \otimes G_g$ and H^* is the composition

$$G_f \otimes G_g \xrightarrow{\psi^{-1}} G_{f\otimes g}/(A + A^\perp) \xrightarrow{l} H^*,$$

where the isomorphism on the right is

$$l : x \bmod (V \otimes W) \mapsto \lambda_{f\otimes g}(x, -)|_H.$$

Here is an alternative argument to show that $G_f \otimes G_g$ and $G_{f\otimes g}/(A + A^\perp)$ are isomorphic. Define a natural map

$$G_f \otimes G_g \to G_{f\otimes g}/(A + A^\perp)$$

by

$$(x \bmod V) \otimes (y \bmod W) \mapsto [(x \otimes y) \bmod (V \otimes W)]$$

where $[-]$ denotes the element in $G_{f\otimes g}$ considered modulo $A + A^\perp$. It follows also from $A = \mathrm{Im}\ j_f$ and $A^\perp = \mathrm{Im}\ j_g$ that this map is injective. (Suppose that $u \in G_f \otimes G_g$ is sent to $0 \in G_{f\otimes g}/(A + A^\perp)$. Then the image of u is represented by a sum of elements in $A + A^\perp$. Since $A = \mathrm{Im}\ j_f$ and $A^\perp = \mathrm{Im}\ j_g$, all these elements are of the form $(x \otimes y) \bmod (V \otimes W)$ where either $x \in V$ or $y \in W$. Therefore $u = 0 \in G_f \otimes G_g$.) Surjectivity also follows from the definitions. It is easily seen to be ψ^{-1}.

LEMMA 2.12. *The map $j_f : G_f \otimes W \to G_{f\otimes g}$ restricts to an isomorphism*

$$j_f|_{\mathrm{Ker}(\widehat{\lambda_f}\otimes\widehat{g})} : \mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g}) \xrightarrow{\cong} H.$$

Similarly, the map $j_g : W \otimes G_g \to G_{f\otimes g}$ restricts to an isomorphism

$$j_g|_{\mathrm{Ker}(\widehat{f}\otimes\widehat{\lambda_g})} : \mathrm{Ker}(\widehat{f} \otimes \widehat{\lambda_g}) \xrightarrow{\cong} H.$$

PROOF. We prove the first isomorphism – the second one is similar. Since j_f is injective, it suffices to prove that $j_f(\mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g})) = H$.

First $j_f(\mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g})) \subseteq j_f(G_f \otimes W) = A$. Next, let $u \in \mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g})$ and $j_f(v) \in A$. We have

$$\lambda_{f\otimes g}(j_f(u), j_f(v)) = \lambda_{f\otimes g} \circ j_f(u, v) = (\lambda_f \otimes g)(u, v) = 0.$$

Hence $j_f(\mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g}))$ and A are orthogonal, that is,

$$j_f(\mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g})) \subseteq A^\perp.$$

Therefore,

$$j_f(\mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g})) \subseteq A \cap A^\perp = H.$$

Conversely, let $j_f(x) = j_g(y) \in A \cap A^\perp = H$. Let $z \in G_f \otimes W$. Then

$$(\lambda_f \otimes g)(x, z) = \lambda_{f\otimes g}(j_f(x), j_f(z)) = \lambda_{(f\otimes g)}(j_g(y), j_f(z)) = 0.$$

(The first equality results from Lemma 2.7 and the third one from Lemma 2.9.) This proves that $x \in \mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g})$. Hence $H \subseteq j_f(\mathrm{Ker}(\widehat{\lambda_f} \otimes \widehat{g}))$. This achieves the proof. $\square$

Remark. The following diagram with exact rows and columns is commutative:

$$
\begin{tikzcd}
\mathrm{Ker}(\widehat{f}\otimes \mathrm{id}_W) \arrow[r, hook] \arrow[d, hook, dashed] & \mathrm{Ker}(\widehat{\lambda}_f \otimes \mathrm{id}_{W^*}) \arrow[r, two heads] \arrow[d, hook, dashed] & \mathrm{Ker}(\widehat{f}\otimes \mathrm{id}_{G_g}) \arrow[d, hook, dashed] \\
V\otimes W \arrow[r, hook] \arrow[d] & V\otimes W^* \arrow[r, two heads] \arrow[d] & V\otimes G_g \arrow[d] \\
V^*\otimes W \arrow[r, hook] \arrow[d, two heads, dashed] & V^*\otimes W^* \arrow[r, two heads] \arrow[d, two heads, dashed] & V^*\otimes G_g \arrow[d, two heads, dashed] \\
\mathrm{Coker}(\widehat{f}\otimes \mathrm{id}_W) \arrow[r, hook] & \mathrm{Coker}(\widehat{\lambda}_f \otimes \mathrm{id}_{W^*}) \arrow[r, two heads] & \mathrm{Coker}(\widehat{f}\otimes \mathrm{id}_{G_g})
\end{tikzcd}
$$

According to the "snake lemma", there is a Bockstein map

$$\beta : \mathrm{Ker}(\widehat{f}\otimes \mathrm{id}_{G_g}) \to \mathrm{Coker}(\widehat{f}\otimes \mathrm{id}_W)$$

connecting the exact sequence made of the maps of the first and the last row. Since $\mathrm{Ker}(\widehat{\lambda}_f \otimes \mathrm{id}_{W^*}) = 0$, the Bockstein map β is injective. Hence there is an exact sequence $0 \to \mathrm{Ker}(\widehat{f}\otimes \mathrm{id}_{G_g}) \to \mathrm{Coker}(\widehat{f}\otimes \mathrm{id}_W) \to \mathrm{Coker}(\widehat{\lambda}_f\otimes \mathrm{id}_{W^*})$. Since $G\otimes W = \mathrm{Coker}(\widehat{f}\otimes \mathrm{id}_W)$, β induces an isomorphism

$$\bar{\beta} : \mathrm{Ker}(\widehat{f}\otimes \widehat{\lambda}_g) = \mathrm{Ker}(\widehat{f}\otimes \mathrm{id}_{G_g}) \to \mathrm{Ker}(\mathrm{id}_{G_f}\otimes \widehat{g}) = \mathrm{Ker}(\widehat{\lambda}_f \otimes \widehat{g}).$$

It follows from definitions that

$$\bar{\beta} = j_f^{-1}|_H \circ j_g|_{\mathrm{Ker}(\widehat{f}\otimes \mathrm{id}_{G_g})}.$$

2.3. Tensor product of linkings. Denote by l the bilinear pairing $G/H^\perp \times H \to \mathbb{Q}/\mathbb{Z}$ defined by $l([x], y) = \lambda_{f\otimes g}(x, y)$ for all $x \in G, y \in H$. The goal of this section is to relate this bilinear pairing l to the linking pairing $\lambda_f \otimes \lambda_g$ defined in §**??**.

It will be convenient in this paragraph to identify $G/H^\perp$ and $\frac{V^\sharp\otimes W^\sharp}{V^\sharp\otimes W + V\otimes W^\sharp}$. Our first goal is to define an isomorphism between $G_f \otimes G_g$ and H.

Let $K = \frac{V^\sharp\otimes W_{\mathbb{Q}}}{V^\sharp\otimes W + V\otimes W^\sharp}$. We first define a map

$$V^\sharp \times W \to K$$

by the assignment

$$m : (\xi, w) \mapsto \left[\xi \otimes \frac{w}{n}\right],$$

where n is the smallest nonnegative integer such that $n\,\xi \in \mathbb{Z}$.

LEMMA 2.13. *This map induces a homomorphism $m : G_f \otimes W \to K$.*

PROOF. We show that m is $\mathbb{Z}$-bilinear. Note that any element $\xi \in V^\sharp$ can be written as $\xi = \frac{\xi'}{n}$ where $\xi' \in V$ and $n \in \mathbb{Z}$. We further require ξ' to be indivisible: $\xi' \notin k\ V$ for all $k > 1$. This condition is equivalent to n being the smallest nonnegative integer such that $n\ \xi \in V$. We have

$$m\left(\frac{\xi'}{n}, w\right) = \left[\frac{\xi' \otimes w}{n}\right].$$

It follows that

$$m\left(\frac{\xi'}{n}, w + w'\right) = m\left(\frac{\xi'}{n}, w\right) + m\left(\frac{\xi'}{n}, w'\right).$$

We now verify linearity on the left: let $\zeta \in V^\sharp$ that we write as $\zeta = \frac{\zeta'}{p}$ where $\zeta \in V$ is indivisible. Write

$$n = k \cdot n',\ p = k \cdot p',\ \text{with } k = \gcd(n, p).$$

Then

$$\begin{aligned} m\left(\frac{\xi'}{n} + \frac{\zeta'}{p}, w\right) = m\left(\frac{p'\ \xi'}{n'p'k} + \frac{n'\ \zeta'}{n'p'k}, w\right) &= \left[\frac{(p'\xi' + n'\zeta') \otimes w}{n'p'k}\right] \\ &= \left[\frac{p'\ \xi' \otimes w}{n'p'k} + \frac{n'\ \zeta' \otimes w}{n'p'k}\right] \\ &= \left[\frac{\xi' \otimes w}{n}\right] + \left[\frac{\zeta' \otimes w}{p}\right] \\ &= m\left(\frac{\xi'}{n}, w\right) + m\left(\frac{\zeta'}{p}, w\right). \end{aligned}$$

(We used in the second equality the fact that n' and p' are coprime, so that $p'\ \xi' + n'\ \zeta'$ is again indivisible in V.) Therefore m induces a group homomorphism (still denoted m) $V^\sharp \otimes W \to K$.

It follows from the definition that $V \otimes W \subseteq \mathrm{Ker}(m)$. Hence m actually induces a homomorphism $G_f \otimes W \to K$. □

LEMMA 2.14. $\mathrm{Ker}(\widehat{\lambda}_f \otimes \widehat{g}) = \mathrm{Ker}(\mathrm{id}_{G_f} \otimes \widehat{g})$ *is generated by all elements* $x \otimes w \in G_f \otimes W$ *such that* $\widehat{g}(w) \in nW^*$ *with* $n\ x = 0$ *for some* $n \in \mathbb{Z}$.

PROOF. The subgroup identifies to $\mathrm{Tor}_1^{\mathbb{Z}}(G_f, G_g)$. In particular, it is independent of (G_f, λ_f) and (G_g, λ_g) and depends only on G_f and G_g. The result is clear if G_f is a finite cyclic group. In the general case, G_f is a sum of finite cyclic groups and we use the fact that $\mathrm{Tor}_1^{\mathbb{Z}}(A \oplus B, G_g) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(A, G_g) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(B, G_g)$. □

LEMMA 2.15. *The map* $m : G_f \otimes W \to K$ *restricts to a map* $m|_{\mathrm{Ker}(\mathrm{id}_{G_f} \otimes \widehat{g})}$ *whose image lies in* $G_{f\otimes g}/H^\perp = \frac{V^\sharp \otimes W^\sharp}{V^\sharp \otimes W + V \otimes W^\sharp}$.

PROOF. Let $[\xi] \otimes w$ be a generator of $\mathrm{Ker}(\mathrm{id}_{G_f} \otimes \widehat{g})$ as in Lemma 2.14: there is $n \in \mathbb{Z}$ such that $\widehat{g}(w) \in n\ W^*$ and $n\ \xi \in V$. Thus $\widehat{g}_{\mathbb{Q}}(\frac{w}{n}) = \frac{1}{n}\widehat{g}_{\mathbb{Q}}(w) \in W^*$, that is $\frac{w}{n} \in W^\sharp$. Hence $\xi \otimes \frac{w}{n} \in V^\sharp \otimes W^\sharp$ and

$$m([\xi] \otimes w) = \left[\xi \otimes \frac{w}{n}\right] \in \frac{V^\sharp \otimes W^\sharp}{V^\sharp \otimes W + V \otimes W^\sharp} = G_{f\otimes g}/H^\perp.$$

□

Set $\mu' = \mu'_f = m|_{\mathrm{Ker}(\mathrm{id}_{G_f}\otimes\widehat{g})}$.

LEMMA 2.16. *The map μ' is an isomorphism*

$$\mathrm{Ker}(\mathrm{id}_{G_f}\otimes\widehat{g}) \to G_{f\otimes g}/H^\perp = \frac{V^\sharp\otimes W^\sharp}{V^\sharp\otimes W + V\otimes W^\sharp}.$$

PROOF. The two groups are finite and isomorphic (Lemma 2.11 and Lemma 2.12). Hence it suffices to prove that μ' is onto. Choose orthogonal bases $e = (e_1,\dots,e_n)$ and $\varepsilon = (\varepsilon_1,\dots,\varepsilon_p)$ for $(V_\mathbb{Q}, f_\mathbb{Q})$ and $(W_\mathbb{Q}, g_\mathbb{Q})$ respectively, so that there exist $a_i, b_j \in \mathbb{Z}-\{0\}$ $(1\leqslant i\leqslant n,\ 1\leqslant j\leqslant p)$, such that

$$V^\sharp = \bigoplus_i \frac{1}{a_i}\mathbb{Z}\, e_i,\quad W^\sharp = \bigoplus_j \frac{1}{b_j}\mathbb{Z}\, \varepsilon_j.$$

We have

$$V^\sharp\otimes W + V\otimes W^\sharp = \bigoplus_{i,j}\left(\frac{1}{a_i}\mathbb{Z}+\frac{1}{b_j}\mathbb{Z}\right)(e_i\otimes\varepsilon_j) = \bigoplus_{i,j}\frac{1}{\mathrm{lcm}(a_i,b_j)}\mathbb{Z}(e_i\otimes\varepsilon_j).$$

Therefore

$$\frac{V^\sharp\otimes W^\sharp}{V^\sharp\otimes W + V\otimes W^\sharp} = \frac{\bigoplus_{i,j}\frac{1}{a_ib_j}\mathbb{Z}(e_i\otimes\varepsilon_j)}{\bigoplus_{i,j}\frac{1}{\mathrm{lcm}(a_i,b_j)}\mathbb{Z}(e_i\otimes\varepsilon_j)} = \bigoplus_{i,j}\frac{1}{\mathrm{gcd(a_i,b_j)}}\mathbb{Z}/\mathbb{Z}[e_i\otimes\varepsilon_j].$$

Now we verify the identity:

$$\frac{1}{\gcd(a_i,b_j)}[e_i\otimes\varepsilon_j] = \left[\frac{e_i}{\gcd(a_i,b_j)}\otimes\frac{\varepsilon_j}{\gcd(a_i,b_j)}\right] = \mu'\left(\left[\frac{e_i}{\gcd(a_i,b_j)}\right]\otimes\varepsilon_j\right).$$

□

We define an isomorphism $\mu_f : H\to G_{f\otimes g}/H^\perp$ as the composition

$$\nu_f = \mu'_f\circ j_f^{-1}|_H :\ H \xrightarrow{j_f^{-1}} \mathrm{Ker}(\mathrm{id}_{G_f}\otimes\widehat{g}) \xrightarrow{\mu'_f} G/H^\perp.$$

There is a similar isomorphism $\nu_g : H\to G_{f\otimes g}/H^\perp$ defined as the composition

$$\nu_g = \mu'_g\circ j_g^{-1}|_H :\ H \xrightarrow{j_g^{-1}} \mathrm{Ker}(\widehat{f}\otimes\mathrm{id}_{G_g}) \xrightarrow{\mu'_g} G/H^\perp.$$

It follows from definitions that

$$\nu_f = \nu_g.$$

Recall the isomorphism $\psi : G/H^\perp\to G_f\otimes G_g$ we defined in the previous paragraph. We are now ready to define an isomorphism $\mu : H\to G_f\otimes G_g$ as the composition

$$\mu = \psi\circ\nu.$$

THEOREM 2.3. *For all $x\in G_{f\otimes g}/H^\perp, y\in H$,*

$$l(x,y) = (\lambda_f\otimes\lambda_g)(\psi(x),\psi\circ\nu(y)) = (\lambda_f\otimes\lambda_g)(\psi(x),\mu(y)).$$

PROOF. Let $x = [\xi],\ x' = [\xi'] \in G_f = V^\sharp/V$ and $y = [\zeta],\ y' = [\zeta'] \in G_g = W^\sharp/W$. We have to show the equality $l(\psi^{-1}(x \otimes y), \mu^{-1}(x' \otimes y')) = (\lambda_f \otimes \lambda_g)(x \otimes y,\ x' \otimes y')$. We have $\psi^{-1}(x \otimes y) = [[\xi \otimes \zeta]] = [\xi \otimes \zeta] \bmod H^\perp$.

With no loss of generality, we may assume that x' generates a cyclic (direct) summand of G_f of order n. Hence we may assume that $x' \otimes y' = x' \otimes y''$ with $y'' = [\zeta'']$ of order dividing n. Thus $\mu^{-1}(x' \otimes y') = \mu(x' \otimes y'') = j_f(x' \otimes n\zeta'') = [\xi' \otimes n\zeta'']$.

We compute

$$\begin{aligned} l(\psi^{-1}(x \otimes y), \mu^{-1}(x' \otimes y')) = l([[\xi \otimes \zeta]], [\xi' \otimes n\zeta'']) &= \lambda_{f\otimes g}([\xi \otimes \zeta], [\xi' \otimes n\zeta'']) \\ &= (f_\mathbb{Q} \otimes g_\mathbb{Q})(\xi \otimes \zeta, \xi' \otimes n\zeta'') \bmod 1 \\ &= f_\mathbb{Q}(\xi, \xi') \cdot g_\mathbb{Q}(\zeta, n\zeta'') \bmod 1 \\ &= \lambda_f(x, x') \cdot \underbrace{g_\mathbb{Q}(\zeta, n\zeta'')}_{\in \mathbb{Z}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\lambda_f \otimes \lambda_g)(x \otimes y, x' \otimes y') &= \underbrace{\lambda_f(x, x')}_{\in \frac{1}{n}\mathbb{Z}/\mathbb{Z}} \otimes \underbrace{\lambda_g(y, y')}_{\in \frac{1}{n}\mathbb{Z}/\mathbb{Z}} \\ &= \lambda_f(x, x') \otimes (g_\mathbb{Q}(\zeta, \zeta'') \bmod 1) \\ &= \lambda_f(x, x') \cdot n\ g_\mathbb{Q}(\zeta, \zeta'') \\ &= \lambda_f(x, x') \cdot g_\mathbb{Q}(\zeta, n\zeta''). \end{aligned}$$

Here we used again the fact that $(\frac{a}{n} \bmod 1) \otimes (\frac{b}{n} \bmod 1) = \frac{ab}{n} \bmod 1$ in $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \otimes \frac{1}{n}\mathbb{Z}/\mathbb{Z} = \frac{1}{n}\mathbb{Z}/\mathbb{Z}$. This finishes the proof.

□

COROLLARY 2.3. *The isomorphism class of $l : G/H^\perp \times H \to \mathbb{Q}/\mathbb{Z}$ does not depend on the particular presentations (V, f) and (W, g) and depends only on the linking pairings (G_f, λ_f) and (G_g, λ_g) respectively.*

2.4. Wu classes and quadratic functions. We keep notations from the previous paragraph. We extend the discriminant construction to lattices endowed with a special element called a Wu class.

A *Wu class* $v \in V^\sharp$ is any element $v \in V^\sharp$ such that

$$f(x, x) - f_\mathbb{Q}(x, v) \in 2\mathbb{Z}, \quad \text{for all } x \in V.$$

A Wu class is *integral* if it lies in V. A bilinear lattice (V, f) is said to be *even* if $0 \in \mathrm{Wu}(f)$. Any bilinear lattice has an integral Wu class ([**57**]); the set $\mathrm{Wu}(f)$ of Wu classes is freely and transitively acted on by $V^\sharp$. The action is given by the formula

$$z \cdot s = z + 2s, \quad z \in \mathrm{Wu}(f),\ s \in V^\sharp.$$

This action restricts to an action of V on the set $\mathrm{Wu}^V(f)$ of integral Wu classes.

To a bilinear lattice (V, f, v) equipped with a Wu class, one associates a quadratic linking pairing $\varphi_{f,v} : G_f \to \mathbb{Q}/\mathbb{Z}$ over the linking pairing λ_f by

$$\varphi_{f,v}(x+V) = \frac{1}{2}\left(f_{\mathbb{Q}}(x,x) - f_{\mathbb{Q}}(x,v)\right) \bmod 1, \quad x \in V^\sharp. \tag{2.6}$$

The quadratic function $\varphi_{f,v}$ is homogeneous if and only if v is an integral Wu class.

We now state two basic results in the theory of discriminant quadratic functions.

THEOREM 2.4. *The assignment* $(V, f, v) \mapsto (G_f, \varphi_{f,v})$ *is surjective onto the monoid of quadratic functions on finite abelian groups. When restricted to even lattices, the assignement is surjective onto the monoid of homogeneous quadratic functions on finite abelian groups.*

The equivalence relation on bilinear lattices can be extended to bilinear lattices equipped with Wu classes as follows. Say that (V, f, v) and (W, g, w) are *strongly stably equivalent* if there exist unimodular lattices (U, h, u) and (U', h', u') equipped with Wu classes $u \in \mathrm{Wu}(h)$ and $u' \in \mathrm{Wu}(h')$ respectively and an isomorphism $\psi : U \oplus V \to U' \oplus W$ such that $(h' \oplus g)(\psi(x), \psi(y)) = (h \oplus f)(x, y)$ for all $x, y \in U \oplus V$ and $\psi_{\mathbb{Q}}(u \oplus v) \in u' \oplus w + 2(U' \oplus W)$. The relation is an equivalence relation. It is verified that two strongly stably equivalent triples (V, f, v) and (W, g, w) give rise to isomorphic discriminant quadratic functions. A fundamental result consists in the converse.

THEOREM 2.5. [**12**, Prop. 3.1] *Two nondegenerate quadratic functions on finite abelian groups are isomorphic if and only if they can be lifted to strongly stably equivalent bilinear lattices equipped with Wu classes.*

Consider the lattice $\mathbb{Z}$ equipped with the unimodular form ± 1, sending $(1, 1)$ to ± 1, and the integral Wu class $1 \in \mathbb{Z}$. It is shown in [**12**, Cor. 3.5] that the strong stabilization in Th. 2.4 can be realized using only these unimodular lattices.

2.5. Tensor products and half-integral Wu classes. Let (V, f) and (W, g) be two nondegenerate bilinear lattices.

LEMMA 2.17. *There is an injective map*

$$\mathrm{Wu}(f) \times \mathrm{Wu}(g) \to \mathrm{Wu}(f \otimes g), \ (v, w) \mapsto v \otimes w.$$

PROOF. Let $v \in \mathrm{Wu}(f), w \in \mathrm{Wu}(g)$. For any $x \in V, y \in W$,

$$\begin{aligned}
&(f \otimes g)(x \otimes y, x \otimes y) - (f \otimes g)(v \otimes w, x \otimes y)\\
&\quad = f(x,x)g(y,y) - f(v,x)g(w,y)\\
&\quad = \underbrace{\Big(f(x,x) - f(v,x)\Big)}_{\equiv\, 0 \bmod 2} \underbrace{g(y,y)}_{\in \mathbb{Z}} + \underbrace{f_{\mathbb{Q}}(v,x)}_{\in \mathbb{Z}} \underbrace{\Big(g(y,y) - g_{\mathbb{Q}}(w,y)\Big)}_{\equiv\, 0 \bmod 2}\\
&\quad \equiv \ 0 \bmod 2.
\end{aligned}$$

□

LEMMA 2.18. *The group* $V^\sharp \otimes W + V \otimes W^\sharp$ *acts freely on* $\mathrm{Wu}(f \otimes g)$.

PROOF. The group $V^\sharp \otimes W + V \otimes W^\sharp$ is a subgroup of the group $V^\sharp \otimes W^\sharp$ acting freely on $\mathrm{Wu}(f \otimes g)$. □

This action is not transitive in general since the inclusion $V^\sharp \otimes W + V \otimes W^\sharp \subseteq V^\sharp \otimes W^\sharp$ is proper in general. Indeed, there is equality if and only if

$$V^\sharp \otimes W \ \cap \ V \otimes W^\sharp = V^\sharp \otimes W \ \cap \ (V^\sharp \otimes W)^\sharp = V \otimes W.$$

It will be convenient for our purpose to consider the action of a slightly bigger subgroup (cf. Cor. 3.8). First we describe a special subset of Wu classes. Consider the set S of Wu classes of the form $v \otimes w$ where $v \in \mathrm{Wu}^V(f)$ or $w \in \mathrm{Wu}^W(g)$ (i.e, at least one of the Wu classes v or w has to be integral). Consider first the difference $\Delta = v \otimes w - v' \otimes w'$ of two elements in S. Then

$$\begin{aligned} v \otimes w - v' \otimes w' &= v \otimes w - v \otimes w' + v \otimes w' - v' \otimes w' \\ &= v \otimes (w - w') + (v - v') \otimes w \\ &= 0 \bmod 2(V \otimes W^\sharp + V^\sharp \otimes W). \end{aligned}$$

This suggests to define the set $\mathrm{Wu}^{1/2}(f \otimes g)$ of "half-integral Wu classes" by setting

$$Z' = \frac{1}{2}(V^\sharp \otimes W + V \otimes W^\sharp) \cap (V^\sharp \otimes W^\sharp)$$

and

$$z \in \mathrm{Wu}^{1/2}(f \otimes g) \iff \text{there are } s \in S,\ t \in Z' \text{ such that } z = s + 2t.$$

This is a subset of $V^\sharp \otimes W^\sharp$. Similarly, we define the set $\mathrm{Wu}_0^{1/2}(f \otimes g)$ of "special half-integral Wu classes" by

$$z \in \mathrm{Wu}_0^{1/2}(f \otimes g) \iff \text{there are } s \in S,\ t \in 2Z' = V^\sharp \otimes W + V \otimes W^\sharp \text{ such that } z = s + 2t.$$

This is also a subset of $V^\sharp \otimes W^\sharp$.

What we have proved is

LEMMA 2.19. *The group Z' (resp. $2Z'$) acts freely and transitively on $\mathrm{Wu}^{1/2}(f \otimes g)$, resp. $\mathrm{Wu}_0^{1/2}(f \otimes g)$, by*

$$x \cdot t = x + 2t, \quad x \in \mathrm{Wu}^{1/2}(f \otimes g),\ t \in Z' (\text{resp. } t \in 2Z')$$

and

$$\mathrm{Wu}^Z(f \otimes g) \subseteq \mathrm{Wu}_0^{1/2}(f \otimes g) \subseteq \mathrm{Wu}^{1/2}(f \otimes g) \subseteq \mathrm{Wu}(f \otimes g).$$

As observed above the inclusions are strict in general.

Remark. It follows from Lemma 2.19 that for any half-integral (resp. special half-integral) Wu class z, there exist a pair $(v, w) \in \mathrm{Wu}^V(f) \times \mathrm{Wu}^W(g)$ of *integral* Wu classes such that

$$z = v \otimes w + 2t, \text{for some unique } t \in Z' (\text{resp. } t \in 2Z').$$

The main motivation for introducing the set of half-integral Wu class lies in Theorem 2.6 and Corollary 3.8.

2.6. The discriminant and the characteristic homomorphism. We keep notation from the previous paragraph. The next lemma is mostly a reminder of the definitions.

LEMMA 2.20. *The image of* $V^\sharp \otimes W$ *under the canonical projection* $Z^\sharp \to G_{f\otimes g} = Z^\sharp/Z$ *is* A.

As a consequence, we have

$$V^\sharp \otimes W \;\cap\; (V^\sharp \otimes W)^\sharp = V \otimes W \quad \Longleftrightarrow \quad A \cap A^\perp = 0.$$

Let $z \in Z \otimes \mathbb{Q}$ be a Wu class for $(Z, f \otimes g)$. The discriminant (eq. (2.6)) of $(Z, f \otimes g, z)$ produces a nondegenerate quadratic function $\varphi_{f\otimes g,z} : G_{f\otimes g} \to \mathbb{Q}/\mathbb{Z}$ where $G_{f\otimes g} = Z^\sharp/Z$.

Recall that the subgroup H in $G_{f\otimes g}$ consists in the intersection of $A = j_f(G_f \otimes W)$ and $A^\perp = j_g(V \otimes G_g)$. Note that H is also the image of $V^\sharp \otimes W \;\cap\; V \otimes W^\sharp$ under the canonical projection $Z^\sharp \to G_{f\otimes g}$.

LEMMA 2.21. $\varphi_{f\otimes g,z}|_H$ *is a homomorphism* $H \to \mathbb{Q}/\mathbb{Z}$.

PROOF. By Lemma 2.2, the associated linking pairing $\lambda_{f\otimes g}$ vanishes on $H \times H$. □

According to [**12**, Th. 2.10], there is an affine isomorphism

$$\frac{\mathrm{Wu}(f \otimes g)}{2Z} \to \mathrm{Quad}(\lambda_{f\otimes g}), \quad [z] \mapsto \varphi_{f\otimes g,z}$$

over the group isomorphism

$$G_{f\otimes g} \to G^*_{f\otimes g}, \;\; [s] \mapsto -\lambda_{f\otimes g}([s], -).$$

Here $G_{f\otimes g} = Z^\sharp/Z$ acts freely and transitively on $\frac{\mathrm{Wu}(f\otimes g)}{2Z}$ by the formula

$$[z] \cdot [s] = [z + 2s], \quad z \in \mathrm{Wu}(f \otimes g), \quad s \in Z^\sharp$$

and $G_{f\otimes g}$ acts freely and transitively on $\mathrm{Quad}(\lambda_{f\otimes g})$ by the usual formula (4.1). The isomorphism is affine in the sense that

$$\varphi_{f\otimes g,z} \cdot [s] = \varphi_{f\otimes g,[z]\cdot[-s]} = \varphi_{f\otimes g,z-2s} \tag{2.7}$$

for any $s \in Z^\sharp$.

LEMMA 2.22. *The group* $G_{f\otimes g}/(A + A^\perp)$ *acts freely and transitively on the quotient set* $\dfrac{\mathrm{Wu}(f \otimes g)}{2(V^\sharp \otimes W + V \otimes W^\sharp)}$.

We now investigate the dependency of the homomorphism of Lemma 2.21 on the Wu class. Recall that $G_{f\otimes g}/(A + A^\perp) = G_{f\otimes g}/H^\perp$ acts freely and transitively on $H^* = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z})$ by the formula

$$[x] \cdot \alpha = \alpha + \lambda_{f\otimes g}(x, -), \;\; x \in G_{f\otimes g}, \;\; \alpha \in H^*.$$

LEMMA 2.23. *The affine map*

$$\mathrm{Wu}(f\otimes g)\to \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Q}/\mathbb{Z}),\ z\mapsto \varphi_{f\otimes g,z}|_H$$

induces an affine isomorphism

$$\frac{\mathrm{Wu}(f\otimes g)}{2(V^\sharp\otimes W\ +\ V\otimes W^\sharp)}\to H^*$$

over the isomorphism

$$G_{f\otimes g}/H^\perp\to H^*,\ [x]\mapsto -\lambda_{f\otimes g}(x,-)|_H.$$

PROOF. To prove that the map is well-defined, it suffices to verify that $\varphi_{f\otimes g,z+2[k]}|_H=\varphi_{f\otimes g,z}|_H$ for $k\in(V^\sharp\otimes W)+(V\otimes W^\sharp)$. This amounts to verifying that for $x\in(V^\sharp\otimes W)\cap(V\otimes W^\sharp)$,

$$(f_{\mathbb{Q}}\otimes g_{\mathbb{Q}})(k,x)\in\mathbb{Z}.$$

This follows by Lemma 2.6.

As noted before, the group $G_{f\otimes g}/H^\perp$ acts freely and transitively on both sets $\frac{\mathrm{Wu}(f\otimes g)}{2(V^\sharp\otimes W\ +\ V\otimes W^\sharp)}$ and H^*. Let us verify that the induced map is affine: for any $x\in Z^\sharp$, $z\in\mathrm{Wu}(f\otimes g)$,

$$\varphi_{f\otimes g,z\cdot x}|_H=\varphi_{f\otimes g,z+2x}|_H=\varphi_{f\otimes g,z}|_H-\lambda_{f\otimes g}([x],-)|_H=\varphi_{f\otimes g,z}|_H\cdot[x].$$

We conclude by using the fact that $G_{f\otimes g}/H^\perp$ acts freely and transitively on both sets.

□

To sum up the results so far, we proved that the following diagram is commutative:

$$\begin{tikzcd}
\frac{\mathrm{Wu}(f\otimes g)}{2Z} \arrow[r, "\varphi_{f\otimes g,-}", "\simeq"'] \arrow[d] & \mathrm{Quad}(\lambda_{f\otimes g}) \arrow[d, "\mathrm{res_H}(-)"] \\
\frac{\mathrm{Wu}(f\otimes g)}{2(V^\sharp\otimes W+V\otimes W^\sharp)} \arrow[r, "\mathrm{res}_H(\varphi_{f\otimes g,-})", "\simeq"'] & H^*
\end{tikzcd}$$

Here the downward left arrow is the natural epimorphism induced by the inclusion $Z\subseteq V^\sharp\otimes W\ +\ V\otimes W^\sharp$ and res_H denotes the restriction to the subgroup H.

COROLLARY 2.4. *The quotient set*

$$\frac{\mathrm{Wu}(f\otimes g)}{2(V^\sharp\otimes W+V\otimes W^\sharp)}$$

has the structure of an Abelian group isomorphic to $G_f\otimes G_g$. In particular, the zero element is the unique class $[z]$ such that $\varphi_{f\otimes g,z}|_H=0$.

Remark. The quotient set

$$\frac{\mathrm{Wu}_0^{1/2}(f\otimes g)}{2(V^\sharp\otimes W+V\otimes W^\sharp)}$$

is a singleton. In other words, under the affine map of Lemma 2.23, all special half-integral Wu classes correspond to the same element in H^*. Furthermore, this element has order at most 2 since $2\mathrm{Wu}_0^{1/2}(f\otimes g)\subseteq 2(V^\sharp\otimes W+V\otimes W^\sharp)$.

THEOREM 2.6. *Let $z\in\mathrm{Wu}_0^{1/2}(f\otimes g)$. The map $\varphi_{f\otimes g,z}|_H$ has order at most 2 in H^* and depends only on λ_f and λ_g.*

The subgroup generated by $\varphi_{f\otimes g,z}|_H$ in H^* therefore only depends on λ_f and λ_g. We call this subgroup the *characteristic* subgroup associated to λ_f and λ_g. It is either trivial or has order 2.

COROLLARY 2.5. *If $G_f\otimes G_g$ has odd order, then the characteristic subgroup is trivial.*

PROOF. By Lemma 2.11, H^* has odd order; so has any subgroup of H^*, in particular the characteristic subgroup. By Th. 2.6, it must have order dividing 2, hence it is trivial. □

A proof of Th. 2.6 relying on structural properties of the discriminant (§2.7) is given in §2.8.

Alternatively, an explicit expression for the characteristic map $\varphi_{f\otimes g,z}|_H$ is derived in §2.9. (However, the proof relies partially on Th. 2.6.)

2.7. More on the discriminant. This paragraph is devoted to properties of the discriminant. They are used to prove Th. 2.6 in the next paragraph.

We begin with the following observations. There is the natural right action of the group $\mathrm{Aut}(V)$ of automorphisms of V on the set of nondegenerate symmetric bilinear pairings on V:

$$f\cdot\alpha=\alpha^*f=f\circ(\alpha^{\otimes 2}),\ \alpha\in\mathrm{Aut}(V).$$

Similarly, for a finite abelian group G, the group $\mathrm{Aut}(G)$ of automorphisms of G acts on the set $\mathrm{Quad}(G)$ of quadratic functions on G by the formula

$$q\cdot\beta=\beta^*q=q\circ\beta,\ q\in\mathrm{Quad}(G),\ \beta\in\mathrm{Aut}(G).$$

There is also an action of $\mathrm{Aut}(G)$ on the set of all linking pairings on G by a similar formula.

Let $\mathrm{O}(f)$ denote the automorphism group of f, that is, the isotropy subgroup of $\mathrm{Aut}(V)$ consisting of automorphisms fixing f. Let $\mathrm{O}(\lambda_f)$ denote the automorphism group of λ_f, that is the isotropy subgroup of $\mathrm{Aut}(G_f)$ consisting of automorphisms fixing λ_f. Then $\mathrm{O}(\Lambda_f)$ acts on the set of quadratic functions over λ_f, $\mathrm{Quad}(\lambda_f)\subseteq\mathrm{Quad}(G_f)$, by the same formula as above. Recall that any automorphism α of V (resp. fixing f) induces an automorphism $[\alpha]$ of G_f (resp. fixing λ_f). Hence the assignment

$$\alpha \mapsto [\alpha]$$

yields natural maps

$$\mathrm{Aut}(V) \to \mathrm{Aut}(G_f), \quad \mathrm{O}(f) \to \mathrm{O}(\lambda_f)$$

making the following diagram commutative

$$\begin{tikzcd} \mathrm{O}(f) \arrow[r] \arrow[d, "i"] & \mathrm{O}(\lambda_f) \arrow[d, "i"] \\ \mathrm{Aut}(V) \arrow[r] & \mathrm{Aut}(G_f) \end{tikzcd}$$

where the vertical arrows are canonical inclusions.

LEMMA 2.24. *For any* $\alpha \in \mathrm{Aut}(V)$,

$$v \in \mathrm{Wu}(f) \iff \alpha_{\mathbb{Q}}^{-1} v \in \mathrm{Wu}(\alpha^* f).$$

PROOF. Direct computation. □

Consider now the set $L_{\mathrm{Wu}}(V)$ of all pairs (f, v) where $f : V \times V \to \mathbb{Z}$ is a lattice pairing as before and $v \in \mathrm{Wu}(f)$. As a consequence of Lemma 2.24, the group $\mathrm{Aut}(V)$ acts on $L_{\mathrm{Wu}}(V)$ by the formula:

$$(f, v) \cdot \alpha = (\alpha^* f, \alpha_{\mathbb{Q}}^{-1} v). \tag{2.8}$$

In particular, $\mathrm{O}(f)$ acts on $\mathrm{Wu}(f)$. Let $\mathrm{O}(f_{\mathbb{Q}})$ denote the automorphism group of $f_{\mathbb{Q}}$. There is a restriction map on $\mathrm{O}(f_{\mathbb{Q}})$ defined by $\alpha \mapsto \alpha|_{\mathrm{Wu}(f)}$. Denote by $\mathrm{O}(\mathrm{Wu}(f))$ the image. The action of $\mathrm{O}(f)$ on $\mathrm{Wu}(f)$ yields a map

$$\mathrm{O}(f) \to \mathrm{O}(\mathrm{Wu}(f)).$$

For $(f, v) \in L_{\mathrm{Wu}}(V)$, let $\mathrm{O}(f, v)$ denote the isotropy subgroup of $\mathrm{Aut}(V)$ consisting of automorphisms fixing (f, v) under the action (2.8). Observe that there are natural embeddings

$$\mathrm{O}(f, v) \to \mathrm{O}(f), \ \mathrm{O}(\varphi_{f,v}) \to \mathrm{O}(\lambda_f)$$

fitting in the commutative diagram

$$\begin{tikzcd} \mathrm{O}(f, v) \arrow[r] \arrow[d] & \mathrm{O}(\varphi_{f,v}) \arrow[d] \\ \mathrm{O}(f) \arrow[r] & \mathrm{O}(\lambda_f) \end{tikzcd}$$

Finally denote by $\mathrm{Aut}(L_{\mathrm{Wu}}(V))$ the symmetric group over the set $L_{\mathrm{Wu}}(V)$, by $\mathrm{Aut}(\mathrm{Quad}(\lambda_f))$ the symmetric group over the set $\mathrm{Quad}(\lambda_f)$ and by $\mathrm{Aut}(\mathrm{Quad}(G_f))$ the symmetric group over the set $\mathrm{Quad}(G_f)$.

There are natural maps between the various automorphism groups described above. The canonical inclusions

$$\mathrm{Wu}(f) \to L_{\mathrm{Wu}}(V), \ v \mapsto (f, v), \quad \mathrm{Quad}(\lambda_f) \subseteq \mathrm{Quad}(G_f)$$

induce maps

$$\mathrm{O}(\mathrm{Wu}(f)) \to \mathrm{Aut}(L_{\mathrm{Wu}}(V)), \quad \mathrm{Aut}(\mathrm{Quad}(\lambda_f)) \to \mathrm{Aut}(\mathrm{Quad}(G_f))$$

respectively.

THEOREM 2.7. *Let G be a finite abelian group. There exists a lattice V such that the formula*

$$\varphi_{f,v} \cdot \alpha = \varphi_{(f,v)\cdot\alpha}, \quad (f,v) \in L_{\mathrm{Wu}}(V),\ \alpha \in \mathrm{Aut}(V)$$

defines an action of $\mathrm{Aut}(V)$ *on* $\mathrm{Quad}(G)$.

PROOF. According to [**46**], there is a lattice V such that the map

$$L_{\mathrm{Wu}}(V) \to \mathrm{Quad}(G), \quad (f,v) \mapsto \varphi_{f,v}$$

is surjective. The point that requires to be proved is that the formula for the action is independent of the particular choice (f,v) over $q = \varphi_{f,v}$.

Let $f, g : V \times V \to \mathbb{Z}$ be two bilinear lattices equipped with a Wu classe $v \in \mathrm{Wu}(f)$ and $w \in \mathrm{Wu}(g)$ respectively. Assume that both discriminant quadratic functions $\varphi_{f,v}$ and $\varphi_{g,w}$ lie in $\mathrm{Quad}(G)$. Let $\psi \in \mathrm{Aut}(G)$. We say that (f,v) and (g,w) are *strongly stably equivalent over* ψ, denoted

$$(f,v) \underset{\psi}{\sim} (g,w),$$

if there is a strong stable equivalence between (f,v) and (g,w) that is realized by a lattice automorphism inducing the automorphism $\psi : G \to G$. (See §2.4 and Th. 2.5.)

PROPOSITION 2.1. *With the notation above: $\psi^*\varphi_{g,w} = \varphi_{f,v}$ if and only if $(f,v) \underset{\psi}{\sim} (g,w)$.*

A proof is easily derived from [**12**, Prop. 3.1].

LEMMA 2.25. *For three symmetric bilinear pairings on a lattice V equipped with Wu classes: $(f,v) \underset{\psi}{\sim} (f',v')$, $(f',v') \underset{\psi'}{\sim} (f'',v'') \implies (f,v) \underset{\psi'\circ\psi}{\sim} (f'',v'')$.*

PROOF. Direct computation or consequence of Prop. 2.1. □

Let K be a subgroup of $\mathrm{Aut}(G)$. We say that two quadratic functions $q, q' : G \to \mathbb{Q}/\mathbb{Z}$ are *K-isomorphic* (written $q \underset{K}{\sim} q'$) if there exists $\psi \in K$ such that $\psi^*q' = q$. Similarly, for two bilinear lattices $f, g : V \times V \to \mathbb{Z}$ equipped with Wu classes v, v' respectively, we say that (f,v) and (g,v') are *K-isomorphic* (written $(f,v) \underset{K}{\sim} (g,v')$) if there exists $\psi \in K$ such that $(f,v) \underset{\psi}{\sim} (g,v')$. An immediate consequence of Th. 2.5 is the following observation.

LEMMA 2.26. *Let K be a subgroup of* $\mathrm{Aut}(G)$. *We have*

$$\varphi_{f,v} \underset{K}{\sim} \varphi_{g,v'} \iff (f,v) \underset{K}{\sim} (g,v').$$

Recall that $\mathrm{Aut}(V)$ acts on $\mathrm{Quad}(V^\sharp/V)$ via the natural map $\mathrm{Aut}(V) \to \mathrm{Aut}(V^\sharp/V)$. The next observation is a sufficient condition for the equivalence relation $\underset{K}{\sim}$ to be compatible with the action of $\mathrm{Aut}(V)$.

LEMMA 2.27. *Let $q, q' : G \to \mathbb{Q}/\mathbb{Z}$ be two quadratic functions on G and let $K \lhd \mathrm{Aut}(G)$ be a normal subgroup in $\mathrm{Aut}(G)$. Then $q \underset{K}{\sim} q' \iff q \cdot \alpha \underset{K}{\sim} q' \cdot \alpha$ for any $\alpha \in \mathrm{Aut}(V)$.*

PROOF. Suppose $q \underset{K}{\sim} q'$: there exists $\psi \in K$ such that $q' \circ \psi = q$. Since K is normal in $\mathrm{Aut}(G)$, $\psi' = \alpha^{-1} \circ \psi \circ \alpha \in K$ and $(q' \cdot \alpha) \circ \psi' = q \circ \alpha$. $\square$

End of proof of Theorem 2.7. Let $K \lhd \mathrm{Aut}(G)$. Applying Lemmas 2.26 and 2.27, we have

$$(f, v) \underset{K}{\sim} (g, w) \implies (f, v) \cdot \alpha \underset{K}{\sim} (g, w) \cdot \alpha \implies \varphi_{(f,v)\cdot\alpha} \underset{K}{\sim} \varphi_{(g,w)\cdot\alpha}$$

for any $\alpha \in \mathrm{Aut}(V)$. The result follows by taking $K = \{\mathrm{id}_G\}$. $\square$

Remark. The proof above shows that Theorem 2.7 generalizes as follows.

THEOREM 2.8. *Let G be a finite abelian group and let $K \lhd \mathrm{Aut}(G)$. There exists a lattice V such that the formula*

$$\varphi_{f,v} \cdot \alpha = \varphi_{(f,v)\cdot\alpha}, \quad (f, v) \in L_{\mathrm{Wu}}(V),\ \alpha \in \mathrm{Aut}(V)$$

induces an action of $\mathrm{Aut}(V)$ *on the equivalence classes in* $\mathrm{Quad}(G)$ *for the relation* $\underset{K}{\sim}$.

Theorem 2.7 is the case when K is trivial and equivalence classes are singletons. The other extreme case is when $K = \mathrm{Aut}(G)$ and the equivalence classes consist of isomorphic quadratic functions on G. There are other nontrivial intermediate cases since the automorphism group of a finite abelian group is nonsimple in general [**52**]. As an example, the automorphism group of $\mathbb{Z}/3 \times \mathbb{Z}/3$ is $\mathrm{GL}_2(\mathbb{Z}/3)$: both the subgroup $\mathrm{SL}_2(\mathbb{Z}/3)$ of matrices of determinant 1 and the subgroup of diagonal 2×2 matrices with coefficients in $\{\pm 1\}$ are normal.

Since the map

$$\mathrm{Wu}(f) \to \mathrm{Quad}(\lambda_f),\ z \mapsto \varphi_{f,z}$$

is surjective, the action of $\mathrm{Aut}(L_{\mathrm{Wu}}(V))$ on $\mathrm{Quad}(G_f)$ restricts to an action of $\mathrm{O}(\mathrm{Wu}(f))$ on $\mathrm{Quad}(\lambda_f)$ defined by

$$\varphi_{f,z} \cdot \alpha = \varphi_{f,\alpha^{-1}z},\ z \in \mathrm{Wu}(f),\ \alpha \in \mathrm{O}(\mathrm{Wu}(f)).$$

Compatibility of the various actions is expressed by the commutative diagram

$$
\begin{tikzcd}
& \mathrm{Aut}(V) \arrow[rr] \arrow[dd] & & \mathrm{Aut}(L_{\mathrm{Wu}}(V)) \arrow[dd] \\
O(f) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{O}(\mathrm{Wu}(f)) \arrow[ur] \arrow[dd] & \\
& \mathrm{Aut}(G_f) \arrow[rr] & & \mathrm{Aut}(\mathrm{Quad}(G_f)) \\
\mathrm{O}(\lambda_f) \arrow[ur] \arrow[rr] & & \mathrm{Aut}(\mathrm{Quad}(\lambda_f)) \arrow[ur] &
\end{tikzcd}
$$

To verify that the cube is indeed commutative, the main point consists in verifying that for $\alpha \in \mathrm{O}(f)$,

$$\varphi_{f,z} \circ [\alpha] = \varphi_{\alpha^* f, \alpha^{-1} z} = \varphi_{(f,z)\cdot\alpha} = \varphi_{f,z} \cdot \alpha,$$

which follows from our discussion above. In particular, there is an isomorphism

$$\varphi_{f,z} \cdot \alpha \simeq \varphi_{f,z}.$$

Next, we consider the tensor product $f \otimes g$. The group $\mathrm{Aut}(V)$ acts on $L_{\mathrm{Wu}}(V \otimes W)$ via the natural map

$$\mathrm{Aut}(V) \to \mathrm{Aut}(V \otimes W),\ \alpha \mapsto (\alpha \otimes 1_W).$$

Explicitly, the action is given by

$$(k,z) \cdot \alpha = ((\alpha \otimes 1_W)^* k, (\alpha \otimes 1_W)^{-1} z),\quad (k,z) \in L_{\mathrm{Wu}}(V \otimes W),\ \alpha \in \mathrm{Aut}(V).$$

It follows that $\mathrm{Aut}(V)$ acts on $\mathrm{Quad}(G_f)$.

Similarly, there is a natural inclusion map

$$\mathrm{O}(f) \to \mathrm{O}(f \otimes g),\ \alpha \mapsto \alpha \otimes 1_W.$$

It follows that $\mathrm{O}(f)$ acts on $\mathrm{O}(\mathrm{Wu}(f \otimes g))$ and on $\mathrm{O}(\lambda_{f\otimes g})$. Finally, the composition

$$G_f \xrightarrow{-\otimes 1_W} G_f \otimes W \xrightarrow{j_f} G_{f\otimes g}$$

enables to define an action of $\mathrm{Aut}(G_f)$ on $\mathrm{Quad}(G_{f\otimes g})$. This action restricts to an action of $\mathrm{O}(\lambda_f)$ on $\mathrm{Quad}(\lambda_{f\otimes g})$.

The cube above is still commutative if we replace f by $f\otimes g$ in all occurrences of f in the right face.

We note that $\mathrm{Aut}(V)$ also acts on the set of subgroups of $G_{f\otimes g}$ via the map $\mathrm{Aut}(V) \to \mathrm{Aut}(V \otimes W)$. Explicitly,

$$K \cdot \alpha = [\alpha \otimes 1_W]^{-1}(K),\ \alpha \in \mathrm{Aut}(V),\ K \subseteq G_{f\otimes g},$$

where $[\alpha \otimes 1_W]$ denotes the automorphism on $G_{f\otimes g}$ induced by the map $\alpha \otimes 1_W \in \mathrm{Aut}(V \otimes W)$. (The action is a right action so as to be consistent with the previous actions.)

LEMMA 2.28. *The subgroup* $H = j_f(G_f \otimes W) \cap j_g(V \otimes G_g)$ *introduced in §2.2 is invariant under the action of* $\mathrm{Aut}(V)$.

PROOF. Let $\alpha \in \mathrm{Aut}(V)$. Let

$$[x \otimes w] = j_f([x] \otimes w) \in j_f(G_f \otimes W),$$

with $x \in V^\sharp$, $w \in W$. We have

$$[\alpha \otimes 1_W][x \otimes w] = [\alpha_{\mathbb{Q}} x \otimes w] = j_f([\alpha_{\mathbb{Q}} x] \otimes w) \in j_f(G_f \otimes W).$$

Hence $j_f(G_f \otimes W)$ is invariant under α. A similar argument shows that $j_g(V \otimes G_g)$ is invariant under α. The lemma follows. ☐

2.8. Proof of Theorem 2.6. We already know that $\varphi_{f\otimes g,z}|_H$ has order at most 2 for $z \in \mathrm{Wu}_0^{1/2}(f \otimes g)$ and that it is independent of the particular choice of $z \in \mathrm{Wu}_0^{1/2}(f \otimes g)$. For a fixed bilinear lattice g, we shall prove that $\varphi_{f\otimes g,z}|_H$ only depends on λ_f. (The argument is completely symmetric in g.)

First step: action of $\mathrm{Aut}(V)$ and $\mathrm{O}(f)$ on the homomorphism $\varphi_{f\otimes g,z}|_H$.

Recall the (right) action of $\mathrm{Aut}(V)$ on $\mathrm{Quad}(G_f)$ and on subgroups of G_f. Let $\alpha \in \mathrm{Aut}(V)$ act on $\varphi_{f\otimes g,z}$ and H. We have

$$(\varphi_{f\otimes g,z} \cdot \alpha)|_{H\cdot\alpha} = (\varphi_{f\otimes g,z} \cdot \alpha)|_H,$$

according to Lemma 2.28. Thus if $\alpha \in O(f)$, then

$$\varphi_{f\otimes g,z} \cdot \alpha|_{H\cdot\alpha} = \varphi_{f\otimes g,z\cdot\alpha}|_H.$$

Second step: if $z \in \mathrm{Wu}_0^{1/2}(f \otimes g)$ then $\varphi_{f\otimes g,z}|_H$ is invariant under $\mathrm{O}(f)$.

The subset $\mathrm{Wu}_0^{1/2}(f \otimes g)$ is invariant under the action of $\mathrm{O}(f)$ on $\mathrm{Wu}(f \otimes g)$. The claim follows.

Third step: stabilization of f.

Let (U, u) be an unimodular lattice. We show that replacing f by $f \oplus u$ does not affect the homomorphism $\varphi_{f\otimes g,z}|_H$.

First, $\lambda_{f\oplus u} = \lambda_f \oplus \lambda_u = \lambda_f \oplus 0 = \lambda_f$ and $\lambda_{(f\oplus u)\otimes g} = \lambda_{f\otimes g} \oplus \lambda_{u\otimes g}$. Next, the monomorphism $j_{f\oplus u} : G_f \otimes W \to G_{f\otimes g} \oplus G_{u\otimes g}$ factors through the monomorphism $j_f : G_f \otimes W \to G_{f\otimes g}$ and the canonical inclusion $G_{f\otimes g} \to G_{f\otimes g} \oplus G_{u\otimes g}$ sending $x \in G_{f\otimes g}$ to $(x, 0)$. In particular, the images of $j_{f\oplus u}$ and j_f coincide and are contained in $G_{f\otimes g} \oplus 0 \subseteq G_{f\otimes g} \oplus G_{u\otimes g}$. Denote by H' the new subgroup when f is replaced by $f \oplus u$. It follows that $H' \subseteq G_{f\otimes g} \oplus 0 \subseteq G_{f\otimes g} \oplus G_{u\otimes g}$ and is equal to H once $G_{f\otimes g} \oplus 0$ is identified to $G_{f\otimes g}$. Let z' be an arbitrary Wu class of $(f \oplus u) \otimes g = (f \otimes g) \oplus (u \otimes g)$ such that its restriction on $V \otimes W$ is z. Then

$$\varphi_{(f\oplus u)\otimes g,z'}|_{H'} = \varphi_{(f\otimes g)\oplus(u\otimes g),z'}|_{H'} = \varphi_{f\otimes g,z}|_H \oplus 0.$$

Therefore, we have proved that $\varphi_{f\otimes g,z}|_H$ is invariant under $O(f)$ and stabilization of f by unimodular lattices. It follows from [**?**] (see also [**46**] [**14**]) that $\varphi_{f\otimes g,z}|_H$ only depends on λ_f as claimed. □

Remarks. Another proof results from the following observations. First,

$$\text{for all } v \in \mathrm{Wu}^V(f), \quad \varphi_{f\otimes g,v\otimes w} \circ j_f = \varphi_{f,v} \otimes g$$

and similarly

$$\text{for all } w \in \mathrm{Wu}^W(g), \quad \varphi_{f\otimes g,v\otimes w} \circ j_g = f \otimes \varphi_{g,w}.$$

Second, let $z \in \mathrm{Wu}^{1/2}(f \otimes g)$ written as $z = v \otimes w + 2s$ with $v \in \mathrm{Wu}^V(f)$ or $w \in \mathrm{Wu}^W(g)$, and $s \in V^\sharp \otimes W + V \otimes W^\sharp$. Then

$$\varphi_{f\otimes g,z} \circ j_g|_{j_g^{-1}H} = f \otimes \varphi_{g,w}|_{j_g^{-1}H} = \varphi_{f,v} \otimes g|_{j_f^{-1}H} = \varphi_{f\otimes g,z} \circ j_f|_{j_f^{-1}H}.$$

A slightly more explicit expression is given by the formula:

$$\varphi_{f\otimes g,v\otimes w-2t} \circ j_f = \varphi_{f,v} \otimes g + (\widehat{\lambda}_f \otimes \widehat{g}_{\mathbb{Q}})([t])|_{G_f\otimes W}, \tag{2.9}$$

where $v \in \mathrm{Wu}^V(f)$, $t \in V^\sharp \otimes W^\sharp$, $[t] \in G_f \otimes W^\sharp$. Here the map $(\widehat{\lambda}_f \otimes \widehat{g}_{\mathbb{Q}})([t])|_{G_f\otimes W}$ denotes the homomorphism induced by the map adjoint to the bilinear pairing

$$\lambda_f \otimes g_{\mathbb{Q}}|_{W^\sharp\times W} : (G_f \otimes W^\sharp) \times (G_f \otimes W) \to \mathbb{Q}/\mathbb{Z}$$

at $[t] \in G_f \otimes W^\sharp$([1]).

Similarly,

$$\varphi_{f\otimes g,v\otimes w-2t} \circ j_g = f \otimes \varphi_{g,w} + (\widehat{f}_{\mathbb{Q}} \otimes \widehat{\lambda}_g)([t])|_{V\otimes G_g}, \tag{2.10}$$

where $w \in \mathrm{Wu}^W(g)$, $t \in V^\sharp \otimes W^\sharp$, $[t] \in V^\sharp \otimes G_g$. The map $(\widehat{f}_{\mathbb{Q}} \otimes \widehat{\lambda}_g)([t])|_{V\otimes G_g}$ denotes the homomorphism induced by the map adjoint to the bilinear pairing

$$f_{\mathbb{Q}}|_{V^\sharp\times V} \otimes \lambda_g : V^\sharp \otimes G_g \times V \otimes G_g \to \mathbb{Q}/\mathbb{Z}$$

[1]One should note at this point that the bilinear pairing $(\lambda_f \otimes g_{\mathbb{Q}})|_{G_f\otimes W^\sharp\times G_f\otimes W}$ is well defined, as $g_{\mathbb{Q}}(W^\sharp, W) \subseteq \mathbb{Z}$ acts by multiplication on $\lambda_f(G_f, G_f) \subset \mathbb{Q}/\mathbb{Z}$. A similar observation applies to the bilinear pairing $(f_{\mathbb{Q}} \otimes \lambda_g)|_{V^\sharp\otimes G_g\times V\otimes G_g}$ considered below.

at $[t] \in V^\sharp \otimes G_g$.

In the cases when t lies in the smaller subgroup $V^\sharp \otimes W$, the formula simplifies

$$\varphi_{f\otimes g, v\otimes w-2t} \circ j_f = \varphi_{f,v} \otimes g + (\lambda_f \otimes g)([t], -), \tag{2.11}$$

where $v \in \mathrm{Wu}^V(f)$, $t \in V^\sharp \otimes W$, $[t] \in G_g \otimes W$. Similarly,

$$\varphi_{f\otimes g, v\otimes w-2t} \circ j_g = f \otimes \varphi_{g,w} + (f \otimes \lambda_g)([t], -), \tag{2.12}$$

where $w \in \mathrm{Wu}^W(g)$, $t \in V \otimes W^\sharp$, $[t] \in V \otimes G_f$.

2.9. The characteristic homomorphism: explicit form. Let (G, λ) be a nondegenerate linking pairing.

Let (G, λ) and (G', λ') two (nondegenerate) linking pairings on finite abelian groups G and G' respectively. We define a map

$$\chi : G \times G' \to \mathbb{Z}/2$$

as follows: we set $\chi(x, y) = 1$ if x and y both generate an *orthogonal summand* of the *same even order* in G and G' respectively; we set $\chi(x, y) = 0$ otherwise. Note that the map χ depends on the linking pairings λ and λ'.

As an example, if G or G' has odd order, then $\chi = 0$. For G and G' are both cyclic of order a power of 2, $\chi(x, y) = 1$ if and only if x and y are generators.

PROPOSITION 2.2. *The map $\chi : G \times G' \to \mathbb{Z}/2$ is bilinear.*

Therefore χ induces a homomorphism $G \otimes G' \to \mathbb{Z}/2$, still denoted χ.

PROOF. Although the proof is a consequence of the general theory of linking pairings in torsion Dedekind modules (cf. [**13**, Chap. 2]), we give an elementary proof based on the following observation ([**8**, Lemma 28]).

LEMMA 2.29. *Let $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ be a linking pairing and let $x \in G$. The subgroup generated by x in G is an orthogonal summand if and only if x and $\lambda(x, x)$ have the same order in G and $\mathbb{Q}/\mathbb{Z}$ respectively.*

First, it is obvious that χ only depends on unordered pairs $(x, y) \in G \times G'$. Secondly, it is not hard to see that it is sufficient to consider 2-groups. Let $x, y \in G$ and $z \in G'$. Suppose first $\chi(x, z) = \chi(y, z) = 1$. We have to prove that $\chi(x + y, z) = 0$. By hypothesis, both x and y generate an orthogonal summand of even order 2^k in G and similarly z in G'. By Lemma 2.29, the order of $\lambda(x, x)$ and the order of $\lambda(y, y)$ in $\mathbb{Q}/\mathbb{Z}$ coincide with the order of x (resp. of y) in G. Thus there are odd integers $a, b \in \mathbb{Z}$ such that $\lambda(x, x) = \frac{a}{2^k}$ mod 1 and $\lambda(y, y) = \frac{b}{2^k}$ mod 1. Hence for some $c \in \mathbb{Z}$,

$$\begin{aligned} \lambda(x + y, x + y) &= \lambda(x, x) + 2\lambda(x, y) + \lambda(y, y) \\ &= \frac{a}{2^k} + \frac{2c}{2^k} + \frac{b}{2^k} \\ &= \frac{a + 2c + b}{2^k} \bmod 1. \end{aligned}$$

Since $a + 2c + b$ is even, $\lambda(x + y, x + y)$ is of order strictly less than 2^k. By Lemma 2.29 again, we conclude that $x + y$ does not generate an orthogonal summand of order 2^k. Hence $\chi(x + y, z) = 0$.

Suppose next that $\chi(x,z) = \chi(y,z) = 0$. We have to show that $\chi(x+y,z) = 0$. Suppose the contrary. Then $x+y$ generates an orthogonal summand of order 2^k in G. So $\lambda(x+y,x+y) = \frac{a}{2^k} \bmod 1$ for some odd integer a. By our hypotheses, there are even integers $b,c,d \in 2\mathbb{Z}$ such that

$$\begin{aligned} \lambda(x+y,x+y) = \frac{a}{2^k} &= \lambda(x,x) + 2\cdot\lambda(x,y) + \lambda(y,y) \\ &= \frac{b}{2^k} + \frac{c}{2^k} + \frac{d}{2^k} \\ &= \frac{b+c+d}{2^k}. \end{aligned}$$

Since $b+c+d$ is even, $\lambda(x+y,x+y)$ is of order strictly less than 2^k. This is a contradiction. Hence $\chi(x+y,z) = 0$.

Suppose finally that $\chi(x,z) = 1$ and $\chi(y,z) = 0$. Assume that the order of y divides the order of x. By Lemma 2.29, there exists an odd integer $a \in \mathbb{Z}$ and integers $b,c \in 2\mathbb{Z}$ such that $\lambda(x,x) = \frac{a}{2^k} \bmod 1$, $\lambda(y,y) = \frac{2b}{2^k} \bmod 1$ and $\lambda(x,y) = \frac{c}{2^k} \bmod 1$. Hence

$$\begin{aligned} \lambda(x+y,x+y) &= \lambda(x,x) + 2\cdot\lambda(x,y) + \lambda(y,y) \\ &= \frac{a}{2^k} + 2\frac{c}{2^k} + \frac{2b}{2^k} \\ &= \frac{a+2b+2c}{2^k} \bmod 1. \end{aligned}$$

Since $a+2b+2c$ is odd, we conclude that the order of $\lambda(x+y,x+y)$ in $\mathbb{Q}/\mathbb{Z}$ equals the order of $x+y$ in G, hence by Lemma 2.29, $x+y$ generates an orthogonal summand of order 2^k in G, as z does in G'. This implies $\chi(x+y,z) = 1$. □

We now identify the characteristic homomorphism by means of the tensor product of linking pairings. Recall that for a group G, we denote by $G[n]$ the subgroup of elements of order dividing n.

COROLLARY 2.6. *Let (G,λ) be a linking pairing. Let n be an integer and $(\mathbb{Z}/n,\lambda')$ be a cyclic linking pairing. Let $h : G[n] \to G\otimes\mathbb{Z}/n\mathbb{Z}$ be the isomorphism defined by $h(x) = x\otimes(1 \bmod n)$. The characteristic homomorphism $\chi : G\otimes\mathbb{Z}/n\mathbb{Z} \to \frac{1}{2}\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}/2$ is given by*

$$\chi(x) = \frac{n}{2}\lambda(h^{-1}(x),h^{-1}(x)), \quad x \in G\otimes\mathbb{Z}/n\mathbb{Z}. \tag{2.13}$$

Furthermore, let $\lambda_0 : \mathbb{Z}/n\times\mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ be the linking pairing uniquely determined by

$$\lambda_0(1 \bmod n, 1 \bmod n) = \begin{cases} \frac{1}{2} & \text{if } n = 0 \bmod 2; \\ 0 & \text{if } n = 1 \bmod 2. \end{cases}.$$

Then

$$\chi_{\lambda,\lambda'}(x) = (\lambda\otimes\lambda_0)(x,x), \quad x \in G\otimes\mathbb{Z}/n\mathbb{Z}. \tag{2.14}$$

Remark. Note that the linking pairing λ_0 in the statement of Cor. 2.6 is degenerate if $n \neq 2$.

This result provides a way to compute easily χ for any pair of linking pairings: decompose λ' into an orthogonal sum of indecomposable linking pairings; disregard the noncyclic ones and the cyclic ones of odd order; using the remark above, the homomorphism χ is the orthogonal sum of the restriction of χ to the remaining components which is computed by Cor. 2.6.

We now relate our previous construction (§2.6) to the characteristic homomorphism. Recall the isomorphism $\mu : H \to G_f \otimes G_g$ defined in §2.3.

THEOREM 2.9. *Let $z \in \mathrm{Wu}_0^{1/2}(f \otimes g)$. Then $\varphi_{f\otimes g,z}|_H = \chi \circ \mu$.*

PROOF. It is sufficient to verify the statement with $z = v \otimes w$ where v and w are integral Wu classes of f and g respectively. Then we verify that they coincide on the generators of the orthogonal summands of an orthogonal splitting of (G_f, λ_f) and (G_g, λ_g). □

We can now write down a general formula for the homomorphism $\varphi_{f\otimes g,z}|_H$ for an arbitrary Wu class $z = z_0 + 2t$, $z_0 \in \mathrm{Wu}^Z(f \otimes g)$, $t \in Z^\sharp$:

$$\varphi_{f\otimes g,z}|_H = \chi \circ \mu(-) - (\lambda_f \otimes \lambda_g)(\psi([t]), \mu(-)), \tag{2.15}$$

where $\psi : G/H^\perp \to G_f \otimes G_g$ is the natural isomorphism defined in §2.2. We deduce

THEOREM 2.10. *With the notation above, the homomorphism $\varphi_{f\otimes g,z_0+2t}|_H$ is zero if and only if*

$$\chi = (\lambda_f \otimes \lambda_g)(\psi([t]), -).$$

In other words, $\varphi_{f\otimes g,z_0+2t}|_H$ is zero if and only if $\psi([t])$ is the characteristic element in $G_f \otimes G_g$, corresponding to the characteristic homomorphism $\chi = \chi_{\lambda_f,\lambda_g}$ under the map adjoint to $\lambda_f \otimes \lambda_g$.

3. Classification results

3.1. Gauss sums. Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on a finite abelian group. The complex number

$$\Gamma(G, q) = \sum_{x\in G} \exp(2\pi i q(x))$$

is the *unnormalized Gauss sum* associated to (G, q). It is convenient to define also

$$\gamma(G, q) = |G|^{-\frac{1}{2}}|G^\perp|^{-\frac{1}{2}} \sum_{x\in G} \exp(2\pi i q(x)),$$

be the *normalized Gauss sum* associated to q. These sums are clearly invariants of the isomorphism class of q. Two basic properties, immediate from the definition, are the behavior with respect to the orthogonal sum of quadratic functions and the natural involution (opposite) of quadratic functions

$$\gamma((G, q) \oplus (G', q')) = \gamma(G \oplus G', q \oplus q') = \gamma(G, q) \cdot \gamma(G', q'),$$
$$\gamma(G, -q) = \overline{\gamma(G, q)}.$$

These properties are interpreted in the classical context of the Witt group of quadratic functions (See §4). Another useful property is the behaviour of the Gauss sum when a homomorphism is added to the quadratic function:

$$\gamma(G, q + \widehat{b}_q(\alpha)) = \gamma(G, q)\ e^{-2\pi i q(\alpha)}. \tag{3.1}$$

Gauss sums play a fundamental rôle in the classification of pointed linking pairings.

3.2. Classification of pointed linking pairings. The use of Gauss sums in the classification of isomorphism classes of linking pairings goes back to Minkowski; it is actually proved in [**33**] that Gauss sums form a complete system of invariants. Using a different approach, we now extend this result to the classification of isomorphism classes of pointed linking pairings.

A *pointed linking pairing* is a pair formed by a linking pairing (G, λ) and an element $c = (c_1, \ldots, c_n) \in G^n$ for some natural integer n. Two pointed linking pairings (G, λ, c) and (G', λ', c') are *isomorphic* if there is an isomorphism of linking pairings that sends c onto c'.

In the sequel of this paragraph, we fix a nondegenerate pointed linking pairing (G, λ, c), with $c = (c_1, \ldots, c_n) \in G^n$, $n \geqslant 1$. Consider now a triple (V, h, s) where $h : V \to \mathbb{Z}$ is a homogeneous nondegenerate quadratic function on a lattice V and $s = (s_1, \ldots, s_n) \in (V^*)^n$. We form a new quadratic function on $V \otimes G$ defined by

$$h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c) \tag{3.2}$$

where $h \otimes \lambda$ is the usual tensor product of a homogeneous quadratic function and a linking pairing and $s \otimes c = \sum_j s_j \otimes c_j \in V^* \otimes G$. Explicitly

$$(h \otimes \lambda)\left(\sum_j x_j \otimes y_j\right) = \sum_j h(x_j)\lambda(x_j, x_j) + \sum_{j<k} b_h(x_j, x_k)\lambda(x_j, x_k), \quad x_j \in V,\ y_j \in G$$

where b_h is the bilinear symmetric pairing associated to h. Here $(\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c)$ is the homomorphism $V \otimes G \to \mathbb{Q}/\mathbb{Z}$ defined by

$$(\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c)(x \otimes y) = \sum_i s_i(x)\lambda(c_i, y), \quad x \in V,\ y \in G.$$

Let $\Gamma_{h,s}(\lambda, c) = \Gamma\left(V \otimes G, h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c)\right)$ be the (unnormalized) Gauss sum associated to the quadratic function defined above by (3.2). It is convenient to consider as well the normalized Gauss sum $\gamma_{h,s}(\lambda, c) = \gamma\left(V \otimes G, h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c)\right)$.

LEMMA 3.1. *If $s \otimes c = (\widehat{b}_h \otimes \mathrm{id}_G)(y)$ for some $y \in V \otimes G$, then*

$$\gamma_{h,s}(\lambda, c) = \gamma(V \otimes G, h \otimes \lambda) \cdot e^{-2\pi (h\otimes\lambda)(y)}. \tag{3.3}$$

PROOF. According to the hypothesis,

$$\begin{aligned}\gamma_{h,s}(\lambda, c) &= \gamma\left(V \otimes G, h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c)\right)\\ &= \gamma\left(V \otimes G, h \otimes \lambda + (b_h \otimes \widehat{\lambda})(y)\right)\\ &= \gamma(V \otimes G, h \otimes \lambda) \cdot e^{-2\pi(h\otimes\lambda)(y)}.\end{aligned}$$

□

Also we need to define invariants extracted from the group G. Recall that every linking pairing (G, λ) splits in an orthogonal decomposition into linking pairings

$$(G, \lambda) = \bigoplus_p (G_p, \lambda_p)$$

on p-groups, where p describes a finite subset of primes. Furthermore, every linking pairing (G_p, λ_p) splits in an orthogonal decomposition into linking pairings

$$(G_p, \lambda_p) = \bigoplus_{k \geqslant 1} (G_p^k, \lambda_p^k)$$

where each G_p^k is a free $\mathbb{Z}/p^k$-module and hence has a well defined *rank*. Set

$$\rho_p^k(\lambda) = \mathrm{rank}\ G_p^k \in \mathbb{N}.$$

Clearly the ranks $\rho_p^k(\lambda)$ depend only on the underlying group G, are additive under direct sums and only finitely many of them are non zero.

THEOREM 3.1. *Two pointed linking pairings (G, λ, c) and (G', λ', c') with distinguished n-tuples $c \in G^n$ and $c' \in (G')^n$ are isomorphic if and only if the following conditions are satisfied:*

(1) *$\rho_p^k(\lambda) = \rho_p^k(\lambda')$ for all prime p and all $k \geqslant 1$;*
(2) *$\gamma_{h,s}(\lambda, c) = \gamma_{h,s}(\lambda', c')$ for all triples (V, h, s) of lattices V equipped with a homogeneous quadratic function h and a multiform $s \in (V^*)^n$.*

We make a few observations on Th. 3.1. Condition (1) is purely group-theoretic and does not involve the pairings nor the distinguished elements.

In condition (2), only a finite number of Gauss sums is required. However, it can be shown that one needs, in the most general case, to consider at least one rank 2 lattice (V, f).

If we replace the Gauss sums $\gamma_{h,s}(\lambda, c)$ by *unnormalized* Gauss sums $\Gamma_{h,s}(\lambda, c)$ in condition (2), then condition (1) becomes redundant. Taking the absolute value of appropriately chosen unnormalized Gauss sums yield the invariants of condition (1).

The classification of linking pairings without distinguished point (see [**33**, Th. 4.1]) is recovered from Th. 3.1 by taking $n = 1$ and setting c and c' to be the zero element of G and G' respectively. The Gauss sum $\gamma_{h,s}(\lambda, 0)$ then is just $\gamma(V \otimes G, h \otimes \lambda)$.

PROOF. We give an abridged proof here, referring to [**13**] for details. The proof is based on two lemmas.

The first lemma is a "reduction to linear algebra" based on the classification of linking pairings. Let $N \geqslant 1$. Recall that n denotes the number of distinguished elements. Denote by $\mathcal{R}_{N,n}$ (resp. $\mathcal{R}_N$) the vector space of matrices with N rows and n columns (resp. the vector space of square symmetric matrices of size N) with entries in $\mathbb{Q}/\mathbb{Z}$. For $r = (r_{jk})_{1\leqslant j,k\leqslant N} \in \mathcal{R}_N$ and $r' \in \mathcal{R}_{N,n}$, set

$$S_{r,r'}(\lambda, c) = \{(x_1, \ldots, x_N) \in G^N \mid \lambda(x_j, x_k) = r_{jk} \text{ and } \lambda(x_j, c_k) = r'_{jk}\}.$$

This set is clearly finite and we denote its cardinality by $|S_{r,r'}(\lambda, c)|$.

LEMMA 3.2. *Two pointed linking pairings (G, λ, c) and (G', λ', c') with n distinguished elements are isomorphic if and only if $\rho_p^k(\lambda) = \rho_p^k(\lambda')$ for all prime p and all $k \geqslant 1$ and $|S_{r,r'}(\lambda, c)| = |S_{r,r'}(\lambda', c')|$ for all matrices $r \in \mathcal{R}_N$ and $\mathcal{R}_{N,n}$ for N large enough.*

The second lemma is classical.

LEMMA 3.3. *A family of distinct characters is free over $\mathbb{C}$.*

We interpret the unnormalized Gauss sums $\Gamma_{h,s}(\lambda, c)$ as characters and related them to the invariants $|S_{r,r'}(\lambda, c)|$. Fix a basis of V and identify h with a square symmetric matrix of size N, each $s_j \in V^*$, $1 \leqslant j \leqslant N$, with a vector $(s_{jk})_{1\leqslant k\leqslant N} \in \mathbb{Z}^N$. Then

$$\Gamma_{h,s}(\lambda, c) = \sum_{\substack{r\in\mathcal{R}_N \\ r'\in\mathcal{R}_{N,n}}} |S_{r,r'}(\lambda, c)| \ \exp\Big(2\pi i \mathrm{Trace}\Big(hr + sr'\Big)\Big). \tag{3.4}$$

The sum is finite since only finitely many terms are non zero. The maps

$$\kappa_{r,r'} : (h, s) \mapsto \exp\Big(2\pi i \mathrm{Trace}\Big(fr + sr'\Big)\Big)$$

are distinct characters, hence the family $(\kappa_{r,r'})_{r,r'}$ is free over $\mathbb{C}$. On the other hand, only finitely many sets $S_{r,r'}(\lambda, c)$ are non empty. Therefore there is a finite number M of homogeneous quadratic functions equipped with multiforms $(h^1, s^1), \ldots, (h^M, s^M)$ such that the matrix $\Phi = (\kappa_{r_j,r'_j}(h^k, s^k))_{1\leqslant j,k\leqslant M}$ is invertible over $\mathbb{C}$. Set

$$\Gamma = (\Gamma_{h^j,s^j}(\lambda, c))_{1\leqslant j\leqslant M}, \quad S = (|S_{r_j,r'_j}(\lambda, c)|)_{1\leqslant j\leqslant M}.$$

We deduce from (3.4) the identity

$$\Gamma = \Phi \cdot S.$$

Since Φ is invertible, Γ determines S and conversely. The result then follows from Lemma 3.2. $\square$

The Gauss sum $\gamma_{h,s}(\lambda, c)$ and the quadratic function (3.2) can be interpreted using the discriminant construction as follows. First,

$$h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes a) = (b_h \otimes q) + (\widehat{b_h}_{\mathbb{Q}} \otimes \widehat{\lambda})(\xi \otimes c), \tag{3.5}$$

where on the right hand side:

- q is a homogeneous quadratic refinement of λ;
- $b_h : V \times V \to \mathbb{Z}$ denotes the symmetric bilinear pairing associated to the quadratic form $h : V \to \mathbb{Z}$;
- $\xi \in (V^\sharp)^n$ is defined by $\widehat{f}_{\mathbb{Q}}(\xi_j) = s_j$, $1 \leqslant j \leqslant n$;
- $\xi \otimes c = \sum_j \xi_j \otimes c_j$;
- The map $(\widehat{b_h}_{\mathbb{Q}} \otimes \widehat{\lambda})(\xi \otimes c) : V \times G \to \mathbb{Q}/\mathbb{Z}$ is induced by the map adjoint to $b_h \otimes \lambda$ at $\xi \otimes c$. It is defined by

  $$(\widehat{b_h}_{\mathbb{Q}} \otimes \widehat{\lambda})(\xi \otimes c)(x \otimes y) = (b_h)_{\mathbb{Q}}(\xi, x)\ \lambda(c, y) \quad x \in V,\ y \in G.$$

It follows from (2.10) that the quadratic function on the right hand side identifies to $\varphi_{b_h \otimes g, v \otimes w - 2t} \circ j_g$ where (W, g, w) is a bilinear lattice equipped with an integral Wu class $w \in \mathrm{Wu}(g)$ such that $(G, q) = (W^\sharp/W, \varphi_{g,w})$, $v \in \mathrm{Wu}(b_h)$ is a Wu class for b_h and $t \in V^\sharp \otimes W^\sharp$ is a lift of $\xi \otimes c \in V^\sharp \otimes G$. We conclude that

$$\gamma_{h,s}(\lambda, c) = \gamma(V \otimes G, \varphi_{b_h \otimes g, v \otimes w - 2t} \circ j_g). \tag{3.6}$$

3.3. The classification of pointed quadratic functions. The results of the previous paragraph are generalized to pointed quadratic functions. A *pointed quadratic function* on a finite abelian group G consists of a quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$ equipped with $c = (c_1, \ldots, c_n) \in G^n$ for some integer $n \geqslant 0$. Two pointed quadratic functions (G, q, c) and (G', q', c') are *isomorphic* if there is an isomorphism of linking pairings that sends c_j onto c'_j, $1 \leqslant j \leqslant n$.

Before stating the theorem of this paragraph, we recall two simple definitions. Given a quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$, the difference $d_q(x) = q(x) - q(-x)$, $x \in G$, defines a homomorphism $G \to \mathbb{Q}/\mathbb{Z}$, the *homogeneity defect*. This map is zero if and only if q is homogeneous. Recall that a quadratic function canonically induces an associated linking pairing b_q. Hence there is a well defined surjective ("forgetful") homomorphism

$$(G, q, c) \mapsto (G, b_q, c)$$

from the monoid of pointed (nondegenerate) quadratic functions (with distinguished n-tuples) to the monoid of pointed (nondegenerate) linking pairings (with distinguished n-tuples). We shall use a related but distinct homomorphism

$$(G, q, c) \mapsto (G, b_q, c \oplus \widehat{b}_q^{-1} d_q)$$

from the monoid of pointed (nondegenerate) quadratic functions with distinguished n-tuples to the monoid of pointed (nondegenerate) linking pairings with distinguished $(n+1)$-tuples. Here the adjoint map $\widehat{b}_q : G \to G^*$ is bijective hence $\widehat{b}_q^{-1} d_q$ is a well defined element in G and $c \oplus \widehat{b}_q^{-1} d_q$ denotes the $(n+1)$-tuple obtained by adjoining the form $d_q \in G^*$ to the n-tuple

$c = (c_1, \dots, c_n)$ on the right. This latter map is not onto. (It is onto if we restrict the image to pointed linking pairings with $(n+1)$-tuples of distinguished points whose last distinguished point lies in $2G$.)

THEOREM 3.2. *Two pointed quadratic functions (G, q, c) and (G', q', c') with distinguished n-tuples are isomorphic if and only if the following conditions are satisfied:*

(1) $\rho_p^k(b_q) = \rho_p^k(b_{q'})$ *for all prime p and all $k \geqslant 1$;*
(2) $\gamma_{h,s}(b_q, c \oplus \widehat{b}_q^{\,-1} d_q) = \gamma_{h,s}(b_{q'}, c' \oplus \widehat{b}_q^{\,-1} d_{q'})$ *for all triples (V, h, s) of lattices V equipped with a homogeneous quadratic function h and a multiform $s \in (V^*)^{n+1}$;*
(3) $\gamma(G, q) = \gamma(G', q')$ *and* $\gamma(G, q + \widehat{b}_q(c_i)) = \gamma(G, q' + \widehat{b}_{q'}(c_i'))$, $i = 1, \dots, n$.

Remarks similar to those to Th. 3.1 also apply to Th. 3.2.

PROOF. Clearly if the pointed quadratic functions are isomorphic then the conditions are verified. Let us prove the converse. Suppose the conditions are satisfied. Then the conditions of Th. 3.1 are satisfied. Therefore, $(G, b_q, c \oplus \widehat{b}_q^{\,-1} d_q)$ and $(G', b_{q'}, c' \oplus \widehat{b_{q'}}^{\,-1} d_{q'})$ are isomorphic pointed linking pairings. Explicitly, let $\phi : G \to G'$ an isomorphism such that $\phi^* b_{q'} = b_q$ and $\phi(c) = c'$ and $\phi^* d_{q'} = d_q$. Replacing the triple (G', q', c') by the isomorphic triple $(G, \phi^* q', \phi^* c') = (G, \phi^* q', c)$, it is enough to show that (G, q, c) and $(G, \phi^* q', c)$ are isomorphic. Note that $d_{\phi^* q'} = \phi^* d_{q'} = d_q$. So we may assume that (G, q, c) and (G, q', c) are two pointed quadratic functions over the same associated bilinear linking pairing, with the same homogeneity defect and the same distinguished elements satisfying the conditions (1), (2) and (3). Let us construct an isomorphism between (G, q, c) and (G, q', c). Since q and q' are quadratic functions over the same nondegenerate linking pairing, they differ by some $\alpha \in G$: $q' = q + \widehat{b}_q(\alpha)$. The equality $d_q = d_{q'}$ implies that $2\alpha = 0$. Since $\gamma(G, q) = \gamma(G, q') = \gamma(G, q)\; e^{2\pi i q(\alpha)}$, we deduce that $q(\alpha) = 0$. Define a map $[\mathfrak{n}] : G \to \mathbb{Z}/2\mathbb{Z}$ by $b_q(\alpha, x) = \frac{[\mathfrak{n}](x)}{2} \bmod 1$ for all $x \in G$. Clearly $[\mathfrak{n}]$ is a homomorphism. Since $q(a) = 0$,

$$0 = q(2\alpha) = q(\alpha) + q(\alpha) + b_q(\alpha, \alpha) = b_q(\alpha, \alpha),$$

hence $[\mathfrak{n}](\alpha) = 0$. Consider the map

$$\psi : G \to G, \;\; x \mapsto x + \mathfrak{n}(x)\alpha$$

where $\mathfrak{n}(x) \in \mathbb{Z}$ is an arbitrary lift of $[\mathfrak{n}](x) \in \mathbb{Z}/2\mathbb{Z}$. Since

$$\begin{aligned} \psi^2(x) = \psi(x + \mathfrak{n}(x)\alpha) &= x + \mathfrak{n}(x)\alpha + \mathfrak{n}\left(x + \mathfrak{n}(x)\alpha\right)\alpha \\ &= x + \mathfrak{n}(x)\alpha + \mathfrak{n}(x)\alpha + \mathfrak{n}(x)\mathfrak{n}(\alpha)\alpha \\ &= x + \mathfrak{n}(x)\; 2\alpha + 0 \\ &= x + 0 \\ &= x, \end{aligned}$$

ψ is an involutive automorphism of G. Furthermore,

$$q(\psi(x)) = q(x + \mathfrak{n}(x)\alpha) = q(x) + \mathfrak{n}(x)\; b_q(x, \alpha) + q(\mathfrak{n}(x)\alpha) = q(x) + b_q(x, \alpha) + 0 = q'(x)$$

for any $x \in G$. Therefore $\psi^* q' = q$. Since

$$\begin{aligned}\gamma(G,q)\ e^{2\pi i q(c_i)} = \gamma(G, q + \widehat{b}_q(c_i)) = \gamma(G, q' + \widehat{b}_{q'}(c_i)) &= \gamma(G,q')\ e^{2\pi i q'(c_i)}\\ &= \gamma(G,q)\ e^{2\pi q'(c_i)},\end{aligned}$$

we deduce that $q(c_i) = q'(c_i) = q(c_i) + b_q(\alpha, c_i)$, hence $b_q(\alpha, c_i) = 0$. Hence

$$\psi(c_i) = c_i + \mathfrak{n}(c_i)\alpha = c_i$$

for all $i = 1, \ldots, n$. Therefore ψ is an isomorphism between (G, q, c) and (G, q', c) as desired. □

Remark 3.1. The system of invariants of Th. 3.2 is minimal in the sense that if one equality among the equalities of conditions (1)–(3) is not satisfied then there is a pair of nonisomorphic pointed quadratic functions satisfying all the other equalities.

Example 3.1. As an illustration of the previous remark, we point out that there exist nonisomorphic pointed quadratic functions (G, q, c) and (G, q', c') such that (G, q) and (G', q') are isomorphic and the associated pointed linking pairings (G, b_q, c) and $(G, b_{q'}, c')$ are isomorphic. Such an example is provided by

$$(\mathbb{Z}/16\mathbb{Z}, q(k \bmod 16) = \frac{x^2+4x}{32} \bmod 1, c = 1 \bmod 16)$$

$$\text{and}\quad (\mathbb{Z}/16\mathbb{Z}, q(k \bmod 16) = \frac{-7x^2+20x}{32} \bmod 1, c' = 3 \bmod 16).$$

The map $x \mapsto 3x$ provides the isomorphism between the associated pointed linking pairings, the map $x \mapsto 5x$ provides the isomorphism between the quadratic functions, but there is no isomorphism between the pointed quadratic functions. In terms of invariants, one checks that $\gamma(G, q + \widehat{b}_q(c)) \neq \gamma(G, q' + \widehat{b_{q'}}(c'))$. All other equalities in the statement of Th. 3.2 are satisfied.

Corollary 3.1. *Suppose that $c \in (2G)^n$. Two pointed quadratic functions (G, q, c) and (G', q', c') with distinguished n-tuples are isomorphic if and only if the following conditions are satisfied:*

(1) $\rho_p^k(b_q) = \rho_p^k(b_{q'})$ *for all prime p and all $k \geqslant 1$;*
(2) $\gamma_{h,s}(b_q, c \oplus \widehat{b}_q^{\,-1} d_q) = \gamma_{h,s}(b_{q'}, c' \oplus \widehat{b}_q^{\,-1} d_{q'})$ *for all triples (V, h, s) of lattices V equipped with a homogeneous quadratic function h and a multiform $s \in (V^*)^{n+1}$;*
(3) $\gamma(G, q) = \gamma(G', q')$.

Proof. Conditions (1) and (2) imply that $c' \in (2G)^n$. In the proof of Th. 3.2, the last equality of condition (3) is used only to ensure that $b_q(\alpha, c_i) = 0$. But the condition $c_i \in 2G$ already implies that equality for all $i = 1, \ldots, n$. □

Remark 3.2. Cor. 3.1 applies in particular if the underlying group has odd order.

Another special case worth considering is the case of pointed *homogeneous* quadratic functions: it turns out that this case is analogous to the case of pointed linking pairings. Fix a nondegenerate pointed homogeneous quadratic function (G, q, c) with a distinguished n-tuple $c \in G$. Consider a triple (V, f, s) where (V, f) is a bilinear lattice and s an element in $(V^*)^n$. We form the quadratic function on $V \otimes G$

$$f \otimes q + (\mathrm{id}_{V^*} \otimes \widehat{b}_q)(s \otimes c)$$

and denote by

$$\gamma_{f,s}(q, c) = \gamma\Big(V \otimes G, f \otimes q + (\mathrm{id}_{V^*} \otimes \widehat{b}_q)(s \otimes c)\Big) \tag{3.7}$$

the corresponding Gauss sum.

Corollary 3.2. *Two pointed homogeneous quadratic functions (G, q, c) and (G', q', c') with distinguished n-tuples are isomorphic if and only if the following conditions are satisfied:*

(1) $\rho_p^k(b_q) = \rho_p^k(b_{q'})$ *for all prime p and all $k \geqslant 1$;*
(2) $\gamma_{f,s}(q, c) = \gamma_{f,s}(q', c')$ *for all triples (V, f, s) of bilinear lattices (V, f) equipped with a multiform $s \in (V^*)^n$.*

Proof. Since q is homogeneous, $d_q = 0$, hence $\gamma_{h,s\oplus s_{n+1}}(b_q, c \oplus 0) = \gamma_{h,s}(b_q, c)$ for all triples (V, h, s) of lattices V equipped with a homogeneous quadratic function h and a multiform $s \in (V^*)^n$. Observe that

$$h \otimes b_q = b_h \otimes q$$

for any homogeneous quadratic function $h : V \to \mathbb{Z}$ and homogeneous quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$. Hence $\gamma_{h,s}(b_q, c) = \gamma_{b_h,s}(q, c)$. The second observation is that $\gamma(G, q) = \gamma_{f,0}(q, c)$ and $\gamma(G, q + \widehat{b}_q(c_i)) = \gamma_{f,1}(q, c_i)$ with the pointed bilinear lattice $V = \mathbb{Z}$, $f(1, 1) = 1$, $s = 0$ and $s = 1_{\mathbb{Z}}$ respectively. The result follows from Th. 3.2. $\square$

Remark 3.3. It is possible to give a proof of Corollary 3.2 along the lines of Th. 3.1. We leave it to the reader to prove directly that the isomorphism class of (G, q, c) is classified by condition (1) and the Gauss sums $\gamma_{f,s}(q, c)$ for all triples (V, f, s). Since $\gamma(G, q) = \gamma_{f,0}(q, c)$ for $V = \mathbb{Z}$, $f(1, 1) = 1$, this easily implies the result. As mentioned above, this line of proof is valid for homogeneous quadratic functions only.

As the particular case of quadratic functions with no distinguished element (or with trivial element), we recover the classification of quadratic functions [**12**, Th. 4.1].

Corollary 3.3. *Two quadratic functions (G, q) and (G', q') are isomorphic if and only if $\gamma(G, q) = \gamma(G', q')$ and there is an isomorphism ϕ of their associated linking pairings such that $d_{q'} \circ \phi = d_q$.*

Remark 3.4. As illustrated in Example 3.1, the classification of pointed quadratic functions cannot be recovered by Corollary 3.3 and Theorem 3.1 alone.

3.4. Linking groups and pointed linking groups. We generalize the notion of linking pairings to allow non torsion elements in the underlying group.

Let $n \geqslant 1$. First we extend slightly the definition linking pairings. A *linking group* is a pair (G, λ) where G is a finitely generated abelian group and $\lambda : \text{Tors } G \times \text{Tors } G \to \mathbb{Q}/\mathbb{Z}$ is a linking pairing. According to our terminology, a linking pairing is a torsion linking group (i.e., the underlying group G in the definition is a torsion group). If G has no torsion element, then the linking group (G, λ) reduces to the underlying group G. An *isomorphism* between linking groups (G, λ) and (G', λ') is an isomorphism $\phi : G \to G'$ of groups such that $\phi|_{\text{Tors } G} : \text{Tors } G \to \text{Tors } G'$ verifies $(\phi|_{\text{Tors } G})^*(\lambda') = \lambda$. In other words, an isomorphism of linking groups is a group isomorphism whose restriction to torsion induces an isomorphism of linking pairings.

Example 3.2. . Any bilinear lattice (V, f) induces a linking group $(G^f = \text{Coker } \widehat{f}, \lambda^f)$ by formula (**??**). (See §2.2.) This linking group is called the *discriminant linking group*.

Two equivalent lattices (V, f) and (V', f') induce isomorphic discriminant linking pairings if and only the induced nondegenerate bilinear lattices $(\bar{V}, \bar{f})$ and $(\bar{V}', \bar{f}')$ are stably equivalent; however the induced discriminant linking groups may be non isomorphic. A simple example is provided by $V = \mathbb{Z}$, $f(x, y) = 2xy$ and $V' = \mathbb{Z} \oplus \mathbb{Z}$ and $f'(x \oplus x', y \oplus y') = 2xy$. We see on this example that $\text{Coker}\widehat{f} = \mathbb{Z}/2\mathbb{Z}$ while $\text{Coker } \widehat{f'} = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Hence the linking groups are not isomorphic while the linking pairings are.

A *pointed linking group* is a triple (G, λ, c) where (G, λ) is a linking group and $c \in G^n$ is a distinguished n-tuple. (In contrast with the definition of pointed linking pairings, G is now allowed to have non torsion - including distinguished - elements.) An *isomorphism* of pointed linking groups is an isomorphism of the underlying linking groups sending the distinguished n-tuple to the distinguished n-tuple.

Example 3.3. The first integral homology of a 3-manifold endowed with distinguished elements provides a fundamental example of pointed linking group.

We now derive a lemma in order to deal with isomorphism of pointed linking groups.

The classification of pointed linking groups can be essentially reduced to the classification of pointed linking pairings by means of the following lemma. For a map $f : A \to B$, we denote by $f^{\otimes n} : A^n \to B^n$ the n-ary cartesian product map induced by f.

Lemma 3.4. *Let (G, λ, c) and (G', λ', c') be two pointed linking groups with distinguished n-tuples $c \in G^n$ and $c \in G'^n$ respectively. The following assertions are equivalent:*

(1) *There is an isomorphism of pointed linking groups*
$$(G, \lambda, c) \simeq (G', \lambda', c')$$

(2) *There are*
  (i) *a group isomorphism* $\nu : G/\mathrm{Tors}\ G \to G'/\mathrm{Tors}\ G'$ *such that* $\nu^{\otimes n}([c]) = [c']$;
  (ii) *two retractions* $r : G \to \mathrm{Tors}\ G$ *and* $r' : G' \to \mathrm{Tors}\ G'$ *of the natural inclusions into* G *and* G' *respectively and an isomorphism* ψ *of pointed linking pairings*

$$(\mathrm{Tors}\ G, \lambda, r^{\otimes n}(c)) \simeq (\mathrm{Tors}\ G', \lambda', r'^{\otimes n}(c')).$$

Proof. In order to lighten notation, since it is clear when n-ary cartesian product is meant, we suppress the superscript $^{\otimes n}$. (1) $\Rightarrow$ (2): clearly a pointed linking group isomorphism φ induces a linking group isomorphism $\varphi|_{\mathrm{Tors}\ G}$ between $(\mathrm{Tors}\ G, \lambda)$ and $(\mathrm{Tors}\ G', \lambda')$. Choose any retraction $r : G \to \mathrm{Tors}\, G$ of $i : \mathrm{Tors}\, G \to G$. Then $r' = \varphi|_{\mathrm{Tors}\ G} \circ r \circ \varphi^{-1}$ is a retraction of $i' : \mathrm{Tors}\ G' \to G'$. Then $\varphi|_{\mathrm{Tors}\ G}(r(c)) = r' \circ \varphi(c) = r'(c')$. (2) $\Rightarrow$ (1): let $p : G \to G/\mathrm{Tors}\ G,\ x \mapsto p(x) = [x]$ denote the canonical projection. Define similarly the canonical projection p' onto $G'/\mathrm{Tors}\ G'$. The map

$$(r, p) : G \to \mathrm{Tors}\ G \oplus G/\mathrm{Tors}\ G,\ x \mapsto (r(x), p(x) = [x])$$

is a group isomorphism. There is a similar isomorphism $(r', p') : G' \to \mathrm{Tors}\ G' \oplus G'/\mathrm{Tors}\ G'$. Define an isomorphism $\varphi : G \to G'$ by the following composition

$$G \xrightarrow{(r,p)} \mathrm{Tors}\ G \oplus G/\mathrm{Tors}\ G \xrightarrow{\psi \oplus \nu} \mathrm{Tors}\ G' \oplus G'/\mathrm{Tors}\ G' \xrightarrow{(r',p')^{-1}} G'.$$

Thus $\varphi = (r', p')^{-1} \circ (\psi \oplus \nu) \circ (r, p)$. By construction $\varphi|_{\mathrm{Tors}\ G} = \psi$ and thus it is an isomorphism of pointed linking pairings between $(\lambda, r(c))$ and $(\lambda, r'(c'))$. The isomorphism $\varphi : G \to G'$ induces an isomorphism $[\varphi] : G/\mathrm{Tors}\ G \to G'/\mathrm{Tors}\ G'$ defined by $[\varphi]([x]) = [\varphi(x)]$ for all $x \in G$. By construction $[\varphi] = \nu$. The formula $s([x]) = x - r(x),\ x \in G$ defines unambiguously a section s of the canonical projection $p : G \to G/\mathrm{Tors}\ G$. Define similarly a section s' of the canonical projection $p' : G' \to G'/\mathrm{Tors}\ G'$ by $s'([x']) = x' - r'(x')$, $x' \in G'$. Then

$$\varphi \circ s([x]) = \varphi(x - r(x)) = \varphi(x) - \varphi(r(x)) = \varphi(x) - r' \circ \varphi(x) = s' \circ [\varphi]([x]),$$

thus $\varphi \circ s = s' \circ [\varphi]$. It follows that

$$\begin{aligned} \varphi(c) = \varphi(r(c) + c - r(c)) = \varphi(r(c)) + \varphi(s([c])) &= r'(c') + s'([\varphi]([c])) \\ &= r'(c') + s'(\nu([c])) \\ &= r'(c') + s'([c']) \\ &= r'(c') + c' - r'(c') \\ &= c'. \end{aligned}$$

□

Corollary 3.4. *Two linking groups* (G, λ) *and* (G', λ') *are isomorphic if and only if the groups* G *and* G' *are isomorphic and the linking groups* $(\mathrm{Tors}\ G, \lambda)$ *and* $(\mathrm{Tors}\ G', \lambda')$ *are isomorphic.*

3.5. The isomorphism problem for pointed lattices. We now give a necessary and sufficient condition to solve the isomorphism problem for pointed lattices (pointed linking groups with trivial linking pairing and abelian free group). This is the Proposition 3.1 below. This condition will be used in the sequel to manufacture invariants of linking groups.

Let V be a lattice. The linear group $\mathrm{GL}(V)$ acts naturally on V in the usual way. Extend diagonally this action to any n-ary cartesian power of V.

PROPOSITION 3.1. *Let V be a lattice. Two n-tuples $x = (x_1, \ldots, x_n) \in V^n$ and $y = (y_1, \ldots, y_n) \in V^n$ lie in the same orbit of $\mathrm{GL}_n(V)$ if and only if*

(3.8)
$$\text{for any } N \in \mathbb{Z}, \text{ for any } (a_1, \ldots, a_n) \in \mathbb{Z}^n, \quad \sum_j a_j\, x_j \in N{\cdot}V \;\Leftrightarrow\; \sum_{j\in J} a_j\, y_j \in N{\cdot}V.$$

To prove Proposition 3.1, we need a number of lemmas.

LEMMA 3.5. *Let $x_1, \ldots, x_n$ be $\mathbb{Z}$-independent elements in a lattice V. The sublattice S generated by $x_1, \ldots, x_n$ is primitive if and only if*

(3.9)
$$\text{for all } (a_1, \ldots, a_n) \in \mathbb{Z}^n \text{ with } \gcd(a_1, \ldots, a_n) = 1, \; \sum_i a_i x_i \in V \setminus \bigcup_{k>1} k \cdot V.$$

PROOF. Suppose that there is $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $\gcd(a_1, \ldots, a_n) = 1$ such that $\sum_i a_i x_i \in k{\cdot}V$ for some $k > 1$. Let $y = \frac{1}{k}\sum_i a_i x_i \in V$. By hypothesis, $k \cdot y = \sum_i a_i x_i \in S$. We claim that $y \notin S$. Otherwise since the x_i's are independent, $k|a_i$ for all i, which contradicts the fact that $\gcd(a_1, \ldots, a_n) = 1$. Hence S is not primitive.

Conversely, suppose that S is not primitive. There exists $y \in V \setminus S$ such that $k \cdot y \in S$ for some $k > 1$. Consider the smallest integer $k > 1$ realizing this condition. Then there exist $b_1, \ldots, b_n$ such that $k \cdot y = \sum_i b_i x_i$. Let $l = \gcd(b_1, \ldots, b_n)$. Since $y \notin S$ and by minimality of k, the integers k and l are coprime. Let $a_i = b_i/l$, $i = 1, \ldots, n$. By construction, $\gcd(b_1, \ldots, b_n) = 1$. We have
$$ky = l \underbrace{\sum_i a_i x_i}_{\in S}.$$
Since k and l are coprime, $\sum_i a_i x_i \in k \cdot V$. This is the desired result. ☐

LEMMA 3.6. *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two families of independent elements in V satisfying the condition* (3.8). *Let S and S' be the sublattices generated by $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. Let $c_{ij} \in \mathbb{Q}$ be rational numbers, $1 \leqslant i, j \leqslant n$. The primitive hull of S is generated by $x'_i = \sum_j c_{ij} x_j$ if and only the primitive hull of S' is generated by $y'_i = \sum_j c_{ij} y_j$, $1 \leqslant i \leqslant n$.*

PROOF. Consequence of the previous lemma. ☐

LEMMA 3.7. *With the same hypothesis and notation as in the previous lemma, there exists an automorphism $\varphi : V \to V$ such that $\varphi(x_i) = y_i$ for all i.*

PROOF. Define $\varphi(x'_i) = y'_i$, $1 \leqslant i \leqslant n$. This defines an isomorphism between the respective primitive hulls $\widetilde{S}$ and $\widetilde{S}'$. By Lemma 2.2, this isomorphism extends to an automorphism $\widetilde{\varphi} : V \to V$. Let x'_i, $1 \leqslant i \leqslant n$, be the generators of $\widetilde{S}$. There are rational numbers c_{ij} $(1 \leqslant i, j \leqslant n)$ such that $x'_i = \sum_j c_{ij} x_j$. The matrix $C = (c_{ij})_{1 \leqslant i,j \leqslant n}$ is invertible over $\mathbb{Q}$. Since some multiple of each x'_j lies in S, we deduce that the inverse matrix $C^{-1} = (d_{ij})_{1 \leqslant i,j \leqslant n}$ has integral coefficients. By the hypothesis, $y'_i = \sum_j c_{ij} x'_j$, $1 \leqslant i, j \leqslant n$, form a $\mathbb{Z}$-basis of generators for $\widetilde{S}'$. Therefore,

$$\begin{aligned} \varphi(x_i) &= \varphi\left(\sum_j d_{ij} x'_j\right) \\ &= \sum_j d_{ij} \varphi(x'_j) \\ &= \sum_j d_{ij} y'_j \\ &= y_i. \end{aligned}$$

This is the desired result. □

PROOF OF PROPOSITION 3.1. Necessity is clear. Let J be a maximal subset of $\{1, \ldots, n\}$ such that the elements x_j, $j \in J$, are independent over $\mathbb{Z}$. Then the relation for $N = 0$ shows that the elements y_j, $j \in J$, are also independent over $\mathbb{Z}$. Applying the previous lemma yields an automorphism $\varphi \in \mathrm{GL}(V)$ such that $\varphi(x_i) = y_i$ for all $i \in J$. Let $k \notin J$. There is a relation

$$\sum_{j \in J} a_j \; x_j + a_k \; x_k = 0,$$

for some $a_k \neq 0$. Then

$$\sum_{j \in J} a_j \; y_j + a_k \; y_k = 0. \tag{3.10}$$

Since $y_j = \varphi(x_j)$ for $j \in J$, we deduce that

$$0 = \varphi\left(\sum_{j \in J} a_j \; x_j + a_k \; x_k\right) = \sum_{j \in J} a_j \; y_j + a_k \; \varphi(x_k).$$

Comparing this equality to (3.10), we deduce that

$$a_k \; y_k = a_k \; \varphi(x_k).$$

Since $a_k \neq 0$ and since V is torsion free, $y_k = \varphi(x_k)$. □

3.6. The stable classification. We begin by recalling the classical results on the stable classification of lattices. Recall from §2.2 that by definition, two bilinear lattices are stably equivalent if they become isomorphic after adding to them some unimodular orthogonal summands. Furthermore, the map

$$(V, f) \mapsto (G_f, \lambda_f)$$

induces a bijective correspondence between stable equivalence classes of nondegenerate bilinear lattices and isomorphism classes of nondegenerate linking pairings (Th. 2.2).

The aim of this paragraph is to extend this result to the more general setting of pointed linking groups.

A *pointed bilinear lattice* V is a bilinear lattice equipped with a finite ordered collection c of elements $c_1, \ldots, c_n \in V^*$. Pointed bilinear lattices form a monoid for the orthogonal sum $\oplus$. Let (V, f, c) and (V, f, c) be two pointed bilinear lattices. A *weak isomorphism* between them is an isomorphism $\psi : (V, f) \to (V', f')$ of bilinear lattices such that $\psi^*(c') = c \bmod \widehat{f}(V)$. For instance, if the bilinear lattices are isomorphic in the usual sense and if the distinguished elements c and c' lie in $\widehat{f}(V)$ and $\widehat{f}(V')$ respectively, then the pointed bilinear lattices are weakly isomorphic.

Example 3.4. Let (U, h) and (U', h') be unimodular bilinear lattices. If $\varphi : (U, h) \to (U', h')$ is an isomorphism of bilinear lattices, then $\varphi : (U, h, u) \to (U', h', u')$ is a weak isomorphism of pointed lattices for any $u \in U^n$ and any $u' \in U'^n$.

We say that (V, f, c) and (V', f', c') are *stably equivalent* if there exist pointed unimodular lattices (U, h, u) and (U', h', u') such that $(V, f, c) \oplus (U, h, u)$ and $(V', f', c') \oplus (U', h', u')$ are weakly isomorphic as pointed bilinear lattices.

Clearly stably equivalent pointed lattices induce isomorphic pointed linking groups. The main observation of this section lies in the converse and generalizes Th. 2.2.

Theorem 3.3. *Two pointed bilinear lattices (V, f, c) and (V', f', c') are stably equivalent if and only if there is an isomorphism*

$$\phi : (\mathrm{Coker}\ \widehat{f}, \lambda^f, [c]) \to (\mathrm{Coker}\ \widehat{f}', \lambda^{f'}, [c'])$$

of the induced discriminant linking groups. In fact, any isomorphism $\phi : (H, \lambda, x) \to (H', \lambda', x')$ of linking groups lifts to a stable equivalence of pointed bilinear lattices.

Proof. Let $\mathfrak{s} : (V, f, c) \oplus (U, h, u) \to (V', f', c') \oplus (U', h', u')$ be an explicit weak isomorphism realizing the two stable equivalence between the pointed bilinear lattices (V, f, c) and (V', f', c'). Since unimodular pointed lattices are sent via the discriminant construction to trivial linking pairing (with trivial distinguished elements), this isomorphism induces an isomorphism

$$\mathrm{Coker}\ (\widehat{f' \oplus h'}) = \mathrm{Coker}\ \widehat{f}' \to \mathrm{Coker}\ \widehat{f} = \mathrm{Coker}\ \widehat{f \oplus h}$$

whose restriction to $G^{f'} = \mathrm{Tors\ Coker}\ \widehat{f}$ is an isomorphism of the linking pairings $\lambda^{f'}$ and λ^f.

Conversely, it suffices to prove the last statement. Let $\phi : (H, \lambda, x) \to (H', \lambda', x')$ be an isomorphism of linking groups. Set $G = \mathrm{Tors}\ H$ and $G' = \mathrm{Tors}\ H'$. Using [**14**, Proof of Th. 4.1], lift $\phi|_G : G \to G'$ to a stable equivalence $\mathfrak{s}$ between nondegenerate bilinear lattices (V, f) and (V', f'). Consider the following commutative diagram of extension of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r, "i"] \arrow[d, "\phi|_G"] & H \arrow[r, "p"] \arrow[d, "\phi", "\simeq"'] & F \arrow[r] \arrow[d, "{[\phi]}"] & 0 \\ 0 \arrow[r] & G' \arrow[r, "i"] & H' \arrow[r, "p'"] & F' \arrow[r] & 0 \end{tikzcd}$$

where i is the canonical inclusion and $[\phi] : F \to F'$ is the isomorphism induced by ϕ. The two horizontal short exact sequences are split. Choose a section $s : F \to H$ of p. Then $s' = \phi \circ s \circ [\phi]^{-1} : F' \to H'$ is a section of p'. Define

$$\tilde{V} = V \oplus s(F)^*, \quad \tilde{f} = f \oplus 0$$

and similarly $\tilde{V'} = V' \oplus s'(F')^*, \tilde{f'} = f' \oplus 0$. We see that $\tilde{\phi} = \mathfrak{s} \oplus \phi|_{s(F)}$ is an isomorphism between the lattices $(\tilde{V}, \tilde{f})$ and $(\tilde{V'}, \tilde{f'}$ inducing the isomorphism $\phi : (H, \lambda) \to (H', \lambda')$. Furthermore, let $\tilde{x} \in \tilde{V}^* = V^* \oplus s(F)$ and $\tilde{x'} \in \tilde{V'}^* = V'^* \oplus s'(F')$ be lifts of $x \in H = G \oplus s(F)$ and $x \in H' = G' \oplus s'(F')$ respectively. Clearly $\tilde{\phi}^*$ sends $\tilde{x'}$ to $\tilde{x}$ mod $\widehat{\tilde{f}}(\tilde{V})$. Therefore, $\tilde{\phi}$ is a suitable stable equivalence between pointed bilinear lattices lifting the isomorphism between group linkings (H, λ, x) and (H', λ', x'). □

In practice, the following corollary is useful. It shows that in the process of stabilization, one can restrict to a particular pointed unimodular lattice. Denote by ± 1 the bilinear lattice on $\mathbb{Z}$ sending $(1, 1)$ to ± 1.

Corollary 3.5. *Two n-pointed bilinear lattices (V, f, c) and $V', f', c')$ are stably equivalent if and only if they are related by a finite sequence*

$$(V, f, c) \overset{o_1}{\dashrightarrow} \cdots \overset{o_r}{\dashrightarrow} \cdots \overset{o'_r}{\dashleftarrow} \cdots \overset{o'_1}{\dashleftarrow} (V', f', c')$$

of the following two operations:

(i) *lattice isomorphisms;*
(ii) *orthogonal sum with* $(\mathbb{Z}, \pm 1, 0)$.

Let us state the particular case of torsion groups.

Corollary 3.6. *Two pointed nondegenerate bilinear lattices (V, f, c) and (V', f', c') are stably equivalent if and only if the discriminant linking pairings (G_f, λ_f) and $(G_{f'}, \lambda_{f'})$ are isomorphic. Furthermore, any isomorphism between two nondegenerate pointed linking pairings (G, λ, x) and (G', λ', x') can be lifted to a stable equivalence of pointed bilinear lattices $(V, f, \tilde{x})$ and $(V', f', \tilde{x'})$.*

Remark 3.5. The notion of stable equivalence generalizes the notion of stable equivalence for bilinear lattices (without distinguished element) and is weaker than the notion of strongly stable equivalence defined for bilinear lattices equipped with Wu classes. It is indeed clear from the definition that if two triples (V, f, c) and (V', f', c') of bilinear lattices equipped with distinguished elements $v \in V^*$ and $v' \in V'^*$ that happen to be image of Wu classes (by $\widehat{f}_{\mathbb{Q}}$ and $\widehat{f'}_{\mathbb{Q}}$ respectively) are strongly stably equivalent, then they are stably equivalent. See Th. 2.1 and Th. 2.5 respectively.

3.7. The vanishing condition. We study precisely the condition for a generalized Gauss sum to vanish, in terms of characteristic elements. We keep notation from the previous paragraphs.

LEMMA 3.8. *$\gamma(G, q)$ is nonzero if and only if $q(G^{\perp}) = 0$. If this condition is satisfied then $|\gamma(G, q)| = 1$.*

See for example [**6**, Lemma 1.1, §2.3] for a proof. In particular:

COROLLARY 3.7. *If q is nondegenerate ($G^{\perp} = 0$), then $\gamma(G, q)^{-1} = \overline{\gamma(G, q)}$.*

The study of possibly degenerate quadratic functions is necessary, however, in many contexts. For instance, the tensor product of a nondegenerate homogeneous quadratic function and a nondegenerate bilinear lattice may be degenerate.

EXAMPLE 3.5. The tensor product of the nondegenerate quadratic form defined by $x \mapsto x^2/4$, $x \in \mathbb{Z}/2$ and the nondegenerate bilinear lattice defined by $(x, y) \mapsto 2xy$, $x \in \mathbb{Z}$, is degenerate.

Note that, in the general case, $q|_{G^{\perp}}$ is a homomorphism $G \to \mathbb{Q}/\mathbb{Z}$.

Let $f : V \times V \to \mathbb{Z}$ and $g : W \times W \to \mathbb{Z}$ be two nondegenerate bilinear lattices. Endow $f \otimes g$ with a Wu class $z \in (V \otimes W)^{\sharp}$. Recall that the subgroups $A = j_f(G_f \otimes W)$ and $B = j_g(V \otimes G_g)$ are mutually orthogonal in $G_{f\otimes g}$ with respect to the discriminant linking pairing $\lambda_{f\otimes g}$ (See §2.2, Lemma 2.9). As before, we set $H = A \cap B = A \cap A^{\perp}$.

We apply Lemma 3.8 to the quadratic function $\varphi_{f\otimes g,z} \circ j_f$ on $G_f \otimes W$. We deduce from Lemma 2.8 that $(G_f \otimes W, \varphi_{f\otimes g,z} \circ j_f) \simeq (A, \varphi_{f\otimes g,z}|_A)$. The next result is an immediate consequence:

THEOREM 3.4. *The following assertions are equivalent:*

(1) $\gamma(G_f \otimes W, \varphi_{f\otimes g,z} \circ j_f) \neq 0$.
(2) $\gamma(V \otimes G_g, \varphi_{f\otimes g,z} \circ j_g) \neq 0$.
(3) $\varphi_{f\otimes g,z}|_H = 0$.
(4) *$\psi([t])$ is the characteristic element associated to $\lambda_f \otimes \lambda_g$ for $z = z_0 + 2t$ with $z_0 \in \mathrm{Wu}^Z(f \otimes g)$, $t \in V^{\sharp} \otimes W^{\sharp}$, $[t] \in G_{f\otimes g}/H^{\perp}$ and ψ is the natural isomorphism $G_{f\otimes g}/H^{\perp} \to G_f \otimes G_g$.*
(5) *The Wu class $z \in \mathrm{Wu}(f\otimes g)$ is sent to 0 under the natural projection $\mathrm{Wu}(f \otimes g) \to \mathrm{Wu}(f \otimes g)/2(V^{\sharp} \otimes W + V \otimes W^{\sharp})$.*

PROOF. Since j_f is injective (Lemma 2.8), $(G_f \otimes W, \varphi_{f\otimes g,z} \circ j_f) \simeq (A, \varphi_{f\otimes g,z}|_A)$. The annihilator of $\varphi_{f\otimes g,z}|_A$ is $A \cap A^{\perp} = H$. Therefore, applying Lemma 3.8 to the quadratic function $\varphi_{f\otimes g,z}|_A$ gives the equivalence (1) $\iff$ (3). A similar argument yields (2) $\iff$ (3). The equivalence (3) $\iff$ (4) is Theorem 2.10. The equivalence (3) $\iff$ (5) follows from Lemma 2.23 (§2.4). $\square$

Fix a Wu class $z \in \mathrm{Wu}(f \otimes g)$. Recall (§2.4) that $(V \otimes W)^{\sharp}$ acts freely and transitively on $\mathrm{Wu}(f \otimes g)$. Therefore, there exists a unique triple (v, w, ξ)

such that $v \in \mathrm{Wu}^V(f)$, $w \in \mathrm{Wu}^W(g)$ and $\xi \in (V \otimes W)^\sharp$ such that

$$z = v \otimes w + 2\xi.$$

It follows from (2.9) that

$$\varphi_{f\otimes g,z}|_H = (\varphi_f \otimes g) \circ j_f^{-1}|_H - \lambda_{f\otimes g}([\xi], -)|_H$$

where $\lambda_{f\otimes g}$ denotes the (induced) bilinear pairing $G/H^\perp \times H \to \mathbb{Q}/\mathbb{Z}$ (see Lemma 2.23). Suppose now that z verifies one of the conditions of Th. 3.4. Then $\varphi_{f\otimes g,z}|_H = 0$. Thus

$$(\varphi_{f,v} \otimes g) \circ j_f^{-1}|_H = \lambda_{f\otimes g}([\xi], -)|_H.$$

Observe that the homomorphism $H \to \mathbb{Q}/\mathbb{Z}$, $x \mapsto (\varphi_{f,v} \otimes g) \circ j_f^{-1}(x)$ has order 2 in H^*: this follows from Theorem 2.6. [Alternative proof: it follows from Lemma 2.12 that $(\varphi_{f,v} \otimes g) \circ j_f^{-1}|_H = (\varphi_{f,v} \otimes g)|_{\mathrm{Ker}(\widehat{\lambda}_f \otimes \widehat{g})}$. Note that $2 \cdot \varphi_{f,v}(x) = \lambda_f(x,x)$ since $\varphi_{f,v}$ is homogeneous. Hence

$$\begin{aligned} 2 \cdot (\varphi_{f,v} \otimes g) \circ j_f^{-1}(H) &= 2 \cdot (\varphi_{f,v} \otimes g)(\mathrm{Ker}(\widehat{\lambda}_f \otimes \widehat{g})) \\ &= (2\varphi_{f,v} \otimes g)(\mathrm{Ker}(\widehat{\lambda}_f \otimes \widehat{g})) = (\tilde{\lambda}_f \otimes g)(\mathrm{Ker}(\widehat{\lambda}_f \otimes \widehat{g})) = 0 \end{aligned}$$

where $\tilde{\lambda}_f$ denotes the map $x \mapsto \lambda_f(x,x)$.] Therefore, since the bilinear pairing $G/H^\perp \times H \to \mathbb{Q}/\mathbb{Z}$ is nondegenerate, $2[\xi] = 0$ in $G/H^\perp \simeq G_f \otimes G_g$. This is equivalent to $2\xi \in V^\sharp \otimes W + V \otimes W^\sharp$.

Hence we have proved:

Corollary 3.8. *If a Wu class $z \in \mathrm{Wu}(f \otimes g)$ satisfies $\varphi_{f\otimes g,z}|_H = 0$ then $z \in \mathrm{Wu}^{1/2}(f \otimes g)$.*

4. Reciprocity

We derive a formula in the Witt group of torsion quadratic functions; presented as an alternative, it generalizes all previously known formulas of reciprocity. We keep notation from the previous paragraphs. The formula can be regarded as a far-reaching generalization of the classical Van der Blij formula. Hence we begin the Van der Blij formula first.

4.1. The van der Blij formula. In 1959, F. van der Blij stated a formula relating the bilinear lattice to its discriminant function. In short, this is a computation of the Gauss sum associated to a discriminant quadratic function. Since any quadratic function is a discriminant quadratic function (Th 2.4), this computation applies to any (nondegenerate) finite quadratic function. The computation is explicit in terms of the bilinear lattice lying over the quadratic function and yields a fundamental invariant of the quadratic function. The applications of this beautiful formula, many times rediscovered, lie in algebra and topology.

Theorem 4.1. *Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate quadratic function on a finite abelian group. Let (V, f, v) be any (nondegenerate) bilinear lattice* *equipped with a Wu class $v \in V^\sharp$ such that $(G_f, \varphi_{f,v}) = (G, q)$. Then*

$$\gamma(G_f, \varphi_{f,v}) = \exp\Big(\frac{2\pi i}{8}(\operatorname{sign}(f) - f_{\mathbb{Q}}(v,v))\Big). \tag{4.1}$$

Here $\operatorname{sign}(f)$ denotes the signature of the lattice $(V, f) \otimes \mathbb{R}$. It follows that the rational residue $\operatorname{sign}(f) - f_{\mathbb{Q}}(v,v)$ mod 8 is an invariant of (G, q) and will be denoted $\beta(q) \in \mathbb{Q}/8\mathbb{Z}$.

As hinted above, the applications in algebra and topology are just too numerous to list. We shall content ourselves with a few obvious observations and consequences:

1. If q is homogeneous then $\beta(q) \in \mathbb{Z}/8\mathbb{Z}$.
2. If (V, f) is unimodular then $\operatorname{sign}(f) = f(v, v)$ mod 8 for any Wu class $v \in V$.

A complete proof of the formula above can be adapted for instance from the original paper [**3**] or from [**43**].

4.2. Statement. A quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group is *metabolic* if there exists a subgroup $H \subseteq G$ such that $H^\perp = H$ and $q|_H = 0$. Two quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ are *Lagrange-related* if the quadratic function $q \oplus (-q')$ on $G \oplus G'$ is metabolic. This defines an equivalence relation on the monoid of quadratic functions on finite abelian groups; furthermore, the set of all equivalence classes forms a group $\mathfrak{WQ}$, the *Witt group* of quadratic functions on finite abelian groups (see [**55**, Chap. 5, §1]). The addition is induced by orthogonal sum.

It is not hard to see that the Gauss sum $\gamma(G, q)$ associated to a quadratic function is actually an invariant of its Witt class. The general reciprocity formula below is an identity between Witt classes.

Let $f : V \times V \to \mathbb{Z}$ and $g : W \times W \to \mathbb{Z}$ be two nondegenerate bilinear lattices. Endow $f \otimes g$ with a Wu class $z \in (V \otimes W)^\sharp$. Recall that the subgroups $A = j_f(G_f \otimes W)$ and $B = j_g(V \otimes G_g)$ are mutually orthogonal in $G_{f\otimes g}$ with respect to the discriminant linking pairing $\lambda_{f\otimes g}$. As before, we set $H = A \cap B = A \cap A^\perp$.

Theorem 4.2 (Reciprocity). *The following alternative holds: either $\varphi_{f\otimes g,z}$ is not identically zero on H or*

$$[G_{f\otimes g}, \varphi_{f\otimes g,z}] = [G_f \otimes W, \varphi_{f\otimes g,z} \circ j_f] + [V \otimes G_g, \varphi_{f\otimes g,z} \circ j_g] \text{ in } \mathfrak{WQ}. \tag{4.2}$$

A corollary of Th. 4.2 is the following reciprocity formula [**62**, 1.3]. Below bar denotes complex conjugation.

Corollary 4.1.

$$\gamma(G_f \otimes W, \varphi_{f\otimes g,z} \circ j_f) = \gamma(G_{f\otimes g}, \varphi_{f\otimes g,z}) \cdot \overline{\gamma(V \otimes G_g, \varphi_{f\otimes g,z} \circ j_g)}. \tag{4.3}$$

Proof of Corollary 4.1. If $\varphi_{f\otimes g,z}|_H$ is not identically zero on H then Th. 3.4 implies that both sides of (4.3) are zero:

$$\gamma(G_f \otimes W, \varphi_{f\otimes g,z} \circ j_f) = 0 = \gamma(V \otimes G_g, \varphi_{f\otimes g,z} \circ j_g).$$

Suppose that $\varphi_{f\otimes g,z}|_H$ is identically zero. Then the three Gauss sums appearing in (4.3) are nonzero. It is well known that the Gauss map γ induces a homomorphism $\mathfrak{WQ} \to \mathbb{C}^\times$ (e.g., [**55**, §2]). Applying γ to the relation (4.2) and using the fact that $\gamma(G,q)^{-1} = \overline{\gamma(G,q)}$ yields the desired relation. □

Corollary 4.2. *Let $v \in \mathrm{Wu}(f)$ and $w \in \mathrm{Wu}(g)$. For any $v_0 \in \mathrm{Wu}^V(f)$ and $w_0 \in \mathrm{Wu}^W(g)$,*

(4.4)
$$\gamma\left(G_f \otimes W, \varphi_{f,v_0} \otimes g + (\widehat{\lambda}_f \otimes g)(\left[\tfrac{v_0-v}{2}\right] \otimes w)\right) = \gamma(G_{f\otimes g,v\otimes w})\ \overline{\gamma\left(V \otimes G_g, f \otimes \varphi_{g,w_0} + (\widehat{f} \otimes \widehat{\lambda}_g)(v \otimes \left[\tfrac{w_0-w}{2}\right])\right)}.$$

Proof. Apply Cor. 4.1 to the case $z = v \otimes w$. Noting that

$$v \otimes w = v_0 \otimes w + (v - v_0) \otimes w = v \otimes w_0 + v \otimes (w - w_0)$$

with $v - v_0 \in 2V^\sharp$ and $w - w_0 \in 2W^\sharp$, we have, according to (2.9),

$$\varphi_{f\otimes g,v\otimes w} \circ j_f = \varphi_{f,v_0} \otimes g - (\lambda_f \otimes g)(\left[\tfrac{v-v_0}{2}\right] \otimes w, -)|_{G\otimes W}$$

and, according to (2.10),

$$\varphi_{f\otimes g,v\otimes w} \circ j_g = f \otimes \varphi_{g,w_0} - (f \otimes \lambda_f)(v \otimes \left[\tfrac{w-w_0}{2}\right], -)|_{V\otimes G_g}.$$

This yields the desired result. □

Corollary 4.3. [**6**, Th. 3] *For any $v_0 \in \mathrm{Wu}^V(f)$ and $w_0 \in \mathrm{Wu}^W(g)$,*

$$\gamma(G_f \otimes W, \varphi_{f,v_0} \otimes g) = \gamma(G_{f\otimes g,v\otimes w})\ \overline{\gamma(V \otimes G_g, f \otimes \varphi_{g,w_0})}. \tag{4.5}$$

Proof. Cor. 4.2 with integral Wu classes $v = v_0$ and $w = w_0$. □

For the proof of the reciprocity, we refer to [**13**].

CHAPTER 3

The Weil representation of a finite abelian group

In 1964, in a remarkable paper [**65**], André Weil constructed a unitary representation associated to a symplectic locally compact abelian group. In a few decades the Weil representation has appeared to be a central object in mathematics, lying at the crossroads between the theory of theta functions, number theory, harmonic analysis and quantum mechanics.

In the third paragraph of his celebrated paper, André Weil makes in passing the following remark: "qu'il me soit permis, en passant, de signaler l'intérêt qu'il y aurait peut-être à examiner de plus près, du point de vue de la présente théorie, le cas des groupes finis." ("Let me mention in passing the interest that might lie in studying more closely, from the viewpoint of the present theory, the case of finite groups.")

The exact intent of Weil is not immediately clear, aside from a note to a paper by H. Kloosterman. One possible interpretation is that the existence of the Weil representation for finite abelian groups is closely related to the existence of Abelian Topological Quantum Field Theories.

1. The Heisenberg group

1.1. The Heisenberg group associated to a lattice. Let $\omega : V \times V \to \mathbb{Z}$ be a nondegenerate symplectic lattice. We choose a form $\beta : V \times V \to \mathbb{Z}$ such that

$$\beta(x,y) - \beta(y,x) = \omega(x,y),\ \forall x, y \in V. \tag{1.1}$$

Such a form will be called a *Seifert form*. The motivation for this terminology comes from the following example.

Example 1.1. Let $\Sigma \subset S^3$ be an oriented smooth surface. Choose a bicollar $\mathrm{Int}(\Sigma) \times [0,1] \subset S^3 - \partial\Sigma$. For a 1-cycle x representing an element x in $H_1(\Sigma)$, denote by x^+ (resp. x^-) the 1-cycle representative corresponding to $x \times 1$ (resp. $x \times 0$) in the bicollar. The *Seifert form* is a bilinear pairing $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ defined by

$$\beta(x,y) = \mathrm{lk}(x, y^+) \in \mathbb{Z}, \quad x, y \in H_1(\Sigma),$$

where lk denotes the usual symmetric linking pairing of cycles in S^3. We have

$$\beta(x,y) - \beta(y,x) = \mathrm{lk}(x, y^+ - y^-) = x \bullet y$$

where $\bullet$ denotes the intersection pairing on Σ. Hence β satisfies the relation (1.1) where ω is the intersection pairing.

The *Heisenberg group* $\mathscr{H}(V)$ associated to (V, β) is defined the set $V \times \mathbb{Z}$ endowed with the multiplication rule

$$(x, t) \cdot (y, t') = (x + y, t + t' + \beta(x, y)).$$

The short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{H}(V) \longrightarrow V \longrightarrow 0. \tag{1.2}$$

is non-split unless $\omega = 0$. The *symplectic group* $\mathrm{Sp}(V)$ is the subgroup of automorphisms of V leaving ω invariant. The *affine symplectic group* $\mathrm{ASp}(V)$ is the group of automorphisms of $\mathscr{H}(V)$ acting trivially on the center Z. It fits into the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(V, \mathbb{Z}) \longrightarrow \mathrm{ASp}(V) \longrightarrow \mathrm{Sp}(V) \longrightarrow 1 \tag{1.3}$$

which is non-split unless $\omega = 0$.

1.2. The Heisenberg group associated to a finite group. Let A be a finite abelian group. A *symplectic form* on A is a nondegenerate alternate bilinear pairing $\omega : A \times A \to \mathbb{Q}/\mathbb{Z}$. We say that (A, ω) is a symplectic abelian group. We choose a form $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ such that

$$\beta(x, y) - \beta(y, x) = \omega(x, y), \ \forall x, y \in A. \tag{1.4}$$

By analogy with classical knot theory (see Example 1.1 above), we still call the form β above satisfying (1.4) a *Seifert form* associated to the symplectic form ω. Note that there is no uniqueness of the Seifert form β for a given symplectic form ω: adding a symmetric bilinear pairing to a Seifert form produces another Seifert form.

The *Heisenberg group* is the extension $\mathscr{H}(A) = \mathscr{H}_\beta(A)$ associated to (A, β): it is the set $A \times \mathbb{Q}/\mathbb{Z}$ equipped with the group law

$$(x, t) \cdot (y, t') = (x + y, t + t' + \beta(x, y)).$$

The Heisenberg group fits into the short exact sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow \mathscr{H}(A) \longrightarrow A \longrightarrow 0. \tag{1.5}$$

The center Z of $\mathscr{H}(A)$ is $Z = 0 \times \mathbb{Q}/\mathbb{Z}$.

LEMMA 1.1. *The Heisenberg group $\mathscr{H}(A)$ is a split extension of A if and only if $\mathscr{H}(A)$ is a direct product of A and $\mathbb{Q}/\mathbb{Z}$) if and only if $\omega = 0$.*

PROOF. If $\omega = 0$ then $\mathscr{H}(A) = A \times \mathbb{Q}/\mathbb{Z}$ with direct product group structure. Conversely, suppose that the extension $\mathscr{H}(A)$ is split. There exists then a section $s : A \to \mathscr{H}(A), s(a) = (s_1(a), s_2(a))$, with $s_1 : A \to A$ and $s_2 : A \to \mathbb{Q}/\mathbb{Z}$. Since s is a section, we must have $s_1(a) = a$. Since s is a group homomorphism,

$$s_2(a + a') = s_2(a) + s_2(a') + \beta(a, a'), \quad \forall a, a' \in A.$$

This implies that β is symmetric, which in turn implies that $\omega = 0$. $\square$

Let

$$\mathrm{ASp}(A) = \{s \in \mathrm{Aut}(\mathscr{H}(A)) \mid s|_Z = \mathrm{Id}_Z\}. \tag{1.6}$$

Let $\mathrm{Sp}(A)$ denote the symplectic group of (A, ω), that is, the subgroup of automorphisms of A leaving ω invariant. Note that $\mathrm{Sp}(A)$ is finite. It is not hard to see that $\mathrm{ASp}(A)$ is an extension of $\mathrm{Sp}(A)$ that fits into the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{ASp}(A) \longrightarrow \mathrm{Sp}(A) \longrightarrow 1. \tag{1.7}$$

(In particular, $\mathrm{ASp}(A)$ is finite.) More exactly, elements in $\mathrm{ASp}(A)$ can be presented as ordered pairs (s, q), $s \in \mathrm{Aut}(A)$, $q : A \to \mathbb{Q}/\mathbb{Z}$ acting on $H(A)$ by

$$(s, q) \cdot (x, t) = (s(x), t + q(x)), \quad (x, t) \in H(A). \tag{1.8}$$

The group law in $\mathrm{ASp}(A)$ is given by

$$(s, q) \cdot (s', q') = (s \circ s', q \circ s' + q'). \tag{1.9}$$

Furthermore, s and q are related by the formula

$$q(x + y) - q(x) - q(y) = \beta(s(x), s(y)) - \beta(x, y), \ \forall x, y \in A. \tag{1.10}$$

Since the left hand side is symmetric, it follows that $s \in \mathrm{Sp}(A)$. Since the right hand side is bilinear, it follows that q is a *quadratic function* on A. In particular, each $s \in \mathrm{Sp}(A)$ determines a linking pairing $\lambda_s : A \times A \to \mathbb{Q}/\mathbb{Z}$ defined by

$$\lambda_s(x, y) = \beta(s(x), s(y)) - \beta(x, y), \quad x, y \in A. \tag{1.11}$$

LEMMA 1.2. *The following assertions are equivalent:*

(1) *The affine symplectic group* $\mathrm{ASp}(A)$ *is a split extension of* $\mathrm{Sp}(A)$ *and* $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$*;*
(2) *A has odd order.*
(3) *Multiplication by* 2 *defines an automorphism of* A*;*

PROOF. Let $\mathrm{ASp}'(A)$ denote the set of pairs (s, λ_s) where $s \in \mathrm{Sp}(A)$ and $\lambda_s = s^*\beta - \beta$. This set is a group for the operation

$$(s, \lambda_s) \cdot (s', \lambda_{s'}) = (s \circ s', (s')^*\lambda_s + \lambda_{s'}).$$

Let $(s, q) \in \mathrm{ASp}(A)$. According to (1.10), $b_q = \lambda_s$. The projection map $p : \mathrm{ASp}(A) \to \mathrm{Sp}(A), (s, q) \mapsto s$ factors through the map $p'' : \mathrm{ASp}(A) \to \mathrm{ASp}'(A)$, $(s, q) \mapsto (s, b_q)$, so that the diagram

$$\begin{tikzcd}
\mathrm{ASp}(A) \arrow[rr, "p"] \arrow[dr, "p''"'] & & \mathrm{Sp}(A) \\
& \mathrm{ASp}'(A) \arrow[ur, "p'"'] &
\end{tikzcd}$$

commutes. There is a section $\sigma : \mathrm{Sp}(A) \to \mathrm{ASp}(A)$ if and only if there is a section $\sigma'' : \mathrm{ASp}'(A) \to \mathrm{ASp}(A)$. Thus the proof is a relative version of the proof of Prop. 4.1.

Let us prove (1) $\Longrightarrow$ (2). Assume that A has even order. We shall show that there is no group-theoretic section $\sigma'' : \mathrm{ASp}'(A) \to \mathrm{ASp}(A)$. Consider an element $x_0 \in A$ of order 2^k where k is maximal. There exists another element $x_1 \in A$ of order 2^k such that the isotropic subgroups B_0 and B_1 generated by x_0 and x_1 respectively do not intersect nontrivially, the subgroup $B_0 \oplus B_1$ is an orthogonal summand of (A, ω) and

$$\omega(x_0, x_1) = \beta(x_0, x_1) = \frac{x_0 \cdot x_1}{2^k} \pmod 1, \ \beta(x_1, x_0) = \beta(x_0, x_0) = \beta(x_1, x_1).$$

Define a symplectomorphism $s : A \to A$ by setting

$$s(m\ x_0, n\ x_1) = (-n\ x_0, m\ x_1),\ m, n \in \mathbb{Z}$$

and by extending by the identity on the orthogonal complement of $B_0 \oplus B_1$. Then $\lambda_s = s^*\beta - \beta$ is a symmetric linking pairing of order 2^k. For $x = (x_0, x_1)$ and $x' = (x_0', x_1')$, we have

$$\lambda_s(x, x') = -\frac{x_1 x_0' + x_0 x_1'}{2^k} \bmod 1.$$

A quadratic enhancement of λ_s is the quadratic form q defined by $q(x) = -\frac{x_0 x_1}{2^k} \cdots$

Let us prove (3) $\Longrightarrow$ (1). We should prove that the short exact sequence (1.7) is split. Define a map $\sigma : \mathrm{Sp}(A) \to \mathrm{ASp}(A)$ by $\sigma(s) = (s, q_s)$ with

$$q_s(x) = \frac{1}{2}\lambda_s(x, x),\ x \in A.$$

The map $q_s : A \to \mathbb{Q}/\mathbb{Z}$ is clearly a (homogeneous) quadratic function. We have

$$\begin{aligned}
q_{s \circ s'}(x) &= \frac{1}{2}\big(\beta(s \circ s'(x), s \circ s'(x)) - \beta(x, x)\big) \\
&= \frac{1}{2}\big(\beta(s \circ s'(x), s \circ s'(x)) - \beta(s'(x), s'(x))\big) \\
&+ \frac{1}{2}\big(\beta(s'(x), s'(x)) - \beta(x, x)\big) \\
&= q_s \circ s'(x) + q_{s'}(x).
\end{aligned}$$

It follows that

$$\sigma(s \circ s') = (s \circ s', q_{s \circ s'}) = (s \circ s', q_s \circ s' + q_{s'}) = \sigma(s) \cdot \sigma(s').$$

Hence σ is a group-theoretic section of the short exact sequence (1.7).

Let us prove (1) $\Longrightarrow$ (2). Let $\mathrm{ASp}'(A)$ denote the set of pairs (s, λ_s) where $s \in \mathrm{Sp}(A)$ and $\lambda_s = s^*\beta - \beta$. This set is a group for the operation

$$(s, \lambda_s) \cdot (s', \lambda_{s'}) = (s \circ s', (s')^*\lambda_s + \lambda_{s'}).$$

Let $(s, q) \in \mathrm{ASp}(A)$. According to (1.10), $b_q = \lambda_s$.

The projection map $\mathrm{ASp}(A) \to \mathrm{Sp}(A), (s, q) \mapsto s$ factors through the map $\mathrm{ASp}(A) \to \mathrm{ASp}'(A),\ (s, q) \mapsto (s, b_q)$. Hence if the projection map $(s, q) \mapsto s$ has a section, then the projection map $(s, q) \mapsto (s, b_q)$ has a section.

$\square$

REMARK 1.1. By definition the Heisenberg group depends on a Seifert form β for the symplectic form ω. For any two Seifert forms $\beta, \beta' : A \times A \to \mathbb{Q}/\mathbb{Z}$ for ω, the corresponding Heisenberg groups $\mathscr{H}_\beta(A)$ and $\mathscr{H}_{\beta'}(A)$ are isomorphic by an isomorphism that is trivial on the center. But there is no canonical isomorphism. Consider the set $I_{\beta,\beta'}$ of all isomorphisms $\mathscr{H}_\beta(A) \to \mathscr{H}_{\beta'}(A)$ that act trivially on the center Z. Then the group $\mathrm{ASp}(A)$ acts freely and transitively on $I_{\beta,\beta'}$. It is therefore sufficient to fix one Seifert form β for ω and study the corresponding group $\mathrm{ASp}(A)$.

REMARK 1.2. Our presentation of the Heisenberg groups in this paragraph and in the previous one uses the additive notation for the value group (since the applications we have in mind are for lattices and finite groups). Given a symplectic abelian group (A, ω) with Seifert form β, we may regard the symplectic form and the Seifert form as bimultiplicative pairings $A \times A \to \mathrm{U}(1)$ into the multiplicative group $\mathrm{U}(1)$. Then we could equivalently define the Heisenberg group associated to A by the same group law as before but on the underlying set $A \times \mathrm{U}(1)$. This leads to a Heisenberg group $\mathscr{H}(A; \mathrm{U}(1))$ that contains an isomorphic image of the Heisenberg group $\mathscr{H}(A)$.

2. The Heisenberg group and the discriminant

The goal of this paragraph is to study the Heisenberg group associated to the discriminant group. Let (V, ω) be a nondegenerate symplectic lattice. The discriminant construction (see Chap.) associated to (V, ω) a discriminant finite symplectic group $(G_\omega, \lambda_\omega)$ that fits into the exact sequence

$$0 \longrightarrow V \xrightarrow{\hat{\omega}} V^\sharp \longrightarrow G_\omega \longrightarrow 0.$$

The intermediate group $V^\sharp$ has a natural symplectic structure induced by ω, namely $\omega_{\mathbb{Q}}|_{V^\sharp \times V^\sharp} : V^\sharp \times V^\sharp \to \mathbb{Q}$.

LEMMA 2.1. *There is a Seifert form $\beta : V \times V \to \mathbb{Z}$ associated to ω such that $\beta_{\mathbb{Q}}(V, V^\sharp) \subseteq \mathbb{Z}$ and $\beta_{\mathbb{Q}}(V^\sharp, V) \subseteq \mathbb{Z}$.*

In the sequel we fix such a Seifert form β for V. This determines a Heisenberg group $H(V)$ for V. Then $V^\sharp$ has also a Seifert form, namely $\beta_{\mathbb{Q}}|_{V^\sharp \times V^\sharp} : V^\sharp \times V^\sharp \to \mathbb{Q}$. We define the Heisenberg group $H(V^\sharp)$ associated to $(V^\sharp, \beta_{\mathbb{Q}}|_{V^\sharp \times V^\sharp})$ in the same way as before: it is the set $V^\sharp \times \mathbb{Q}$ endowed with the usual multiplication rule.

The last symplectic group $(G_\omega, \lambda_\omega)$ also inherits a Seifert form, namely the Seifert form defined by

$$\beta_\omega([x], [y]) = \beta_{\mathbb{Q}}(x, y) \bmod 1, \quad x, y \in V^\sharp.$$

(Lemma 2.1 ensures that this is well defined.)

LEMMA 2.2. *There is a short exact sequence*

$$0 \longrightarrow H(V) \longrightarrow H(V^\sharp) \longrightarrow H(G_\omega) \longrightarrow 0.$$

PROOF. The inclusion map $H(V) \to H(V^\sharp)$ is the natural one provided by the set-theoretic inclusion. By Lemma 2.1, $H(V) \lhd H(V^\sharp)$. The natural projection map $H(V^\sharp)/H(V) \to H(G_\omega)$, $(w,t) \mapsto ([w],[t])$ is a group isomorphism. $\square$

3. The Schrödinger representation

In this section, (A,ω) is a finite symplectic group.

First we choose a suitable pair (a Seifert form β, a Lagrangian A_1 in A).

COROLLARY 3.1. *There exists a pair of transverse Lagrangians A_0, A_1 for (A,ω) such that $\beta(A_0,A_0) = \beta(A_1,A_1) = \beta(A_0,A_1) = 0$ and the bilinear pairing $\beta|_{A_1\times A_0} : A_1 \times A_0 \to \mathbb{Q}/\mathbb{Z}$ is nonsingular.*

PROOF. By the decomposition result of Prop. 3.2, $(A,\omega) = \oplus_{k\in K}(B_k,\omega_k)$ where B_k is the direct sum of two copies of a cyclic group of order p^{n_k} and $\omega(x_k,y_k) = 1/p^{n_k}$ for some $n_k \in \mathbb{N}^\times$ and some set x_k, y_k of generators of B_k. We may assume that the sets $\{x_k,\ k \in K\}$ and $\{y_k,\ k \in K\}$ generate Lagrangians A_1 and A_0 respectively in A. Define a Seifert form $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ for ω (cf. proof of Cor. 3.4) by the formulas $\beta(x_k,x_k) = \beta(y_k,y_k) = \beta(y_k,x_k) = 0$ and $\beta(x_k,y_k) = \omega(x_k,y_k)$ and bilinear extension. Then $\beta|_{A_1\times A_0} = \omega|_{A_1\times A_0}$. The last statement of the lemma follows. $\square$

In the sequel, (A,ω) is endowed with a triple (β, A_0, A_1) satisfying the properties of Corollary 3.1.

Let $\chi : Z \to \mathrm{U}(1)$ be a character on the center of $\mathscr{H}(A)$. For each $a \in A$, define a character

$$\chi_a : A \to \mathrm{U}(1),\ x \mapsto \chi(\beta(a,x)).$$

Let L^2A_0 be the Hilbert space consisting of $\mathbb{C}$-valued functions over A_0 endowed with the positive definite hermitian product

$$(f,g) \mapsto \sum_{x\in A_0} f(x)\cdot\overline{g(x)}. \tag{3.1}$$

An orthonormal basis for L^2A_0 consists of the set of functions $\delta_x, x \in A_0$ defined by $\delta_x(y) = 1$ if $x = y$ and $\delta_x(y) = 0$ otherwise. Recall that L^2A_0 is actually a commutative and associative algebra for the convolution product

$$f \star g\ (x) = \sum_{y\in A_0} f(x-y)\ g(y),\ x \in A_0.$$

We have $\delta_x \star \delta_y = \delta_{x+y}$, $x, y \in A_0$. The map

$$x \mapsto \delta_x$$

extends linearly to an algebra isomorphism $\mathbb{C}[A_0] \to L^2A_0$.

Consider on L^2A_0 the following two operators:

- Translation: $(T_{a_0}f)(z) = f(z-a_0)$ defined for $a_0 \in A_0$.
- Modulation: $(M_{a_1}f)(z) = \chi_{a_1}(z)\ f(z)$ defined for $a_1 \in A_1$.

REMARK 3.1. Our convention for the translation operator is chosen so that

$$T_{a_0}\delta_x = \delta_{x+a_0}, \quad x \in A, \quad a_0 \in A_0.$$

The modulation operator can also be written

$$M_{a_1}\delta_x = \chi_{a_1}(x)\ \delta_x, \quad x \in A,\ a_1 \in A_1.$$

DEFINITION 3.1. For each $a = (a_0, a_1) \in A$, define the *Weyl operator* $W_a : L^2A_0 \to L^2A_0$ by

$$W_a = T_{a_0}M_{a_1}.$$

It is clear from the definition that $W_0 = \mathrm{Id}_{L^2A_0}$. Furthermore, if $a = (a_0, a_1)$ and $b = (b_0, b_1)$ then

$$W_bW_a = \chi_{b_1}(a_0)\ W_{b+a}. \tag{3.2}$$

It follows that

$$W_bW_a = \chi_{a_1}(b_0)^{-1}\chi_{b_1}(a_0)\ W_aW_b. \tag{3.3}$$

LEMMA 3.1. *For any $a = (a_0, a_1) \in A$,*

$$W_a^* = \chi_{a_1}(a_0)\ W_{-a}. \tag{3.4}$$

In particular, $W_a^*W_a = \mathrm{Id}_{L^2A_0}$.

PROOF. Using the inner product (3.1), one computes

$$\langle f, W_{-a}g\rangle = \chi_{a_1}(a_0)\langle W_af, g\rangle.$$

The first assertion follows. Now by (3.4) and (3.2),

$$W_a^*W_a = \chi_{a_1}(a_0)W_{-a}W_a = \chi_{a_1}(a_0)\chi_{-a_1}(a_0)W_0 = \mathrm{Id}_{L^2A_0}.$$

□

Let $\mathrm{U}(L^2A_0)$ denote the space of unitary operators on L^2A_0. The map

$$A \to \mathrm{U}(L^2A_0),\ a \mapsto W_a,$$

is a projective unitary representation in the sense that there is a cocycle $c(a, b) = \chi_{a_1}(b_0)^{-1} \in \mathrm{U}(1)$ such that $W_{a+b} = c(a, b)W_aW_b$. The (multiplicatively written) group (known as the Mackey obstruction group associated to A, c) that consists of all pairs $(a, t) \in A \times \mathrm{U}(1)$ endowed with the law

$$(a, z) \cdot (a', z') = (aa', zz'c(a, b)^{-1}), \quad a \in A, z \in \mathrm{U}(1)$$

is precisely the Heisenberg group $\mathscr{H}(A; \mathrm{U}(1))$ defined in Remark 1.2. The map

$$\pi : \mathscr{H}(A) \to \mathrm{U}(L^2A_0),\ (a, t) \mapsto \chi(t)\ W_a,$$

is a faithful unitary linear representation of $\mathscr{H}(A)$. This is the Schrödinger representation of the Heisenberg group.

LEMMA 3.2. *The following properties hold:*

(1) *The Weyl operators $W_a, a \in A$, form a basis of* $\mathrm{End}_{\mathbb{C}}(L^2A_0)$.
(2) *The Schrödinger representation $\pi : \mathscr{H}(A) \to \mathrm{U}(L^2A_0)$ is irreducible.*
(3) $\pi|_Z(0, t) = \chi(t)\ \mathrm{Id}$ *for all $t \in \mathbb{Q}/\mathbb{Z}$.*

Proof. We follow [**49**, Lemma 3.2], see also [**29**, p. 823].

(1) Define a representation $A \to \mathrm{End}_{\mathbb{C}}(L^2 A_0)$ by

$$\alpha(x)\phi = W_x \phi W_x^{-1}.$$

If $a = (a_0, a_1)$ and $b = (b_0, b_1)$ then the relation (3.3) implies that

$$\alpha(b)(W_a) = \chi_{a_1}(b_0)^{-1}\chi_{b_1}(a_0)\ W_a.$$

Let

$$\Xi_a(b) = \chi_{a_1}(b_0)^{-1}\chi_{b_1}(a_0).$$

W_a is an eigenvector of α with eigencharacter Ξ_a. Since $a \mapsto \Xi_a$ is an isomorphism of A onto $\mathrm{Hom}(A, S^1)$, the eigenvectors W_a have distinct eigencharacters. It follows that $\{W_a,\ a \in A\}$ is a set of linearly independent elements of $\mathrm{End}_{\mathbb{C}}(L^2 A_0)$. Since the cardinality of this set is

$$|A| = |A_0| \cdot |A_1| = |A_0| \cdot |A_0| = |A_0|^2 = \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(L^2 A_0),$$

the set is a basis of $\mathrm{End}_{\mathbb{C}}(L^2 A_0)$.

(2) A subspace of $L^2 A_0$ invariant under all the Weyl operators W_a is invariant under $\mathrm{End}_{\mathbb{C}}(L^2 A_0)$, according to 1. Thus it is either 0 or $L^2 A_0$.

(3) Follows from definitions. □

Theorem 3.1 (Stone-Von Neumann-Mackey). *For any irreducible unitary representation $\rho : \mathscr{H}(A) \to \mathrm{U}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space such that $\rho|_Z(0,t) = \chi(t)\ \mathrm{Id}_{\mathcal{H}}$, there is an isometry $\Psi : L^2(A_0) \to \mathcal{H}$ such that*

$$\Psi(\pi(h)f) = \rho(h)\Psi(f), \quad \textit{for all } f \in L^2(A_0),\ h \in \mathscr{H}(A).$$

In short: up to unitary equivalence, there is a unique unitary irreducible representation $\pi : \mathscr{H}(A) \to \mathrm{U}(L^2 A_0)$ such that $\pi|_Z(0,t) = \chi(t)\ \mathrm{Id}_{L^2 A_0}$. The Schrödinger representation is essentially unique.

Proof. We follow [**49**, Theorem 3.1] supplying details from [**38**, p.26-27]. Since π is faithful, we regard $\mathscr{H}(A)$ as embedded into $\mathrm{U}(L^2 A_0)$. By the previous lemma, $\mathrm{End}_{\mathbb{C}}(L^2 A_0)$ is freely generated over $\mathbb{C}$ by the Weyl operators W_a, $a \in A$. Therefore $\rho : \mathscr{H}(A) \to \mathrm{U}(\mathcal{H})$ extends linearly to a representation $\tilde{\rho} : \mathbb{C}[\mathscr{H}(A)] = \mathrm{End}_{\mathbb{C}}(L^2 A_0) \to \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ by

$$\tilde{\rho}\left(\sum_a \lambda_a W_a\right) = \sum_a \lambda_a \rho(W_a). \tag{3.5}$$

This turns $\mathcal{H}$ into a $\mathrm{End}_{\mathbb{C}}(L^2 A_0)$-module. Since ρ is irreducible, $\mathcal{H}$ is a simple $\mathrm{End}_{\mathbb{C}}(L^2 A_0)$-module. On the one hand, since ρ is unitary,

$$\rho(W_a)^* \rho(W_a) = \mathrm{Id}_{\mathcal{H}} \text{ for all } a \in A.$$

On the other hand, since W_a is unitary,

$$\rho(W_a^*)\rho(W_a) = \rho(W_a^* W_a) = \rho(\mathrm{Id}_{L^2 A_0}) = \mathrm{Id}_{\mathcal{H}}.$$

Hence $\rho(W_a^*) = \rho(W_a)^{-1} = \rho(W_a)^*$. It follows from (3.5) that

$$\tilde{\rho}(\phi^*) = \tilde{\rho}(\phi)^* \text{ for all } \phi \in \mathrm{End}_{\mathbb{C}}(L^2 A_0).$$

For $a \in A_0$, let p_a be the orthogonal projector onto $\mathbb{C}\delta_a \subseteq L^2 A_0$ defined by $p_a(f) = \langle f, \delta_a \rangle \delta_a$. Then $p_a^2 = p_a$ and $p_a^* = p_a$. Furthermore, $\tilde{\rho}(p_a)^2 =$

$\tilde{\rho}(p_a^2) = \tilde{\rho}(p_a)$ and $\tilde{\rho}(p_a)^* = \tilde{\rho}(p_a^*) = \tilde{\rho}(p_a)$ thus $\tilde{\rho}(p_a)$ is an orthogonal projector. Since $\tilde{\rho}$ is injective, $\tilde{\rho}(p_a)$ has rank 1. In particular,

$$\tilde{\rho}(p_a)\mathcal{H} = \mathbb{C}v_0 \neq 0$$

for some vector $v_0 \in \mathcal{H}$, $||v_0||_{\mathcal{H}} = 1$. Define a map $\Psi : L^2A_0 \to \mathcal{H}$ by setting

$$\Psi(\phi\delta_0) = \tilde{\rho}(\phi)v_0, \quad \phi \in \mathrm{End}_{\mathbb{C}}(L^2A_0)$$

and extending by $\mathbb{C}$-linearity. This map is well defined since

(i) $\mathrm{End}_{\mathbb{C}}(V)\delta_0 = L^2A_0$;
(ii) if $\phi(\delta_0) = \phi'(\delta_0)$ then $\phi p_0 = \phi' p_0$, so $\tilde{\rho}(\phi)\tilde{\rho}(p_0) = \tilde{\rho}(\phi')\tilde{\rho}(p_0)$, hence $(\tilde{\rho}(\phi) - \tilde{\rho}(\phi')) \circ \tilde{\rho}(p_0) = 0$. Since $\tilde{\rho}(p_0) \neq 0$, we have $\tilde{\rho}(\phi)|_{\mathcal{H}_0} = \tilde{\rho}(\phi')|_{\mathcal{H}_0}$ so that $\tilde{\rho}(\phi)h = \tilde{\rho}(\phi')h$.

We claim that Ψ is surjective. Let $\phi_a \in \mathrm{End}_{\mathbb{C}}(L^2A_0)$ such that $\phi_a(\delta_0) = \delta_a$. Then $\phi_a p_0 \phi_a^* = p_a$ and therefore $\mathrm{Id}_{L^2A_0} = \sum_{a\in A_0} \phi_a p_a \phi_a^*$ Applying $\tilde{\rho} : \mathrm{End}_{\mathbb{C}}(L^2A_0) \to \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ yields

$$\mathrm{Id}_{\mathcal{H}} = \sum_{a\in A_0} \tilde{\rho}(\phi_a)\tilde{\rho}(p_0)\tilde{\rho}(\phi_a)^*.$$

Hence any element $h \in \mathcal{H}$ decomposes as $h = \sum_{a\in A_0} \lambda_a \tilde{\rho}(\phi_a)v_0$. Surjectivity of Ψ follows.

Observe that $\Psi(\phi\delta_a) = \Psi(\phi\phi_a\delta_0) = \tilde{\rho}(\phi\phi_a)v_0 = \tilde{\rho}(\phi)\tilde{\rho}(\phi_a)v_0 = \tilde{\rho}(\phi)\Psi(\phi_a\delta_0) = \tilde{\rho}(\phi)\Psi(\delta_a)$. It follows that

$$\Psi(\phi f) = \tilde{\rho}(\phi)\Psi(f) \quad \text{for all } \phi \in \mathrm{End}_{\mathbb{C}}(L^2A_0) \text{ and } f \in L^2A_0. \tag{3.6}$$

Using again that ρ is unitary, we deduce that $\Psi : L^2A_0 \to \mathcal{H}$ is an isometry. Restricting the property (3.6) to $\mathscr{H}(A)$ yields the desired result. □

An alternative incarnation of *Schrödinger representation* is the representation of $\mathscr{H}(A)$ *induced* from a maximal abelian normal subgroup in $\mathscr{H}(A)$.

LEMMA 3.3. *For any Lagrangian A_1 in A, the subset $L_1 = A_1 \times \mathbb{Q}/\mathbb{Z} \subset \mathscr{H}(A)$ is a maximal abelian normal subgroup.*

In particular, L_1 contains the center $Z = 0 \times \mathbb{Q}/\mathbb{Z}$. Extend the character $\chi : Z \to \mathrm{U}(1)$ to a character $\tilde{\chi}$ on L_1 by

$$\tilde{\chi} : L_1 \to \mathrm{U}(1), \quad (x, t) \mapsto \chi(t).$$

We now assume that these choices are fixed throughout the construction to follow.

Let $\mathcal{H}_{A_1}$ be the Hilbert space of functions $f : \mathscr{H}(A) \to \mathbb{C}$ such that

$$f(h \cdot l) = \tilde{\chi}(l)^{-1} f(h), \quad h \in \mathscr{H}(A),\ l \in L_1.$$

The induced representation is given by the action of $\mathscr{H}(A)$ on $\mathcal{H}_{A_1}$ by left translations

$$\pi'(h)[f](x) = f(h^{-1}x), \quad f \in \mathcal{H}_{A_1},\ h \in \mathscr{H}(A).$$

To see that this representation is equivalent to the previous representation π, observe first that any map $f : A_0 \to \mathbb{C}$ extends in a unique way to a map $\widetilde{f} \in \mathcal{H}_{A_1}$ defined by

$$\widetilde{f}(a_0 \cdot l) = \tilde{\chi}(l)^{-1}\ f(a_0), \quad a_0 \in A_0,\ l \in L_1.$$

Proposition 3.1. *The map $f \mapsto \widetilde{f}$ is an isometry*

$$L^2(A_0) \to \mathcal{H}_{A_1},\ f \mapsto \widetilde{f},$$

which is an equivalence between the representations π and π'.

Proof. The first assertion follows from the definition. For the second assertion, we verify the identify $\widetilde{\pi(h)[f]} = \pi'(h)[\tilde{f}]$ for $h \in \mathscr{H}(A)$ and $f \in L^2(A_0)$. □

It follows in particular that the Schrödinger representation only depends on the choice of one *Lagrangian* A_0 in A rather than on a pair of transverse Lagrangians of A. This is also clear from the definition of the Weyl operator (where A_1 can be replaced by A/A_0 throughout the construction).

As our construction is based on the finite group A, there is yet another presentation of the Schrödinger representation as a *quotient* of the group algebra of the Heisenberg group. Let $I = I_{A_1,\chi}$ be the two-sided ideal in $\mathbb{C}[\mathscr{H}(A)]$ generated by the set $\{h \cdot l - \chi(l)h,\ h \in \mathscr{H}(A),\ l \in L_1\}$. Set

$$\mathcal{H}(A) = \mathbb{C}[\mathscr{H}(A)]/I.$$

The Heisenberg group $\mathscr{H}(A)$ acts by translations on the left on the group algebra $\mathbb{C}[\mathscr{H}(A)]$:

$$h \cdot \delta_k = \delta_{hk}, \quad h, k \in \mathscr{H}(A).$$

Since I is stable under $\mathscr{H}(A)$, this action descends to a representation $\mathscr{H}(A) \times \mathcal{H}(A) \to \mathcal{H}(A)$:

$$\pi''(h)[\delta_k] = [h \cdot \delta_k] = [\delta_{hk}], \quad h, k \in \mathscr{H}(A).$$

Proposition 3.2. *The map $\delta_{a_0} \mapsto [\delta_{a_0}]$ defines an equivalence*

$$L^2 A_0 \to \mathcal{H}(A)$$

between the representation π and π''.

4. The Weil representation

4.1. The canonical intertwining operator. Consider the version of the Schrödinger representation constructed in the previous paragraph as an induced functional representation. It is defined as a map $\pi_\Lambda : \mathscr{H}(A) \to \mathrm{U}(\mathcal{H}_\Lambda))$ (noted π' in the previous paragraph) and depends on the choice of a Lagrangian Λ in A (noted A_1 in the previous paragraph). Another choice of Lagrangian Λ' leads to another Schrödinger representation $\pi_{\Lambda'} : \mathscr{H}(A) \to \mathrm{U}(\mathcal{H}_{\Lambda'})$. By the Stone-Von Neumann-Mackey theorem, the two representations are unitary equivalent: there exists an isometry $\rho_{\Lambda',\Lambda} \in \mathrm{Hom}_{\mathscr{H}(A)}(\mathcal{H}_\Lambda, \mathcal{H}_{\Lambda'})$ such that

$$\rho_{\Lambda',\Lambda}(\pi_\Lambda f) = \pi_{\Lambda'}(\rho_{\Lambda',\Lambda}(f)), \quad f \in \mathcal{H}_\Lambda.$$

This relation determines the map $\rho_{\Lambda',\Lambda'}$ up to a scalar of modulus one. Such a map is called an *intertwining operator*.

LEMMA 4.1. *Suppose that Λ and Λ' are transverse in A. Up to a complex unit scalar,*

$$\rho_{\Lambda',\Lambda}[f](h) = \sum_{l'\in\Lambda'} f(h\cdot(l',0)) \tag{4.1}$$

PROOF. We need to verify that $\rho[f] \in \mathcal{H}_{\Lambda'}$. Let $L' = \Lambda' \times \mathbb{Q}/\mathbb{Z}$ the maximal normal subgroup associated to Λ'. Let $x' = (l'', t) = (l'', 0)(0, t) \in L'$. Then $f(hx') = f(h\cdot(l'',0)\cdot(0,t)\cdot(l',0)) = f(h\cdot(l'',0)(l',0)\cdot(0,t)) = \chi(t)^{-1}\cdot f(h\cdot(l'',0)(l',0)) = \tilde{\chi}(x')^{-1} f(h\cdot(l'',0)(l',0))$. Then by summing over $l' \in \Lambda'$, we see that $\rho[f](hx') = \tilde{\chi}(x')^{-1}\cdot\rho[f](h)$. Thus $\rho[f] \in \mathcal{H}_{\Lambda'}$. Next, since the action is by translations, ρ is a $\mathscr{H}(A)$-map. $\square$

Consider now three Lagrangians Λ, Λ' and Λ'' in A. They give rise to three intertwining operators $\rho_{\Lambda',\Lambda}$, $\rho_{\Lambda'',\Lambda'}$ and $\rho_{\Lambda'',\Lambda}$ respectively. Both $\rho_{\Lambda'',\Lambda}$ and $\rho_{\Lambda'',\Lambda'}\circ\rho_{\Lambda',\Lambda}$ are intertwiners of $\mathcal{H}_{\Lambda''}$ and $\mathcal{H}_{\Lambda}$. Since $\mathcal{H}_{\Lambda}$ is irreducible (Lemma 3.2), it follows from Schur's lemma that there exists $C(\Lambda'',\Lambda',\Lambda) \in \mathbb{C}$ such that

$$\rho_{\Lambda'',\Lambda'} \circ \rho_{\Lambda',\Lambda} = C(\Lambda'',\Lambda',\Lambda)\ \rho_{\Lambda'',\Lambda}. \tag{4.2}$$

As noticed by A. Weil, it turns out that the cocycle $C(\Lambda'',\Lambda',\Lambda)$ can be expressed as a Gauss sum. We shall describe it in the case when Λ'',Λ' and Λ are mutually transverse. Since $\Lambda \oplus \Lambda'' = A$, there is a well-defined projection $p_{\Lambda'',\Lambda} : A \to \Lambda''$ on Λ'' with respect to Λ. Consider the restriction $p : \Lambda' \to \Lambda''$ of the map $p_{\Lambda'',\Lambda}$ to Λ'.

LEMMA 4.2. *Suppose Λ'',Λ' and Λ are mutually transverse. Then*

$$C(\Lambda'',\Lambda',\Lambda) = \sum_{l'\in\Lambda'} \chi(-\beta(l', p_{\Lambda'',\Lambda}(l'))).$$

PROOF. Let $\tilde{\delta}_0 \in \mathcal{H}_\Lambda$ the extension of the map δ_0 (defined in Prop. 3.1). Note that the support of $\tilde{\delta}_0$ is $Z\cdot\Lambda = \Lambda\times\mathbb{Q}/\mathbb{Z}$. We apply the identity (4.2) to $\tilde{\delta}_0$ evaluated at $x = 0$. Since Λ and Λ' are transverse, $Z\cdot\Lambda$ and $Z\cdot\Lambda'$ intersect exactly on the center Z. Hence

$$\rho_{\Lambda',\Lambda}[\tilde{\delta}_0](0) = \sum_{l'\in\Lambda'} \tilde{\delta}_0(l',0) = \tilde{\delta}_0(0) = 1.$$

Thus

$$\begin{aligned} C(\Lambda'',\Lambda',\Lambda) &= \rho_{\Lambda'',\Lambda'}(\rho_{\Lambda',\Lambda}[\tilde{\delta}_0])(0) \\ &= \sum_{l''\in\Lambda''}\sum_{l'\in\Lambda'} \tilde{\delta}_0((l'',0)(l',0)) \\ &= \sum_{l''\in\Lambda''}\sum_{l'\in\Lambda'} \tilde{\delta}_0(l''+l', \beta(l'',l')). \end{aligned}$$

Now $\tilde{\delta}_0(a,t) \neq 0$ if and only if $a \in \Lambda$. Thus the only nonzero terms in the sum above occur when $l''+l' \in \Lambda$. This is equivalent to $l'' = -p_{\Lambda'',\Lambda}(l')$. Let us compute this term:

$$\tilde{\delta}_0(\underbrace{l' - p_{\Lambda'',\Lambda}(l')}_{\in\Lambda}, \beta(-p_{\Lambda'',\Lambda}(l'), l')) = \chi(\beta(-p_{\Lambda'',\Lambda}(l'), l')).$$

Summing over all $l' \in \Lambda'$ yields the desired result. $\square$

4.2. Definition of the Weil representation. By definition, the affine symplectic group $\mathrm{ASp}(A)$ acts by automorphisms on the Heisenberg group $\mathscr{H}(A)$:

$$\mathrm{ASp}(A) \times \mathscr{H}(A) \to \mathscr{H}(A),\ (s, h) \mapsto s(h).$$

For each $s \in \mathrm{ASp}(A)$, define a new representation π^s of the Heisenberg group by

$$\pi^s(h) = \pi(s(h))), \quad h \in \mathscr{H}(A).$$

This representation is still irreducible unitary and verifies

$$\pi^s|_Z(0, t) = \pi|_Z((0, t)^s) = \pi|_Z(0, t) = \chi(t)\ \mathrm{Id}_{L^2 A_0}$$

for any $t \in \mathbb{Q}/\mathbb{Z}$. Hence by Theorem 3.1, the representations π and π^s are unitary equivalent: there exists a unitary operator (defined up to a unitary scalar) $\rho_s \in L^2 A_0 \to L^2 A_0$ such that

$$\rho_s(\pi(h)f) = \pi^s(h)(\rho_s f), \quad \forall h \in \mathscr{H}(A),\ \forall f \in L^2 A_0. \tag{4.3}$$

Equivalently,

$$\pi^s = \rho_s\ \pi\ \rho_s^{-1}. \tag{4.4}$$

The *Weil representation* is the map

$$\mathrm{ASp}(A) \to \mathrm{U}(L^2 A_0), s \mapsto \rho_s.$$

This definition depends on the choice, for each $s \in \mathrm{ASp}(A)$, of a unitary operator $\rho_s \in U(L^2 A_0)$ verifying (4.4).

LEMMA 4.3. *The Weil representation is a* projective *representation in the sense that for any $s, s' \in \mathrm{ASp}(A)$, there exists $c(s, s') \in U(1) = S^1$ such that*

$$\rho_{ss'} = c(s, s')\ \rho_s \rho_{s'}. \tag{4.5}$$

The map $(s, s') \mapsto c(s, s')$ is a 2-cocycle satisfying the identity

$$c(s_0, s_1 s_2) c(s_1, s_2) = c(s_0 s_1, s_2) c(s_0, s_1), \quad \forall s_0, s_1, s_2 \in \mathrm{ASp}(A). \tag{4.6}$$

PROOF. For $s, s' \in \mathrm{ASp}(A)$,

$$\begin{aligned}\rho_s \rho_{s'} \pi(h) = \rho_s \pi^{s'}(h) \rho_{s'} = \rho_s \pi(s'(h)) \rho_{s'} = \pi^s(s'(h)) \rho_s \rho_{s'} &= \pi(ss'(h)) \rho_s \rho_{s'} \\ &= \pi^{ss'}(h) \rho_s \rho_{s'}.\end{aligned}$$

Set $C(s, s') = \rho_{ss'}^{-1} \rho_s \rho_{s'} \in \mathrm{U}(L^2 A_0)$. Then

$$C(s, s')\pi(h) = \rho_{ss'}^{-1} \rho_s \rho_{s'} \pi(h) = \rho_{ss'}^{-1} \pi^{ss'}(h) \rho_s \rho_{s'} = \pi(h) \rho_{ss'}^{-1} \rho_s \rho_{s'} = \pi(h) C(s, s').$$

Since π is irreducible, Schur's lemma implies that $C(s, s') = c(s, s')\ \mathrm{Id}_{L^2 A_0}$ for some multiple $c(s, s') \in \mathbb{C}^\times$. Since $C(s, s')$ is unitary, $c(s, s') \in U(1)$.

The cocycle identity is derived from associativity by writing down the equality $\rho_{s_0 \cdot (s_1 s_2)} = \rho_{(s_0 s_1) \cdot s_2}$ and applying (4.5). $\square$

There is the natural question of *linearization*: can one choose the operators $(\rho_s)_{s\in \mathrm{ASp}(A)}$ in such a way that the corresponding Weil representation is linear ? Specifically, is there a map $b : A \to U(1)$ such that $s \mapsto b(s)\ \rho_s$ is linear ? Such a map exists if and only if $b(ss')\ c(s,s') = b(s)\ b(s')$ for all $s, s' \in \mathrm{ASp}(A)$. The cocycle c is a *coboundary* in this case.

In general there is a construction of a central extension of $\mathrm{ASp}(A)$ that has defined on it a linear representation induced by the projective Weil representation and the cocycle c. The set $\mathrm{ASp(A)}_c$ of all pairs $(s,t) \in \mathrm{ASp(A)} \times U(1)$ becomes a group (the Mackey obstruction group associated to c) when endowed with the operation

$$(s,t)\cdot(s',t') = (ss', tt'c(s,s')^{-1}), \quad s \in \mathrm{Asp}(A),\ t \in U(1).$$

Clearly the group $\mathrm{ASp(A)}_c$ is a central extension of $\mathrm{ASp}(A)$ and fits into the short exact sequence

$$1 \to U(1) \to \mathrm{ASp}(A)_c \to \mathrm{ASp}(A) \to 1. \tag{4.7}$$

The map, induced by the projective Weil representation, defined by

$$\mathrm{ASp}(A)_c \to U(L^2A_0),\ (s,t) \mapsto t\ \rho_s$$

is a linear representation.

One can ask for a smallest group U, a map $u : U \to U(1)$ and a group G_c and a map $g : G_c \to \mathrm{ASp}(A)_c$ such that there is a commutative diagram with exact sequences

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & \mathrm{ASp}(A)_c \arrow[r] & \mathrm{ASp}(A) \arrow[r] & 1 \\
1 \arrow[r] & U \arrow[r] \arrow[u, "u"] & G_c \arrow[r] \arrow[u, "g"] & \mathrm{ASp}(A) \arrow[r] \arrow[u, equal] & 1.
\end{tikzcd}$$

By taking $U = 1$, we see the following

PROPOSITION 4.1. *The Weil representation is linearizable if and only if the short exact sequence* (4.7) *splits.*

THEOREM 4.1. *If A had odd order, then the Weil representation is linearizable. If A had even order, then the projective Weil representation lifts to a linear representation of the double cover of* $\mathrm{ASp}(A)$.

5. The Maslov index for finite groups

We define here the Maslov index for an ordered triple of Lagrangians in a finite symplectic group (A,ω). We adapt the original definition by Kashiwara as described by Lion and Vergne in [**38**, I, §1.5] to our setting. For other generalizations see also the work of Kamgarpour and Thomas [**32**].

Let A be a symplectic abelian group. Let A_0, A_1, A_2 be three Lagrangians in A. Consider the abelian group $A_0\oplus A_1\oplus A_2$. The *Maslov index* is defined as the Witt class of the homogeneous quadratic form Q defined on $A_0\oplus A_1\oplus A_2$ by

$$Q(a_0+a_1+a_2) = \omega(a_0,a_1)+\omega(a_1,a_2)+\omega(a_2,a_0), \quad (a_0,a_1,a_2) \in A_0\times A_1\times A_2.$$

In the case when A is a lattice, the Witt group is the Witt group of integral quadratic forms is isomorphic to $\mathbb{Z}$, the isomorphism being given by the signature. In the case when A is a finite abelian group, the Witt group is the Witt group $\mathfrak{W}\mathfrak{Q}$ of finite quadratic forms is isomorphic to $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We denote the Maslov index of (L_0, L_1, L_2) by $\mu(L_0, L_1, L_2) \in \mathfrak{W}\mathfrak{Q}$.

The following two properties are consequences of the definition. The first property states that the Maslov index is invariant under circular permutation:

$$\mu(L_0, L_1, L_2) = -\mu(L_1, L_0, L_2) = -\mu(L_0, L_2, L_1). \tag{5.1}$$

From the classification of finite symplectic pairings, we see that the symplectic group $\mathrm{Sp}(A)$ acts transitively on pairs of transverse Lagrangians. The second property states the Maslov index is invariant under the action of the symplectic group:

$$\forall s \in \mathrm{Sp}(A),\ \mu(s\ L_0, s\ L_1, s\ L_2) = \mu(L_0, L_1, L_2). \tag{5.2}$$

A more subtle property of the Maslov index is the chain relation.

Proposition 5.1. *Let A_0, A_1, A_2, L be four Lagrangians. The Maslov index verifies the relation*

$$\mu(A_0, A_1, A_2) = \mu(A_0, A_1, L) + \mu(A_1, A_2, L) + \mu(A_2, A_0, L). \tag{5.3}$$

The chain relation is a relation in the Witt group. In the classical setting (when A is a lattice or a vector space), the Maslov index is an integer and the chain relation has a geometric interpretation. See [**38**, I, §1.5.8] for a proof. (**Autres références...**)

6. The Weil representation of a finite quadratic form

Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a homogeneous quadratic form on a finite abelian group G with associated linking pairing $b_q : G \times G \to \mathbb{Q}/\mathbb{Z}$. Let (V, ω) be a symplectic lattice equipped with a Seifert form $\beta : V \times V \to \mathbb{Z}$. Then the form

$$b_q \otimes \omega : G \otimes V \times G \otimes V \to \mathbb{Q}/\mathbb{Z}$$

is nondegenerate and alternate, hence is defines a symplectic form on $G \otimes V$.

Lemma 6.1. *The form $b_q \otimes \beta : G \otimes V \times G \otimes V \to \mathbb{Q}/\mathbb{Z}$ is a Seifert form for $b_q \otimes \omega$.*

Proof. Let $x, x' \in G$ and $y, y' \in V$. We compute

$$\begin{aligned}
(b_q \otimes \beta)(x \otimes y, x' \otimes y') - (b_q \otimes \beta)(x' \otimes y', x \otimes y) &= b_q(x, x')\beta(y, y') - b_q(x', x)\beta(y', y) \\
&= b_q(x, x')\beta(y, y') - b_q(x, x')\beta(y', y) \\
&= b_q(x, x')\big(\beta(y, y') - \beta(y', y)\big) \\
&= b_q(x, x')\omega(y, y') \\
&= (b_q \otimes \omega)(x \otimes y, x' \otimes y')
\end{aligned}$$

where we used that b_q is symmetric in the second equality. $\square$

Denote by $\mathrm{O}(q)$ the group of automorphisms of G fixing q. Recall that for $s \in \mathrm{Sp}(V)$, $\lambda_s \in \mathrm{Link}(V)$ denotes the linking pairing defined by $\lambda_s = s^*\beta - \beta$.

PROPOSITION 6.1. *There is a well defined monomorphism*

$$\mathrm{O}(q) \otimes \mathrm{Sp}(V) \to \mathrm{ASp}(G \otimes V), \;\; \alpha \otimes s \mapsto (\alpha \otimes s, q \otimes \lambda_s).$$

In particular, there is a well defined monomorphism

$$\mathrm{Sp}(V) \to \mathrm{ASp}(G \otimes V), \;\; s \mapsto (\mathrm{id}_G \otimes s, q \otimes \lambda_s).$$

PROOF. The point is to verify that $(\alpha \otimes s, q \otimes \lambda_s) \in \mathrm{ASp}(G \otimes V)$. Clearly $\alpha \otimes s \in \mathrm{Sp}(G \otimes V)$. We compute

$$\begin{aligned}
b_{q\otimes\lambda_s}(x \otimes y, x' \otimes y') &= (b_q \otimes \lambda_s)(x \otimes y, x' \otimes y') \\
&= b_q(x,x') \cdot \big(\beta(sy, sy') - \beta(y,y')\big) \\
&= b_q(x,x')\ \beta(sy,sy') - b_q(x,x')\ \beta(y,y') \\
&= b_q(\alpha x, \alpha x')\ \beta(sy,sy') - b_q(x,x')\ \beta(y,y') \\
&= (b_q \otimes \beta)\big((\alpha\otimes s)(x\otimes y), (\alpha \otimes s)(x' \otimes y')\big) - (b_q \otimes \beta)(x \otimes y, x' \otimes y').
\end{aligned}$$

$\square$

Suppose that $V = L_0 \oplus L_1$ is a Lagrangian decomposition of V. Then $G \otimes V = (G \otimes L_0) \oplus (G \otimes L_1)$ is a Lagrangian decomposition of $G \otimes V$. Composing the map of Prop. 6.1 with the Weil representation defined in the previous section gives a projective representation

$$\mathrm{O}(q) \otimes \mathrm{Sp}(V) \to \mathrm{U}(L^2(G \otimes L_0)), (\alpha, s) \mapsto \rho_{\alpha\otimes s, q \otimes \lambda_s}.$$

This is the *Weil representation of the quadratic form q*.

The groups $\mathrm{O}(q)$ and $\mathrm{Sp}(V)$, viewed as subgroups of $\mathrm{ASp}(G \otimes V)$, are mutual centralizers. They form a prominent instance of an *reductive dual pair*.

7. Particular cases and examples

Several particular cases of the Weil representation of the quadratic form q are of interest. The representation $\mathrm{O}(q) \otimes \mathrm{Sp}(V) \to \mathrm{U}(L^2(G \otimes L_0))$ considered in the previous paragraph restricts to a representation

$$\mathrm{Sp}(V) \to \mathrm{U}(L^2(G \otimes L_0)), s \mapsto \rho_{\mathrm{id}_G \otimes s, q \otimes \lambda_s}.$$

For simplicity, we denote this representation by $s \mapsto \rho_s$. We describe this representation in terms of generators for $\mathrm{Sp}(V)$.

Let $g \geqslant 1$. Let $V = \mathbb{Z}^{2g}$ endowed with the canonical symplectic form. Then $\mathrm{Sp}(V)$ identifies with the symplectic group

$$\mathrm{Sp}_{2g}(\mathbb{Z}) = \{M \in \mathrm{GL}_{2g}(\mathbb{Z}),\ M^T \Omega M = \Omega\},\ \Omega = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix},$$

also called Siegel's modular group. (Our convention follows the left action notation: automorphisms act on the left on groups.) The Seifert form is

$$\beta = \begin{bmatrix} 0 & 0 \\ 1_g & 0 \end{bmatrix}.$$

The lattice V has a canonical Lagrangian decomposition

$$V = L_0 \oplus L_1,$$

where $L_0 = \{x \in \mathbb{Z}^{2g} \mid \forall\ g \leqslant j \leqslant 2g,\ x_j = 0\}$ and $L_1 = \{y \in \mathbb{Z}^{2g} \mid \forall\ 1 \leqslant j \leqslant g,\ y_j = 0\}$. For $x = (x_0, x_1),\ y = (y_0, y_1) \in L_0 \times L_1$,

$$\beta(x, y) = \langle x_1, y_0 \rangle$$

where $\langle -, - \rangle$ denotes here the canonical symmetric positive definite product on $\mathbb{Z}^g$. Examples of integral symplectic matrices are

(7.1)

$$\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}, \begin{bmatrix} 1_g & 0 \\ B & 1_g \end{bmatrix} \text{ with } B = B^T \text{ integral}, \begin{bmatrix} A^T & 0 \\ 0 & A^{-1} \end{bmatrix} \text{ with } A \in \mathrm{GL}_g(\mathbb{Z}).$$

Remark 7.1. The set of all matrices of the three types above generates $\mathrm{Sp}_{2g}(\mathbb{Z})$, see [**58**] and [**2**]. Furthermore, the set of each type generates a subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ which has a group theoretic section into $\mathrm{ASp}_{2g}(\mathbb{Z})$.

We explicit below the map

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{U}(L^2(G \otimes L_0)) = \mathrm{U}(L^2(G^g)),\ s \mapsto \rho_s.$$

Proposition 7.1. *The Weil representation $\rho : \mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{U}(L^2(G^g))$ is determined by the following formulas:*

$$\rho \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} f\ (x) = |G \otimes L_0|^{-1/2} \sum_{y \in G \otimes L_0} \chi((b_q \otimes \beta)(y, x))\ f(y). \tag{7.2}$$

$$\rho \begin{pmatrix} 1_g & 0 \\ B & 1_g \end{pmatrix} f\ (x) = \chi(-(q \otimes B)(x))\ f(x). \tag{7.3}$$

$$\rho \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix} f\ (x) = f\big((1_G \otimes A^T)^{-1} x\big). \tag{7.4}$$

Remark 7.2. The proposition has a generalization (with essentially the same proof below) for an arbitrary commutative locally compact group G endowed with some Haar measure (a Borel measure invariant under translations which is unique up to a scalar multiple). In this case, the group $G \otimes L_0$ inherits a Haar measure $d\mu$ and the first relation reads

$$\rho \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} f\ (x) = \int_{G \otimes L_0} \chi((b_q \otimes \beta)(y, x))\ f(y)\ d\mu(y). \tag{7.5}$$

Proof. The three operators defined in the proposition are unitary (for the hermitian product defined by (3.1)). The rest of the proof consists in verifying the identity (4.4). We have to verify that

$$\rho_s(\pi^s(h) f) = \pi(h)(\rho_s f), \quad \forall h \in H(G \otimes V),\ \forall f \in L^2(G \otimes L_0)$$

with s being one of the three symplectomorphisms above. Set $G_0 = G \otimes L_0$ and $G_1 = G \otimes L_1$. Write (s_0, s_1) the image of $a = (a_0, a_1) \in G_0 \oplus G_1$ by $\mathrm{id}_G \otimes s \in \mathrm{Sp}_{2g}(\mathbb{Z})$. Let $h = (a_0, a_1, t) \in H(G \otimes V)$. Recall that s acts on h as

$$s \cdot h = (s_0, s_1, t + (q \otimes \lambda_s)(t)).$$

We have

$$\begin{aligned}
\pi^s(h)f\ (x) &= \pi(s_0, s_1, t + q \otimes \lambda_s(a_0, a_1))f\ (x) \\
&= \chi(t + q \otimes \lambda_s(a_0, a_1))\ W_{s_0,s_1} f\ (x) \\
&= \chi(t + q \otimes \lambda_s(a_0, a_1))\ \chi_{s_1}(x)\ f(x + s_0) \\
&= \chi(t + q \otimes \lambda_s(a_0, a_1))\ \chi(b_q \otimes \beta(s_1, x))\ f(x + s_0).
\end{aligned}$$

Let $g(x)$ be

$$\pi(a_0, a_1, t)f\ (x) = \chi(t)\ \chi((b_q \otimes \beta)(a_1, x))\ f(x + a_0).$$

Consider now each case separately. It will be convenient to denote in this paragraph by $\langle -, - \rangle$ the symmetric bilinear product on $\mathbb{Z}^g$ associated to the $g \times g$ identity matrix.

In the first case: $s_0 = a_1, s_1 = -a_0$. Then

$$\lambda_s(x, y) = -(\langle x_0, y_1 \rangle + \langle x_1, y_0 \rangle).$$

This symmetric bilinear pairing admits a quadratic enhancement defined on V by

$$x \mapsto -\langle x_0, x_1 \rangle = -\beta(x, x).$$

It follows that $q \otimes \lambda_s(a, b) = -(b_q \otimes \beta)(a, b)$ for all $a, b \in G \otimes V$. We compute

$$\begin{aligned}
\rho_s g\ (x) &= |G \otimes L_0|^{-1/2} \sum_{y \in G \otimes L_0} \chi(b_q \otimes \beta(y, x)) g(y) \\
&= |G \otimes L_0|^{-1/2} \chi(t) \sum_{y \in G \otimes L_0} \chi(b_q \otimes \beta(y, x + a_1))\ f(y + a_0) \\
&= |G \otimes L_0|^{-1/2} \chi(t) \sum_{y' \in G \otimes L_0} \chi(b_q \otimes \beta(y' - a_0, x + a_1))\ f(y').
\end{aligned}$$

On the other hand, setting $h = \rho_s f$, we have

$$\begin{aligned}
\pi^s(a, t)h\ (x) &= \pi(a_1, -a_0, t - q \otimes \lambda_s(a_0, a_1))h\ (x) \\
&= \chi(t - b_q \otimes \beta(a_0, a_1))\ \chi(b_q \otimes \beta(-a_0, x))\ h(x + a_1) \\
&= \ |G \otimes L_0|^{-1/2}\ \chi(t)\ \chi(b_q \otimes \beta(-a_0, x + a_1)) \cdot \\
&\qquad \cdot \sum_{y \in G \otimes L_0} \chi(b_q \otimes \beta(y, x + a_1)) f(y) \\
&= \rho_s g\ (x).
\end{aligned}$$

In the second case: $s_0 = a_0,\ s_1 = Ba_0 + a_1$. Then

$$\lambda_s(x, y) = x_0^T B y_0,\ \ x, y \in V.$$

It follows that $(q\otimes\lambda_s)(a_0,a_1)=(q\otimes B)(a_0)$. We compute

$$\begin{aligned}\rho_s g\ (x) &= \chi(-(q\otimes B)(x))\ g(x)\\ &= \chi(t)\ \chi(b_q\otimes\beta(a_1,x))\ \chi(-(q\otimes B)(x))\ f(x+a_0).\end{aligned}$$

Let $h=\rho_s(f)$. We have

$$\begin{aligned}\pi^s(a,t)h\ (x) &= \pi(a_0, Ba_0+a_1, t+(q\otimes\lambda_s)(a_0,a_1))h\ (x)\\ &= \chi(t+(q\otimes B)(a_0))\ \chi(b_q\otimes\beta(Ba_0+a_1,x))\ h(x+a_0)\\ &= \chi(t+(q\otimes B)(a_0))\ \chi(b_q\otimes\beta(Ba_0+a_1,x))\ \chi(-q\otimes B(x+a_0))\ f(x+a_0)\\ &= \chi(t)\ \chi(b_q\otimes\beta(a_1,x))\ \chi(-(q\otimes B)(x))\ f(x+a_0)\\ &= \rho_s g\ (x).\end{aligned}$$

Here we used in the penultimate equality the fact that $(b_q\otimes\beta)(Ba_0,x)=(b_q\otimes\beta)(0\oplus Ba_0, x\oplus 0)=(b_q\otimes B)(a_0,x)$.

In the third case: $s_0=(1_G\otimes A^T)a_0,\ s_1=(1_G\otimes A^{-1})a_1$. Then $\lambda_s=0$. It follows that $q\otimes\lambda_s=0$. We compute

$$\begin{aligned}\rho_s g\ (x) &= g((1_G\otimes A^{-T})x)\\ &= \chi(t)\ \chi(b_q\otimes\beta(a_1,(1_G\otimes A^{-T})x))\ f((1_G\otimes A^{-T})x+a_0).\end{aligned}$$

Let $h=\rho_s f$. We have

$$\begin{aligned}\pi^s(a,t)h\ (x) &= \pi((1_G\otimes A^T)a_0,(1_G\otimes A^{-1})a_1,t)\ h(x)\\ &= \chi(t)\ \chi(b_q\otimes\beta((1_G\otimes A^{-1})a_1,x))\ h(x+(1_G\otimes A^T)a_0).\end{aligned}$$

Since $h(u)=f((1_G\otimes A^{-T})u)$, we deduce that

$$\begin{aligned}\pi^s(a,t)h\ (x) &= \chi(t)\ \chi(b_q\otimes\beta((1_G\otimes A^{-1})a_1,x))\ f((1_G\otimes A^{-T})(x+(1_G\otimes A^T)a_0))\\ &= \chi(t)\ \chi(b_q\otimes\beta((1_G\otimes A^{-1})a_1,x))\ f((1_G\otimes A^{-T})x+a_0)\\ &= \chi(t)\ \chi(b_q\otimes\beta(a_1,(1_G\otimes A^{-T})x))\ f((1_G\otimes A^{-T})x+a_0)\\ &= \rho_s g\ (x).\end{aligned}$$

□

We begin with the Weil representation associated to the group $\mathrm{SL}_2(\mathbb{R})$ (corresponding to the case when the genus of the surface is 1) and then we describe the general case.

Let $\mathrm{SL}_2(\mathbb{R})$, resp. $\mathrm{SL}_2(\mathbb{Z})$, be the multiplicative group of 2 by 2 matrices with real coefficients (resp. with integer coefficients) and determinant equal to one. Let $H=\{\tau=u+iv\in\mathbb{C}\ |v>0\}$ be the upper half plane. The formula

$$(M,\tau)\mapsto M\cdot\tau=\frac{a\tau+b}{c\tau+d},\quad M=\begin{bmatrix} a & b\\ c & d\end{bmatrix},\quad \tau\in H$$

defines a transitive (resp. discontinuous) action of $\mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SL}_2(\mathbb{Z})$) on H. It is well known (see for instance [**57**, Chap. VII; Théorème 2]) that

$$S=\begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}\quad\text{and}\quad T=\begin{bmatrix} 1 & 1\\ 0 & 1\end{bmatrix}$$

generate $\mathrm{SL}_2(\mathbb{Z})$ with relations

$$S^2 = (ST)^3,\ (ST)^6 = 1.$$

The group $\mathrm{SL}_2(\mathbb{R})$ admits a double cover, called the *metaplectic group* $\mathrm{Mp}_2(\mathbb{R})$. This group is realized as the set of pairs

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad \tau \mapsto f_M(\tau)$$

where $f_M(\tau)$ is a holomorphic solution of the equation $c\tau + d = f_M(\tau)^2$. In other words, $\tau \mapsto f_M(\tau)$ is a function defined in H as a holomorphic square root of the holomorphic function $\tau \mapsto c\tau + d$. Elements in $\mathrm{Mp}_2(\mathbb{R})$ obey the associative multiplication law

$$(M, f_M(\tau)) \cdot (N, f_M(\tau)) = (MN, f_M(N \cdot \tau) f_N(\tau)) \tag{7.6}$$

which turns $\mathrm{Mp}_2(\mathbb{Z})$ into a group with unit

$$1_{\mathrm{Mp}_2(\mathbb{Z})} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1 \right).$$

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. The following lemma is a consequence of well known facts about $\mathrm{SL}_2(\mathbb{Z})$.

LEMMA 7.1. *The group* $\mathrm{Mp}_2(\mathbb{Z})$ *is generated by the two elements*

$$\hat{S} = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \sqrt{\tau} \right) \quad \textit{and} \quad \hat{T} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1 \right)$$

with relations

$$\hat{Z} = \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \sqrt{-1} \right) = \hat{S}^2 = (\hat{S}\hat{T})^3,\ \hat{Z}^2 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1 \right),\ \hat{Z}^4 = 1_{\mathrm{Mp}_2(\mathbb{Z})}.$$

The order 4 *element* $\hat{Z}$ *generates the center of* $\mathrm{Mp}_2(\mathbb{Z})$.

Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on a finite abelian group G such that $\gamma(G, q) \neq 0$. There is a *unitary* representation $\rho_q : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[G])$, called the *Weil representation*, associated to (G, q). Let $(\mathfrak{e}_g)_{g \in G}$ be the standard basis of the group ring $\mathbb{C}[G]$ so that $\mathfrak{e}_g \cdot \mathfrak{e}_h = \mathfrak{e}_{g+h}$ (where dot denotes the (convolution) product of $\mathbb{C}[G]$). Then $\rho = \rho_q$ is defined by the action on the generators $\hat{S}, \hat{T} \in \mathrm{Mp}_2(\mathbb{Z})$ by

$$\rho(\hat{S})\mathfrak{e}_g = |G|^{-\frac{1}{2}} \gamma(G, q) \cdot \sum_{h \in G} \exp(-2\pi i\ b_q(g, h))\ \mathfrak{e}_h \tag{7.7}$$

$$\rho(\hat{T})\mathfrak{e}_g = \exp(2\pi i\ q(q))\ \mathfrak{e}_g \tag{7.8}$$

One computes that

$$\rho(\hat{Z})\mathfrak{e}_g = \gamma(G, q)^2\ \mathfrak{e}_{-g}.$$

For a proof using relations of $\mathrm{SL}_2(\mathbb{Z})$ that these formulas indeed define a full projective action of $\mathrm{SL}_2(\mathbb{Z})$, see [**47**].

Let $\hat{A} = (A, f_A)$ be a preimage of $A \in \mathrm{SL}_2(\mathbb{Z})$. Then according to (7.6), the other preimage of A is $\hat{A}\hat{Z}^2 = (A, -f_A)$. Since $\rho(\hat{Z}^2) = \rho(\hat{Z})^2 = \gamma(G, q)^4\ \mathrm{id}_{\mathbb{C}[G]}$, we see that ρ induces a linear representation of $\mathrm{SL}_2(\mathbb{Z})$ if $\gamma(G, q)^4 = 1$ and only a projective representation of $\mathrm{SL}_2(\mathbb{Z})$ otherwise. For a homogeneous quadratic function q, the Gauss sum $\gamma(G, q)$ is an eighth root of unity. It follows that the corresponding cocycle $c(A, B) = c_q(A, B)$ lies in $\{-1, +1\}$ for q is homogeneous. It can be computed as follows. Choose first a canonical preimage $\hat{A}$ in $\mathrm{Mp}_2(\mathbb{Z})$ of each $A \in \mathrm{SL}_2(\mathbb{Z})$ by using a fixed branch cut for the argument of f_A. Then set $\rho(A) = \rho(\hat{A})$. We have by definition

$$\rho(AB) = c(A, B)\ \rho(A)\ \rho(B), \quad A, B \in \mathrm{SL}_2(\mathbb{Z}). \tag{7.9}$$

It follows from this definition and (7.6) that

$$c(A, B) = \rho(AB, f_{AB}(\tau))\ \rho(AB, f_A(B\tau) f_B(\tau))^{-1} = f_{AB}(\tau)\ f_A(B\tau)^{-1}\ f_B(\tau)^{-1}.$$

The second equality follows from the fact that the second term differs only by the choice of the square roots and a different choice introduces only a sign factor. For the same reason, we see that $c(A, B)$ is independent of the actual value of τ. The explicit computation of ρ for an arbitrary element $\hat{A} \in \mathrm{Mp}_2(\mathbb{Z})$ is carried out in [**56**] and [**60**].

For $g \geqslant 1$, let

$$\mathrm{Sp}_{2\mathfrak{g}}(\mathbb{R}) = \{M \in \mathrm{GL}_{2g}(\mathbb{R}) \mid M^T J M = J\}, \quad J = \begin{bmatrix} 0 & 1_g \\ -1_g & 0 \end{bmatrix},$$

the symplectic group over $\mathbb{R}$. This group contains a most important discrete subgroup $\mathrm{Sp}_{2g}(\mathbb{Z})$, consisting of symplectic matrices with integer coefficients, called Siegel's modular group. Exemples of integral symplectic matrices are

$$\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}, \begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix} \text{ with } B = B^T \text{ integral}, \begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix} \text{ with } A \in \mathrm{GL}_g(\mathbb{Z}). \tag{7.10}$$

The set of matrices above generates $\mathrm{Sp}_{2g}(\mathbb{Z})$, see [**58**]. Let H_g denote the set of $g \times g$ symmetric matrices with complex coefficients the imaginary part of which is definite positive (Siegel's half space). The formula

$$(M, Z) \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad Z \in H_g$$

defines a transitive (resp. discontinuous) action of $\mathrm{Sp}_{2g}(\mathbb{R})$ (resp. $\mathrm{Sp}_{2g}(\mathbb{Z})$) on H_g. Since $\pi_1(\mathrm{Sp}_{2g}(\mathbb{R})) = \mathbb{Z}$, the symplectic group admits a double cover $\mathrm{Mp}_{2g}(\mathbb{R})$, called the metaplectic group. This group can be realized as pairs

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2\mathfrak{g}}(\mathbb{R}), \quad Z \mapsto \sqrt{\det(CZ + D)}.$$

Here the map $Z \mapsto \sqrt{\det(CZ + D)}$ is a holomorphic square root of the holomorphic map $Z \mapsto \det(CZ+D)$ (See for instance [**20**, Chap. I, Remarks 2.3 and 3.1]). The group multiplication is given by the same formula as above, except that the action is replaced by the action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on H_g. The metaplectic group over $\mathbb{Z}$ is defined as the inverse image of $\mathrm{Sp}_{2g}(\mathbb{Z})$ under the covering map $\mathrm{Mp}_{2g}(\mathbb{R}) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. The elements

$$\hat{S} = \left(\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}, \sqrt{\det(Z)}\right), \hat{T} = \left(\begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}, 1\right) \text{ and } \hat{U} = \left(\begin{bmatrix} A & 0 \\ 0 & (A^T) - 1 \end{bmatrix}, \sqrt{\det(A^{-1})}\right)$$

lying above (7.10) generate $\mathrm{Mp}_{2g}(\mathbb{Z})$.

Remarkably, the Weil representation extends to the metaplectic group $\mathrm{Mp}_{2g}(\mathbb{Z})$ for any $\mathfrak{g} \geqslant 1$. This is part of the content of the Theorem below. This extension is based on tensor product as follows. Let $(\mathfrak{e}_x)_{x \in G \otimes \mathbb{Z}^g}$ be a basis for $\mathbb{C}[G \otimes \mathbb{Z}^g]$. The symbol 1_g used to denote the $g \times g$ identity matrix shall also be used to denote the canonical positive definite bilinear symmetric pairing

$$(a, b) \mapsto \sum_{1 \leqslant j \leqslant g} a_j b_j, \quad a, b \in \mathbb{Z}^g.$$

Define a map $\rho = \rho_q : \mathrm{Mp}_{2g}(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[G \otimes \mathbb{Z}^g])$ by

$$\rho(\hat{S})\mathfrak{e}_x = |G|^{-\frac{g}{2}} \gamma(G, q) \sum_{y \in G \otimes \mathbb{Z}^g} \exp(-2\pi i (b_q \otimes 1_G)(x, y))\ \mathfrak{e}_y. \tag{7.11}$$

$$\rho(\hat{T})\mathfrak{e}_x = \exp(2\pi i (q \otimes B)(x))\ \mathfrak{e}_x \tag{7.12}$$

$$\rho(\hat{U})\mathfrak{e}_x = \mathfrak{e}_{(1_G \otimes A^T)^{-1} x}. \tag{7.13}$$

These formulas specialize to the case $g = 1$ which is the case considered above. It is not immediately clear that these formulas fit to yield a representation of the metaplectic group. This, however, will be seen below as a consequence of the previous section.

Given a closed oriented surface Σ_g, it will be convenient to endow it with a symplectic basis for $H_1(\Sigma_g)$ so that $\mathrm{Sp}(H_1(\Sigma_g), \bullet)$ is identified with $\mathrm{Sp}_{2g}(\mathbb{Z})$.

THEOREM 7.1. *The map*

$$\rho : \mathrm{Mp}_{2g}(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[G \otimes \mathbb{Z}^g])$$

is a linear representation of the metaplectic group. It induces a projective representation

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[G \otimes \mathbb{Z}^g]),\ A \mapsto \rho(\hat{A})$$

of the symplectic group.

The TQFT *projective representation*

$$\mathcal{M}(\Sigma_g) \to \mathrm{Aut}(\mathcal{T}(\Sigma_g)) = \mathrm{Aut}(\mathbb{C}[G^g]),\ [f] \mapsto \tau(f)$$

defined in the previous section factors through the projective Weil representation

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[G \otimes \mathbb{Z}^g]),\ f_* \mapsto \rho(f_*)$$

so that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Mp}_{2g}(\mathbb{Z}) \arrow[d] \arrow[dr, "\rho"] & \\
\mathrm{Sp}_{2g}(\mathbb{Z}) \arrow[r] & \mathrm{Aut}(\mathbb{C}[G \otimes \mathbb{Z}^g] \arrow[d, equal] \\
\mathcal{M}(\Sigma) \arrow[u, "*"] \arrow[r, "\tau"] & \mathrm{Aut}(\mathbb{C}[G^g])
\end{tikzcd}$$

This theorem achieves our final aim. By means of the previous section (formula 3.2), the Weil representation is now interpreted intrinsically as a topological cobordism invariant. Note that this form of the Weil representation requires a lattice presentation for the quadratic form (G, q).

PROOF. We fix a standard handlebody of genus g whose boundary is the standard closed oriented surface Σ of genus g. The surface Σ is equipped with its standard geometrical symplectic basis $b = ([m_1], \ldots, [m_g], [l_1], \ldots, [l_g])$ (Fig. 1.1). We endow $H = H_1(\Sigma)$ with the basis b. By means of this basis, $\mathrm{Sp}(H, \bullet)$ is identified to $\mathrm{Sp}(2g, \mathbb{Z})$. It suffices to prove that the two representations coincide on a system of generators for the symplectic group. It will be convenient for our purpose to use the following small set of generators. The following lemma follows from Birman's work on the symplectic group [**2**].

LEMMA 7.2. *For $1 \leqslant k, l \leqslant g$, denote by E_{kl} the (k,l)-elementary square matrix of size g. For $1 \leqslant j \leqslant g-1$, let $B_j = E_{jj} - E_{j,j+1} - E_{j+1,j} + E_{j+1,j+1}$. The symplectic group* $\mathrm{Sp}(2g, \mathbb{Z})$ *is generated by the matrices*

$$S = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}, \ T(i) = \begin{bmatrix} 1_g & E_{ii} \\ 0 & 1_g \end{bmatrix}, \ T(j, j+1) = \begin{bmatrix} 1_g & B_j \\ 0 & 1_g \end{bmatrix} \tag{7.14}$$

with $1 \leqslant i \leqslant g$ and $1 \leqslant j \leqslant g-1$.

Hence it suffices to compute $\tau(f)$ where $[f] \in \mathcal{M}(\Sigma)$ is such that $f_* = S$, $T(i)$ and $T(j, j+1)$, for $1 \leqslant i \leqslant g$ and $1 \leqslant j \leqslant g-1$.

Let $\mathrm{Cyl}(\Sigma)$ denote the set of parametrized cylinders over Σ up to parametrization-commuting diffeomorphisms. There is an obvious composition of parametrized cylinders defined by gluing the bottom base of the top cylinder to the top base of the bottom cylinder and composing parametrizations. Then $\mathrm{Cyl}(\Sigma)$ becomes a group, clearly isomorphic to the mapping class group $\mathcal{M}(\Sigma)$.

Therefore, in order to compute $\tau(f)$, we may represent mapping classes as parametrized cylinders over Σ and use §**??**.

The next step consists in presenting such a parametrized cylinder as a special kind of tangle. The class of such special tangles can be made up into a group (see Turaev's presentation in [**61**, IV, §2] and Matveev–Polyak's approach in [**41**][1]). We now sketch the construction, referring to [**61**, IV, §2], [**41**] and [**9**] for details.

Consider a cylinder $M(f) = \Sigma \times [0, 1]$ with the bottom base parametrized by $f \in \mathrm{Diff}^+(\Sigma)$ and the top base by the identity. Glue to $M(f)$ two standard handlebodies along the given parametrizations on $\partial_- M = \Sigma$ and $\partial_+ M = \Sigma$ respectively.

This yields a closed oriented 3-manifold $\widetilde{M}$. Here M is the oriented cylinder $\Sigma \times [0, 1]$, $\Sigma = \Sigma_- = \Sigma_+$. The two sets of longitudes form a link $l^+ \subset M(f)$ and a link $f_*(l^-) \subset M(f)$. Each link is oriented and comes endowed with a framing (from the standard framing on the surface and the parametrization). We denote these links by L^-, L^+ respectively, after the gluing, that is, inside $\widetilde{M}$. Now $\widetilde{M}$ is presented by surgery on a framed link L in S^3. By isotoping

[1] The two approaches are completely equivalent, but distinct. Turaev's approach is geometrically symmetric, but composition introduces new components while Matveev–Polyak's approach leads to a more natural composition but is not symmetric. In order to be compatible with our previous work [**9**], we use here Turaev's presentation. Since we represent the full images of longitudes, our tangles will actually always be links.

$L^+ \cup L^-$, we may push it into the exterior of L in S^3. The combinatorial data that consists of the link $\mathcal{L} = L^- \cup L \cup L^+$ in S^3 determines the parametrized cylinder $\Sigma \times [0,1]$ up to cobordism equivalence. (This presentation of the parametrized cylinder is not unique. See [**41**] for a complete set of moves.)

For each $(c^+, -c^-) \in G^g \times G^g$, there is a $\mathbb{C}$-valued invariant

$$\tau_{c^-,c^+}(\widetilde{M}) = \tau\big(\widetilde{M}, \mathcal{L}; q, (c^+, -c^-)\big) \tag{7.15}$$

defined by (1.1). Following [**9**, §3], we consider the square $|G|^g \times |G|^g$ matrix defined by

$$\tau_M = |G|^{-g/2} \cdot \Big(\tau_{c^-,c^+}(\widetilde{M})\Big)_{c^- \in G^g,\ c^+ \in G^g}.$$

It follows from [**9**, §4, Th. 3] that τ_M is the matrix of the linear map $\tau(M) : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ with respect to the basis specified by Remark 5.1. With the notation of Remark 5.1, $\tau_M = \tau'(M)$.

It remains to perform the computations for three tangles representing three elements in the mapping class group $\mathcal{M}(\Sigma)$ inducing in homology the three elements described in Lemma 7.2.

In the three figures below, components with a small circle have a positive $+1$-framing. All other components have framing 0.

The first link in Fig. 7.1 represents a mapping class f such that $\mathrm{Mat}_b(f_*) = S$.

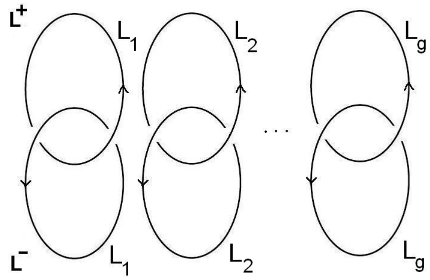


FIGURE 7.1. A special link in S^3 representing a mapping class in $\mathcal{M}(\Sigma_g)$ whose action in homology is S with respect to the symplectic basis b.

The link in Fig. 7.2 represents the positive Dehn twist t_{m_j} relative to the j-th meridian. Note that $\mathrm{Mat}_b(t_{m_j}) = T(j)$.

Recall the simple closed curve k_j on Σ (depicted in Fig. 1.1). The link in Fig. 7.3 represents the positive Dehn twist t_{k_j} relative to the simple closed curve k_j. Note that $\mathrm{Mat}_b(t_{k_j}) = T(j, j+1)$.

It remains to check that (7.15) yields the same formulas in these three cases as described by (7.11) and (7.12). $\square$

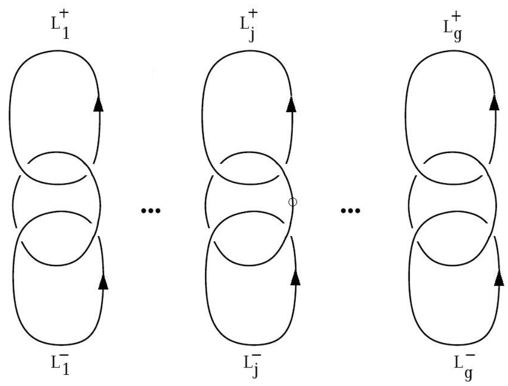


FIGURE 7.2. A special link in S^3 representing the positive Dehn twist relative to the j-th meridian m_j.

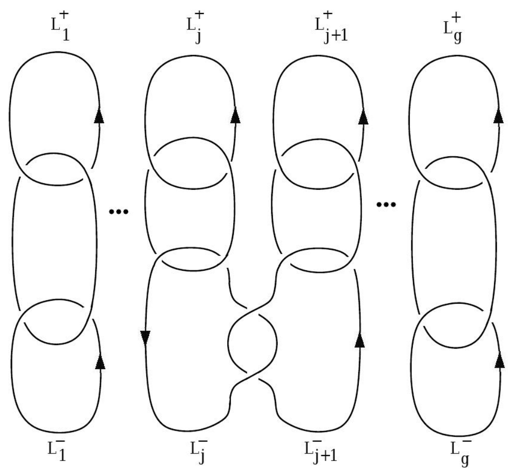


FIGURE 7.3. A special link in S^3 representing the positive Dehn twist relative to the j-th simple closed curve k_j.

Part 2

Abelian topological quantum field theory

CHAPTER 4

Linking and intersection pairings on manifolds

CHAPTER 5

Kirby calculus for oriented framed links in 3-manifolds

Throughout this chapter, M is a closed oriented connected 3-manifold. Consider now an n-component oriented link $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n \subset M$. A *framing* on $\mathcal{L}$ is a trivialization of its unit normal bundle in M. It is completely determined by a vector field $v_{\mathcal{L}}$ on $\mathcal{L}$ of constant length 1. The set of isotopy classes of framings on each component of $\mathcal{L}$ in M (fixed at some basepoint on each component) is freely and transitively acted on by $\mathbb{Z}$ (see e.g., [**25**, Chap. 4]). The *parallel* $\mathcal{L}'_j$ of a component $\mathcal{L}_j$ of $\mathcal{L}$ is obtained by pushing slightly $\mathcal{L}_j$ in the direction of the vector field $v_{\mathcal{L}}$. If $M = S^3$, the map $v_{\mathcal{L}} \mapsto (\mathrm{lk}(\mathcal{L}'_j, \mathcal{L}_j))_j$ gives an explicit correspondence between framings on $\mathcal{L}$ and $\mathbb{Z}^n$. By a theorem of Lickorish and Wallace, any closed oriented connected 3-manifold M is homeomorphic to the manifold M_L obtained by surgery on the 3-sphere S^3 on some framed link $L = L_1 \cup \cdots \cup L_m \subset S^3$ (see e.g. [**25**, Chap. 5]). Another fundamental result due to Kirby asserts that two framed links in S^3 yield homeomorphic 3-manifolds if and only if they are related by a finite sequence of operations called "Kirby moves" [**35**]. (See also Remark 0.3 and Fig. 0.1 below.)

Suppose that M is presented by a framed and oriented link $L = L_1 \cup \cdots \cup L_m$ in S^3. Denote by A_L the integral symmetric matrix whose (j,k) entry is the linking coefficient $\mathrm{lk}_{S^3}(L_j, L_k) \in \mathbb{Z}$ of L_j and L_k in S^3 if $j \neq k$ and whose (j,j) entry is the linking coefficient (framing number) $\mathrm{lk}_{S^3}(L_j, L'_j) \in \mathbb{Z}$ of L_j and its parallel L'_j in S^3. The matrix A_L is called the *linking matrix* of L in S^3. The result of surgery can be regarded as the trace of the following 4-dimensional surgery:

$$M = M_L = \partial X_L, \quad X_L = D^4 \cup \cup_{i=1}^{m} (D^2 \times D^2)_i,$$

where each 2-handle $D^2 \times D^2$ is attached to the 4-ball D^4 by embedding $\partial D^2 \times D^2 = S^1 \times D^2$ into $\partial D^4 = S^3$ in accordance to the orientation and framing of L_i. The group $W_L = H_2(X_L)$ is a finitely generated free abelian and carries an intersection pairing $g_L : W_L \times W_L \to \mathbb{Z}$ which turns it into a bilinear lattice. Choosing a Seifert surface $\Sigma_i \in S^3$ (which can be slightly pushed into the interior of D^4) for each component L_i and closing it by the core of the i-th handle yields a closed surface S_i. The set $([S_1], \ldots, [S_m])$ form a basis of $H_2(X_L)$. With respect to this basis, the matrix of g_L is the linking matrix A_L.

The *linking pairing* of M is an algebraic linking pairing $\lambda_M : \mathrm{Tors}\, H_1(M) \times \mathrm{Tors}\, H_1(M) \to \mathbb{Q}/\mathbb{Z}$ which is a fundamental homotopy invariant of M [**25**, Chap. 4] (see also §**??** below for a definition). It is a key observation that the discriminant construction from (W_L, g_L) yields the linking pairing $(\mathrm{Tors}\ H_1(M), \lambda_M)$ with opposite sign (for the usual convention of orientation of M). With the notation previously introduced, we can state it as

LEMMA 0.3.

$$(G_{g_L}, \lambda_{g_L}) \simeq (\mathrm{Tors}\ H_1(M), -\lambda_M). \tag{0.16}$$

This result will be used several times in the sequel.

The next step is to consider an ordered, oriented and framed n-component link $\mathcal{L}$ in a closed oriented 3-manifold M. Two such pairs $(M, \mathcal{L})$ and $(N, \mathcal{J})$ are diffeomorphic if there is a diffeomorphism $\varphi : M \to N$ such that $\varphi(\mathcal{L}_j) = \mathcal{J}_j$ (sending the j-th component onto the j-th component) and $\varphi^* v_{\mathcal{J}_j} = v_{\mathcal{L}_j}$ (preserving the j-th framing). Note that if two links $\mathcal{J}$ and $\mathcal{K}$ are isotopic in N (as oriented framed ordered links), then $(N, \mathcal{J})$ and $(N, \mathcal{K})$ are diffeomorphic.

The surgery presentation of an oriented link in a closed oriented 3-manifold induces canonically an isomorphism class of a pointed bilinear lattice (see 3.6 for the definition). The construction is as follows. Let $(L = L_1 \cup \cdots \cup L_m, J = J_1 \cup \cdots \cup J_n)$ be a pair of oriented and framed links presenting a link $\mathcal{L}$ in a closed oriented 3-manifold M. Each component J_k when viewed in X_L bounds a surface Σ_k (that can be obtained for instance from a Seifert surface for J_k in S^3). Such a surface Σ_k represents an element $[\Sigma_k] \in H_2(X_L, M_L)$. The intersection pairing $H_2(X_L, M_L) \times H_2(X_L) \to \mathbb{Z}$ is nonsingular (by Poincaré duality and the fact that X_L is simply connected). Hence by the left adjoint map, $[\Sigma_k]$ yields an element $e_k^J \in \mathrm{Hom}(H_2(X_L), \mathbb{Z}) = W_L^*$. The exact sequence

$$\cdots \longrightarrow H_2(X_L) \xrightarrow{j_*} H_2(X_L, M_L) \xrightarrow{\partial} H_1(M_L) \longrightarrow \cdots$$

shows that any other choice for $[\Sigma_k]$ will differ by an element in $\mathrm{Im}\ j_*$. So any other choice for e_k will differ by an element in $\widehat{g}_L(W_L)$. Hence the assignment

$$J_k \mapsto [e_k] \in \mathrm{Coker}\ \widehat{g}_L = G^g$$

is well defined. Therefore the *isomorphism class* of the pointed bilinear lattice $(W_L, g_L, e_1^J, \ldots, e_n^J)$ is well defined. Note that this class is independent of the framing of $\mathcal{L}$.

We make explicit a "Kirby calculus" slightly generalized to this setting. We describe briefly the reversible moves indicated in Fig. 0.1. The first four moves are the classical Kirby moves and involve only the first link: the first two moves are the usual stabilization Kirby moves applied to the first link L (the trivial component added is unlinked from all other component); the third move consists in reversing orientation of the first link L; the fourth move is the usual Kirby handle slide move applied between two components of the first link L. The fifth move is a handle slide of one component of the second link J over one component of the first link L. These handle slides are performed regardless of the orientations of the components by choosing a small band connecting the two components.

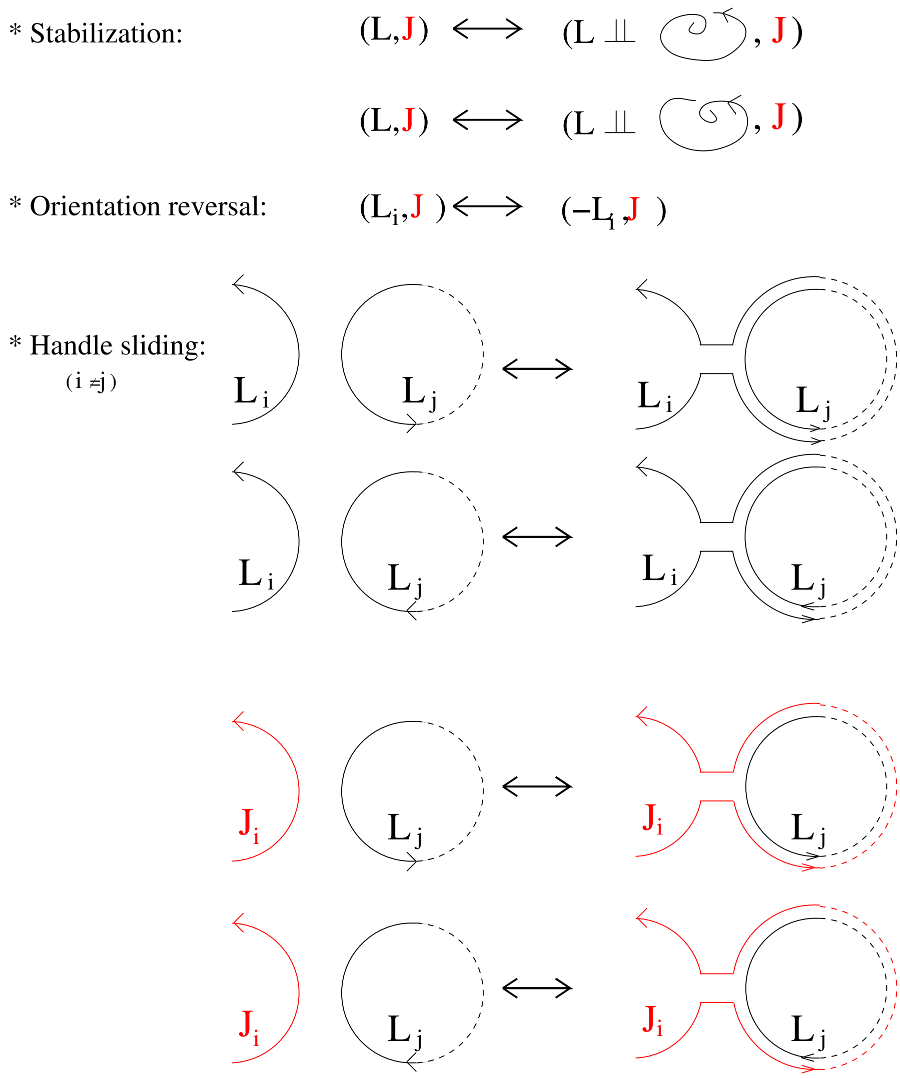


FIGURE 0.1. The Kirby moves for links in 3-manifolds.

THEOREM 0.2. *Two pairs of disjoint oriented and framed links* (L, J) *and* (L', J') *in* S^3 *present diffeomorphic pairs* $(M, \mathcal{L})$ *if and only if, up to re-ordering of the components of* L *and up to isotopy, they are related by a finite sequence of moves indicated in Fig.* 0.1.

REMARK 0.3. The classical Kirby theorem corresponds to the special case $J = J' = \varnothing$.

PROOF. This is straightforward from the usual Kirby's theorem. For each Kirby move $L_1 \to L_2$, we consider the corresponding diffeomorphism $M_{L_1} \to M_{L_2}$ of 3-manifolds and write down combinatorially the image of the link $\mathcal{L}$ in M_{L_2}. $\square$

REMARK 0.4. Theorem 0.2 applies to oriented framed ordered links in oriented closed 3-manifolds. There are actually several versions of Th. 0.2, depending on the exact class of links: ordered/unordered, framed/unframed, oriented/unoriented. In particular, there are:

(1) [unordered, framed, oriented] The set of pairs $(M, \mathcal{L})$ formed by an oriented and framed link $\mathcal{L}$ in an oriented closed connected 3-manifold M. Two such pairs $(M, \mathcal{L})$ and $(N, \mathcal{J})$ are *diffeomorphic* if there is a diffeomorphism $\varphi : M \to N$ such that $\varphi(\mathcal{L}) = \mathcal{J}$ and $\varphi^* v_{\mathcal{J}} = v_{\mathcal{L}}$. Such a pair $(M, \mathcal{L})$ is presented by a pair (L, J) of disjoint oriented and framed links in S^3. The version of Th. 0.2 is in this case:

THEOREM 0.3. *Two pairs of disjoint oriented and framed links* (L, J) *and* (L', J') *in* S^3 *present diffeomorphic pairs* $(M, \mathcal{L})$ *if and only if, up to reordering of the components of each of the individual links (L and J), they are related by a finite sequence of moves indicated in Fig.* 0.1.

(2) [ordered, unframed, oriented] The set of oriented and ordered links in oriented closed 3-manifolds. Such a link is presented by a pair (L, J) of disjoint oriented links where L is framed and J is ordered. Two such links $(M, \mathcal{L})$ and $(N, \mathcal{J})$ are *diffeomorphic* if there exists a diffeomorphism $\varphi : M \to N$ such that $\varphi(\mathcal{L}_j) = \mathcal{J}_j$. The corresponding version of Th. 0.2 is obtained by forgetting the framing of the link J in the surgery presentation (L, J) and in the Kirby moves of Fig. 0.1.

REMARK 0.5. The Kirby moves for pairs of disjoint links (a framed and oriented link in S^3, an oriented link in S^3) preserve the stable equivalence class of the (isomorphism class of the) pointed bilinear lattice defined in the previous remark. This is straightforward from the definitions (Fig. 0.1).

Let $A = A_{L\cup J}$ the linking matrix of the link $L \cup J$ in S^3. This is an integral symmetric matrix of size $m + n$. To distinguish it from the usual linking matrix A_L, we call it the *extended linking matrix*. The extended linking matrix $A_{L\cup J}$ not only represents the stable equivalence class of the pointed bilinear lattice $(W_L, g_L, e_1, \ldots, e_n)$ but also encodes the *framing* of the link $\mathcal{L} \subset M$.

Let us describe how the Kirby moves affect the linking matrix $A_{L\cup J}$. We denote below $A_{L,J}$ the $m \times n$ matrix defined by $(A_{L\cup J})_{st} = \mathrm{lk}(L_s, J_t)$, $1 \leqslant s \leqslant m$, $1 \leqslant t \leqslant n$. We set $A_{J,L} = A^{\mathrm{t}}_{L,J}$.

Stabilization:

$$A_{L\cup J} = \left[\begin{array}{c|c} A_L & A_{L,J} \\ \hline A_{J,L} & A_J \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c|ccc} & & & 0 & & & \\ & A_L & & \vdots & & A_{L,J} & \\ & & & 0 & & & \\ \hline 0 & \cdots & 0 & \pm 1 & 0 & \cdots & 0 \\ \hline & & & 0 & & & \\ & A_{J,L} & & \vdots & & A_J & \\ & & & 0 & & & \end{array}\right]$$

Orientation reversal: for $1 \leqslant i \leqslant m$, let $I_i(-1)$ be the size $m+n$ square matrix whose (j,j) entry is 1 for all $1 \leqslant j \neq i \leqslant m+n$, whose (i,i) entry is -1 and all other entries are zero. Reversing the orientation of one component L_i of L induces the transformation

$$A_{L\cup J} \leftrightarrow I_i(-1) \cdot A_{L\cup J} \cdot A_i(-1)^{\mathrm{t}}.$$

To handle the handle slide cases, we introduce some matrix notation. Let $N, N' \geqslant 1$. For $1 \leqslant s \leqslant N$ and $1 \leqslant t \leqslant N'$, let $E_{ij}(N,N')$ be the $N \times N'$ matrix whose (s,t) entry is 1 and all other entries are zero. If $N = N'$ (square matrices), we write $E_{ij}(N,N') = E_{ij}(N)$. Let $I_N = \sum_k E_{kk}(N)$ the identity matrix of size $N \times N$. Let $T_{ij}(N) = I_N + E_{ij}(N) \in \mathrm{GL}_N(\mathbb{Z})$. Note that $T_{st}(N)^{\mathrm{t}} = T_{ji}$ and $T_{st}(N)^{-1} = I_N - E_{st}(N)$.

Consider the first type of handle sliding. Let $1 \leqslant i \neq j \leqslant m+n$. Denote by $\widetilde{L}_i$ the new component after handle sliding. Sliding the i-th component of L onto the j-th component of L has the following effect.

$$\mathrm{lk}(\widetilde{L}_i, L_k) = \mathrm{lk}(L_i, L_k) \pm \mathrm{lk}(L_j, L_k), \text{ for all } 1 \leqslant k \neq i,j \leqslant m,$$

$$\mathrm{lk}(\widetilde{L}_i, L_j) = \mathrm{lk}(L_i, L_j) \pm \mathrm{lk}(L_j, L_j'),$$

$$\mathrm{lk}(\widetilde{L}_i, J_k) = \mathrm{lk}(L_i, J_k) \pm \mathrm{lk}(L_j, J_k), \text{ for all } 1 \leqslant k \leqslant n$$

and

$$\mathrm{lk}(\widetilde{L}_i, \widetilde{L}_i{}') = \mathrm{lk}(L_i, L_i') + \mathrm{lk}(L_j, L_j') \pm 2\,\mathrm{lk}(L_i, L_j).$$

Consider the case of the sign $+$ in the formulas above. The effect consists in adding to the i-th column of $A_{L\cup J}$ the j-th column of $A_{L\cup J}$ and adding to the i-th row of $A_{L\cup J}$ the j-th row of $A_{L\cup J}$. As is well known, this has a matrix interpretation: the handle slide of L_i over L_j transforms $A_{L\cup J}$ into $T_{ij}(m+n) \cdot A_{L\cup J} \cdot T_{ij}(m+n)^{\mathrm{t}}$. Since there is a block decomposition $T_{ij} = \left[\begin{array}{c|c} T_{ij}(m) & 0 \\ \hline 0 & I_n \end{array}\right]$, the corresponding transformation of $A_{L\cup J}$ is described by

$$A_{L\cup J} = \left[\begin{array}{c|c} A_L & A_{L,J} \\ \hline A_{J,L} & A_J \end{array}\right] \leftrightarrow \left[\begin{array}{c|c} T_{ij}(m) A_L T_{ij}(m)^{\mathrm{t}} & T_{ij}(m) A_{L,J} \\ \hline A_{J,L} T_{ij}(m)^{\mathrm{t}} & A_J \end{array}\right].$$

The case of the sign $-$ is treated similarly and the formula is similar: the matrix $T_{ij}(m+n)$ is replaced by its inverse $T_{ij}(m+n)^{-1} = I_{m+n} - E_{ij}(m+n)$.

The second type of handle sliding is similar, except that this time a component J_i of J slides over a component L_j of L. Here $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Given our ordering, the effect consists in adding to the $(m+i)$-th column of $A_{L\cup J}$ the j-th column of $A_{L\cup J}$ and adding to the $(m+i)$-th row of $A_{L\cup J}$ the j-th row of $A_{L\cup J}$. Hence $A_{L\cup J}$ is replaced by $T_{m+i,j}(m+n) \cdot A_{L\cup J} \cdot T_{m+i,j}(m+n)^{\mathrm{t}}$. Observe that $T_{m+i,j}(m+n) =$

$\left[\begin{array}{c|c} I_m & 0 \\ \hline E_{ij}(n,m) & I_n \end{array}\right]$. We deduce that the transformation of $A_{L\cup J}$ is given by

$$A_{L\cup J} = \left[\begin{array}{c|c} A_L & A_{L,J} \\ \hline A_{J,L} & A_J \end{array}\right] \leftrightarrow \left[\begin{array}{c|c} A_L & A_{L,\tilde{J}} \\ \hline A_{\tilde{J},L} & A_{\tilde{J}} \end{array}\right]$$

with

$$A_{L,\tilde{J}} = A_L E_{ij}(n,m)^{\mathrm{t}} + A_{L,J},\ A_{\tilde{J},L} = A^{\mathrm{t}}_{L,\tilde{J}},$$

and $A_{\tilde{J}} = E_{ij}(n,m)A_L E_{ij}(n,m)^{\mathrm{t}} + E_{ij}(n,m)A_{L,J} + A_{J,L}E_{ij}(n,m)^{\mathrm{t}} + A_J$.

We call the transvections $T_{ij}(m+n)$, $1 \leqslant i \neq j \leqslant m$, $T_{m+i,j}(m+n)$, $1 \leqslant i,j \leqslant n$, the *handle slide transvections* (of first and second kind) respectively.

LEMMA 0.4. *The set of all handle slide transvections generate the subgroup*

$$\left[\begin{array}{c|c} \mathrm{SL}_m(\mathbb{Z}) & 0 \\ \hline \mathrm{Mat}_{n,m}(\mathbb{Z}) & I_n \end{array}\right]$$

of $\mathrm{SL}_{m+n}(\mathbb{Z})$. *The set of all handle slide transvections and all matrices* $I_i(-1)$ *(induced by orientation reversal of one component of* L*),* $1 \leqslant i \leqslant m$, *generate the subgroup*

$$G_{m,n} = \left[\begin{array}{c|c} \mathrm{GL}_m(\mathbb{Z}) & 0 \\ \hline \mathrm{Mat}_{n,m}(\mathbb{Z}) & I_n \end{array}\right] \subseteq \mathrm{GL}_{m+n}(\mathbb{Z}).$$

Let m, n be two nonnegative integers. Consider the set $\mathcal{S}_{m,n}$ of pairs (a nonnegative integer m, an integral symmetric square matrices of size $m+n$). (One should think of elements of $\mathcal{S}_{m+n}$ as an integral symmetric square matrix endowed with a block decomposition parametrized by m and n.)The group $G_{m,n}$ acts naturally on $\mathcal{S}_{m,n}$ by $(g, A) \mapsto g \cdot A \cdot g^t$, for $g \in G_{m,n}$ and $A \in \mathcal{S}_{m,n}$. Declare two such matrices A and B be *equivalent* if they are in the same orbit of the action of $G_{m,n}$. *Stabilization* on $A \in \mathcal{S}_{m,n}$ consists in replacing A by a matrix in $\mathcal{S}_{m+1,n}$ by adding to A one new $(m+1)$-th row and one new $(m+1)$-th column with $(m+1, m+1)$ entry equal to ± 1 and all other entries equal to 0. Let $\mathcal{S} = \cup_{m,n}\mathcal{S}_{m,n}$. Let $A, B \in \mathcal{S}$ be *stably equivalent* if after some finite number of stabilizations on A and B, they become equivalent.

The following result says that stably equivalent extended linking matrices induce stably equivalent pointed bilinear lattices.

PROPOSITION 0.2. *Let* (L, J) *and* $(\tilde{L}, \tilde{J})$ *be two pairs of oriented framed links in* S^3. *If* $A_{L\cup J}$ *and* $A_{\tilde{L}\cup\tilde{J}}$ *are two stably equivalent extended linking matrices, then their associated pointed bilinear lattices* $(W_L, g_L, e_1^J, \ldots, e_n^J)$ *and* $(W_{\tilde{L}}, g_{\tilde{L}}, e_1^{\tilde{J}}, \ldots, e_n^{\tilde{J}})$ *are stably equivalent.*

REMARK 0.6. The converse of Prop. 0.2 does not hold: consider the simplest case of the 3-sphere S^3 equipped with a trivial knot $\mathcal{L}$. A surgery presentation of this knot is the pair $(\varnothing, J = \mathcal{L})$ (no surgery is performed).

The linking matrix is an element in $\mathcal{S}_{0,1}$ (a one-entry matrix) given by the framing integer of $\mathcal{L}$ in S^3. We have $X_L = D^4$ so that the lattice $W_L = H_2(D^4)$ is trivial. The associated pointed bilinear lattice in this case is $(W_L = 0, g_L = 0, e^J = 0)$. If $J' = \mathcal{L}'$ is a trivial knot with a different framing in S^3, then $e^{J'} = e^J = 0$, so that $(\varnothing, \mathcal{L}')$ induces the same associated pointed bilinear lattice $(0, 0, 0)$.

Example 0.1. Consider the two pairs of knots of Fig. 0.2. They differ only by the framing of J in S^3. In both cases, surgery on the framed knot L yields

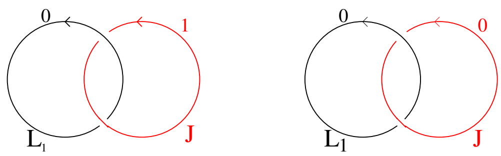


Figure 0.2. Two presentations of knots in $S^1 \times S^2$. The framing of each component in S^3 is indicated by an integer.

$S^1 \times S^2$ equipped with the same knot $\mathcal{L}$ of infinite order in $H_1(S^1 \times S^2) = \mathbb{Z}$. Therefore they give rise to the same stable equivalence class of pointed bilinear lattices, namely the stable equivalence class of $(\mathbb{Z}, 0, [\mathcal{L}])$. We claim that

(1) The oriented and framed knots presented by the two pairs are isotopic.
(2) Any Kirby equivalence between the two pairs of knots requires at least one stabilization.

Proof of the first claim. There is a sequence of Kirby moves transforming the first pair into the second pair indicated in Fig. 0.3. ∎

Remark 0.7. It is instructive to see the sequence of Kirby moves of Fig. 0.3 at the level of the extended linking matrix $A_{L\cup J}$. In the first case, the matrix is $A_0 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ while in the second case, it is $A_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. We have

$$A_1 = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] = A_0$$

The first operation is a stabilization; the second operation is the action of an element in $G_{2,1}$ (corresponding to the composition of the three handle slide moves indicated in Fig. 0.3 – note that these three handle slides commute one with another): indeed $\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{smallmatrix}\right]^{\mathrm{t}}$; the last operation is a destabilization. ∎

Proof of the second claim. To see why the second statement holds, we consider again the extended linking matrices A_0 and A_1. We claim that A_1 and A_2, as matrices in $\mathcal{S}_{1,1}$, are not equivalent. (But Remark 0.7 shows that after each of them is stabilized once, they are become equivalent.) In

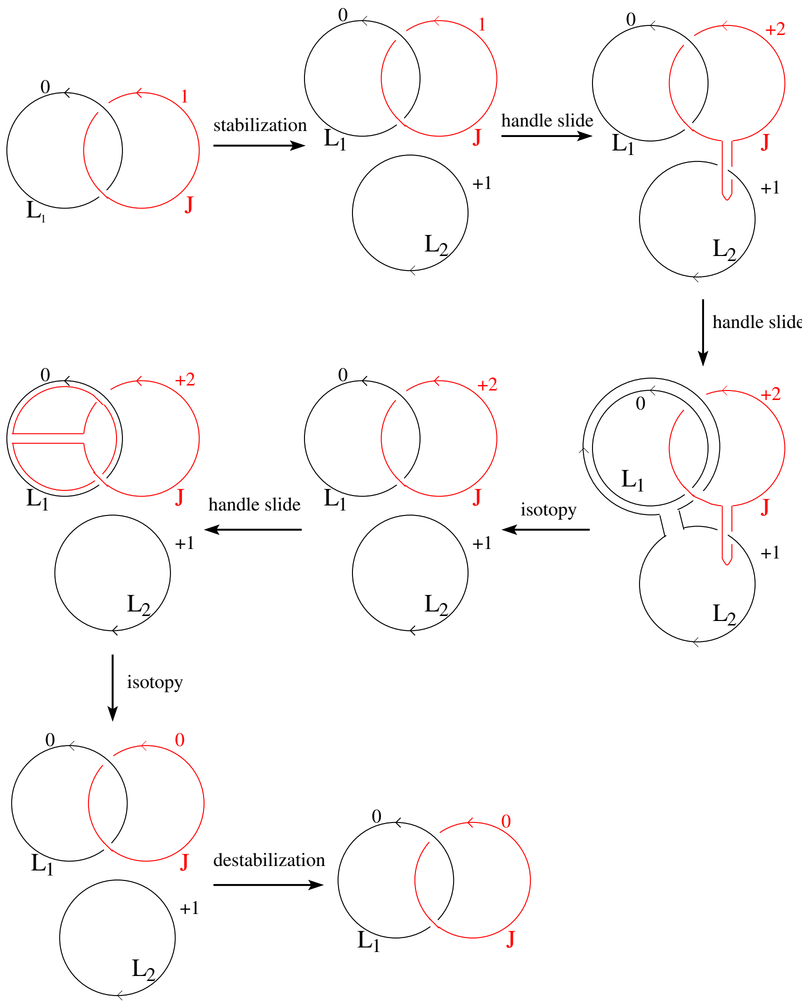


FIGURE 0.3. The two pairs of links are Kirby equivalent.

fact, the parity of the $(2,2)$ entry of A_j $(j = 0, 1)$ is preserved by action of $G_{1,1} = \left[\begin{array}{c|c} \pm 1 & 0 \\ \hline \mathbb{Z} & 1 \end{array}\right]$ on A_0. The claim follows. ■

CHAPTER 6

The invariant τ of a closed 3-manifold

Throughout this chapter, M is a closed oriented connected 3-manifold. We define a topological invariant τ of M and establish a number of properties of τ

1. Definition and first properties

Let $L = L_1 \cup \cdots \cup L_m$ be an oriented and framed link in S^3 presenting M. We denote by A_L the linking matrix of L as defined in the previous chapter. Recall that this is a square symmetric integral matrix of size m. By tensor product over $\mathbb{R}$, this matrix induces a symmetric bilinear pairing $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. Let $\operatorname{sign}(L) \in \mathbb{Z}$ denote its signature. In [**6**], we prove that the number

$$\tau(M, q) = \gamma(G, q)^{-\operatorname{sign}(L)} |G|^{-m/2} \sum_{x \in G \otimes \mathbb{Z}^m} \exp(2\pi i (q \otimes A_L)(x)) \tag{1.1}$$

is invariant under the Kirby moves and is therefore a topological invariant of M.

It is shown in [**6**, Th. 1] that $\tau(M, q)$ depends actually only on the linking pairing λ_M and the first Betti number $b_1(M) \in \mathbb{N}$: more precisely, if two closed oriented connected 3-manifolds M and N are related by an isomorphism $f : H_1(M) \to H_1(N)$ such that $\lambda_N(f(x), f(y)) = \lambda_M(x, y)$ for all $x, y \in \operatorname{Tors} H_1(M)$, then $\tau(M, q) = \tau(N, q)$. Therefore, $\tau(M, q)$ depends on the one hand, on the quadratic form (G, q) up to isomorphism and on the other hand, on the linking group $(H_1(M), \lambda_M)$ up to isomorphism (as defined in §...). The relation is made explicit by means of a reciprocity formula in [**6**, Th. 4].

In particular, we can make the following observation. Define an oriented closed 3-manifold M to be an *integral homology sphere* if $H_*(M) = 0$.

Proposition 1.1. $\tau(M, q) = 1$ *if and only if* M *is an integral homology sphere.*

Proof. Since $H_1(M) = 0$, the linking group $(H_1(M), \lambda_M)$ is trivial. □

This proposition shows that τ detects only nontrivial rational homology spheres. Therefore, one is led to present τ as an algebraic pairing which is the viewpoint adopted in [**10**]. Let $\mathfrak{Q}^0(0)$ denote the set of isomorphism classes of nondegenerate homogeneous quadratic forms on finite abelian groups. We define a related set $\mathfrak{M}(0)$ as follows. An element in $\mathfrak{M}(0)$ is represented by a pair (H, λ) where H is a finitely generated abelian group and λ : Tors $H \times$ Tors $H \to \mathbb{Q}/\mathbb{Z}$ is a linking pairing. Two such pairs represent the same element in $\mathfrak{M}(0)$ if there is an isomorphism between the groups that restrict on the respective torsion subgroups to an isomorphism of the linking pairings. Both sets are monoids for the orthogonal sum $\oplus$, with the trivial form being the neutral element. Define a pairing

$$\langle -, - \rangle : \mathfrak{M}(0) \times \mathfrak{Q}^0 \to \mathbb{C}, \quad (H, \lambda), q \mapsto \langle H, \lambda; q \rangle$$

by

$$\langle H, \lambda; q \rangle = \tau(M, q), \tag{1.2}$$

for any closed oriented 3-manifold M such that $\lambda_M = \lambda$ and $H_1(M) = H$. The existence of such a 3-manifold is ensured by [**33**]. The discussion above on $\tau(M, q)$ ensures that $\langle H, \lambda; q \rangle$ is independent of the particular choice of such a 3-manifold M. One can alternatively present the pairing $\langle -, - \rangle$ in a purely algebraic fashion as follows. Let (W, g) be any bilinear lattice such that $(G_g, \lambda_g) = (H, -\lambda)$ via the discriminant construction. Then

$$\langle H, \lambda; q \rangle = \gamma(G, q)^{-\operatorname{sign}(g)} \ \gamma(G \otimes W, q \otimes g) \ |G \otimes G_g|^{\frac{1}{2}} |G|^{\frac{n}{2}}. \tag{1.3}$$

The pairing $\langle -, - \rangle$ is biadditive in the sense that

$$\langle (H, \lambda) \oplus (H', \lambda'); q \rangle = \langle H, \lambda; q \rangle \cdot \langle H', \lambda'; q \rangle,$$

and

$$\langle H, \lambda; (q \oplus q') \rangle = \langle H, \lambda; q \rangle \cdot \langle H, \lambda; q' \rangle.$$

It follows from [**10**, Th. 1] that the pairing $\langle -, - \rangle : \mathfrak{M}(0) \times \mathfrak{Q}^0(0) \to \mathbb{C}$ is nondegenerate.

2. Extension to Spin structures

A *spin structure* σ on M is a trivialization considered up to homotopy of the tangent bundle over the 1-skeleton M^1 that extends over the 2-skeleton M^2 of M. Spin structures always exist on a closed 3-manifold M. The set of spin structures on M is in bijective correspondence with $H^1(M; \mathbb{Z}/2)$. A spin structure σ on M induces a canonical quadratic refinement

$$q_\sigma : \text{Tors } H_1(M; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

of the linking pairing λ_M. See for instance [**35**], [**25**], [**39**].

Using this fact, a topological invariant τ^{spin} of (M, σ) is defined as follows. Fix an element $(H, \lambda) \in \mathfrak{M}(0)$. The complex number

$$\tau^{\text{spin}}(M, \sigma; H, \lambda) = \langle H, \lambda; q_\sigma \rangle \tag{2.1}$$

is a topological invariant of the pair (M, σ). More precisely: if two closed oriented connected spin 3-manifolds (M, σ_M) and (N, σ_N) are related by an isomorphism $f : H_1(M) \to H_1(N)$ such that $q_{\sigma_N}(f(x)) = q(x)$ for all $x \in$ Tors $H_1(M)$, then $\tau^{\text{spin}}(M, \sigma; H, \lambda) = \tau^{\text{spin}}(N, \sigma_N; H, \lambda)$ for any $(H, \lambda) \in \mathfrak{M}(0)$.

A converse is proved in [**10**, Th. 2]: if $\tau^{\mathrm{spin}}(M,\sigma_M;H,\lambda) = \tau^{\mathrm{spin}}(N,\sigma_N;H,\lambda)$ for any $(H,\lambda) \in \mathfrak{M}(0)$, then (M,σ_M) and (N,σ_N) are related by an isomorphism $f : H_1(M) \to H_1(N)$ such that $q_{\sigma_N}(f(x)) = q_{\sigma_M}(x)$ for all $x \in \mathrm{Tors}\, H_1(M)$. It follows that τ^{spin} classifies Y^{spin} equivalent spin manifolds in Massuyeau's spin refinement of the Goussarov-Habiro theory (see [**39**]).

3. Extension to Spin^{c} structures

The idea of extending the invariant τ to Spin^{c} structures is formally similar to the extension to Spin structures. A Spin^{c} structure on an oriented closed connected 3-manifold M is a complex structure (considered up to homotopy) on the 2-skeleton M^2 that extends to M. For references on Spin^{c} structures, see [**25**] and [**11**]. The set $\mathrm{Spin}^{c}(M)$ of Spin^{c} structures on M is acted on freely and transitively by $H^2(M;\mathbb{Z})$.

The basic observation consists in the interpretation of Spin^{c} structures as quadratic refinements of the linking pairing, as in [**11**, Th. 2.3]. First define a modified linking pairing

$$\lambda'_M : H_2(M;\mathbb{Q}/\mathbb{Z}) \times H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

by the formula

$$\lambda'_M = \lambda_M \circ (\beta_M \times \beta_M),$$

where $\beta_M : H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_1(M)$ denotes the Bockstein homomorphism. This modified linking pairing is nondegenerate if and only if M is a rational homology 3-sphere. There is a natural $H^2(M;\mathbb{Z})$-embedding

$$\mathrm{Spin}^{c}(M) \to \mathrm{Q}(\lambda'_M), \ s \mapsto q^s \tag{3.1}$$

See [**11**, §2] for further details. To the Chern class $c(\sigma) \in H^2(M)$ corresponds the difference $d_{q^\sigma} : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ defined by $d_{q^\sigma}(x) = q^\sigma(x) - q^\sigma(-x) = \langle c(\sigma), x\rangle,\ x \in H_2(M;\mathbb{Q}/\mathbb{Z})$. A spin^{c} structure σ on M is *torsion* if its associated Chern class $c(\sigma) \in H^2(M)$ is torsion. The quadratic refinement q^σ is nondegenerate if and only if q^σ vanishes on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$ if and only if σ is torsion. In this case, the quadratic function q^σ factors through a unique quadratic refinement of the usual linking pairing λ_M. In particular, this is the case if M is a rational homology 3-sphere. As in [**8**], we shall consider only *torsion* spin^{c} structures.

A Spin structure induces naturally a Spin^{c} structure, hence there is a natural map $\mathrm{Spin}(M) \to \mathrm{Spin}^{c}(M)$. This map is injective if M is a rational homology 3-sphere. The image of this map is the set of Spin^{c} structures σ with trivial Chern class $c_1(\sigma) = 0$. This map and the maps above fit into the commutative diagram

$$\begin{tikzcd}
\mathrm{Spin}(M) \arrow[r] \arrow[d] & \mathrm{Spin}^{c}(M) \arrow[d] \\
\mathrm{Quad}(\lambda_M) \arrow[r] & \mathrm{Quad}(\lambda'_M).
\end{tikzcd}$$

The extension of the invariant τ to Spin^c structures is not obvious, however, because a priori one has to define a tensor product involving *non homogeneous* quadratic functions. If we try to mimic the definition of the spin case, then we run into the problem of defining the tensor product $q \otimes g$ where q is a possibly non homogeneous quadratic function[1]. An alternative product (and the corresponding extension) is proposed in [**10**], but the extension in question is shown to fail to have the property of classifying degree 0 invariants of complex spin structures (See [**12**, §3] for the foundations of the theory of finite type invariants of complex spin structures). Another extension suggested at the end of the same paper corresponds to the Gaussian invariant used in the classification of general quadratic functions described here in §3.3 (Th. 3.2).

[1] Lemma 1 of [**8**] applies in fact only to homogeneous quadratic functions.

CHAPTER 7

The invariant τ for an oriented framed link in a closed 3-manifold

The invariant $\tau(M, q)$ of the chapter extends to an invariant of framed links in M. (It was first introduced in [**8**]). We keep the previous notation. This chapter is devoted to the study of the properties of this invariant. In particular, we prove a conjecture stated in [**?**]...

1. Definition and invariance

Let $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$ be an oriented and framed link in a closed oriented 3-manifold M. The pair $(M, \mathcal{L})$ is presented by surgery by a pair

$$L = L_1 \cup \cdots \cup L_m,\ J = J_1 \cup \cdots \cup J_n$$

of disjoint framed oriented links in S^3: the manifold M is obtained by surgery on L as before while the framed link $\mathcal{L}$ in M^3 is the image of the framed link J after the surgery is performed on L. Th. 0.2 asserts that any closed oriented 3-manifold M equipped with an oriented framed link L can be obtained in this fashion. Denote by $A = A_{L\cup J}$ the linking matrix of the link $L \cup J$ in S^3. This is an integral symmetric matrix of size $m+n$. Denote by $\mathrm{sign}(L) \in \mathbb{Z}$ the signature of $A_L \otimes \mathbb{R}$. Associate to each link component $\mathcal{L}_j$ an element $c_j \in G$ (called a *color*) and set $\mathbf{c} = (c_1, \ldots, c_n) \in G^n$. The tensor product $q \otimes A$ can be regarded as a homogeneous quadratic form $G^{m+n} = G^m \times G^n \to \mathbb{Q}/\mathbb{Z}$. The number

$$(1.1)\quad \tau(M, \mathcal{L}; q, \mathbf{c}) = \gamma(G, q)^{-\mathrm{sign}(L)} |G|^{-m/2} \sum_{\mathbf{x} \in G^m} \exp(2\pi i (q \otimes A_{L\cup J})(\mathbf{x}, \mathbf{c}))$$

is a topological invariant of $(M, \mathcal{L})$. We verify that $\tau(M, \mathcal{L}; q, c)$ is invariant under the Kirby moves of Theorem 0.2. Hence $\tau(M, \mathcal{L}; q, c)$ is a topological invariant of $(M, \mathcal{L})$, as claimed.

Clearly, the invariant previously considered in §**??** is recovered if the link is empty or the element $c \in G^n$ is trivial:

$$\tau(M; q) = \tau(M, \varnothing; q, c) = \tau(M, \mathcal{L}; q, 0).$$

REMARK 1.1. For M, q and c fixed, $\tau(M, \mathcal{L}; q, c)$ is an invariant of the *ordered* link $\mathcal{L}$.

REMARK 1.2. For a subset $J \subseteq \{1, \ldots, n\}$, let $\mathcal{L}^J = \cup_{j\in J}$ denote the corresponding sublink of $\mathcal{L}$. If $\mathcal{L}$ is ordered, then $\mathcal{L}^J$ is also ordered. If $c = (c_i)_{1\leqslant i\leqslant n} \in G^n$ is the color vector for the ordered link $\mathcal{L}$, then $c^J =$

$(c_j)_{j\in J} \in G^{|J|}$ is the color vector for $\mathcal{L}^J$. Let $c \in G^n$ be a color vector such that $c_i = 0$ if $i \notin J$. Then

$$\tau(M, \mathcal{L}; q, c) = \tau(M, \mathcal{L}^J; q, c^J). \tag{1.2}$$

In fact, $\tau(M, \mathcal{L}; q, c)$ is more precisely a topological invariant of the pair

$$\left(M, \sum_j c_j \otimes \mathcal{L}_j\right)$$

where we regard $\sum_j c_j \otimes \mathcal{L}_j$ as a framed 1-cycle with coefficients in G ([**8**, Th. 1]).

We will show (as a consequence of Th. 3.1) that it depends on the framed oriented 1-cycle only up to framed cobordism with coefficients in G.

2. The vanishing of the invariant

A necessary and sufficient condition for $\tau(M, \mathcal{L}; q, c)$ to vanish is described in [**8**, Th. 4]. Since this condition plays a decisive rôle in the construction of the topological quantum field theory in §, we relate it explicitly to our previous constructions of the tensor product of linking pairings and the characteristic homomorphism respectively.

THEOREM 2.1. *$\tau(M, \mathcal{L}; q, c)$ is nonzero if and only if the class*

$$\sum_j c_j \otimes [\mathcal{L}_j]$$

in $H_1(M; G)$ is the characteristic element of (G, b_q) and $(\mathrm{Tors}\, H_1(M), \lambda_M)$, i.e., if and only if

$$\theta_{b_q, \lambda_M} = \sum_j c_j \otimes [\mathcal{L}_j].$$

Recall (cf. §2.9) that the characteristic element $\theta = \theta_{b_q, \lambda_M}$ lives in $G \otimes \mathrm{Tors}\, H_1(M)$.

PROOF. First, the fact that $\sum_j c_j \otimes [\mathcal{L}_j]$ must lie in $G \otimes \mathrm{Tors}\, H_1(M)$ is a necessary condition for $\tau(M, \mathcal{L}; q, c)$ to be nonzero is proved in [**8**, Th. 3 (1)].

Next, we observe from (1.1) that $\tau(M, \mathcal{L}; q, c) = 0$ if and only if the last Gauss sum on the right hand of (1.1) is zero. Developping the term $q \otimes A_{L\cup J}(x, c)$ in terms of the block decomposition of the matrix $A_{L\cup J} = \begin{bmatrix} A_L & A_{L,J} \\ A_{J,L} & A_J \end{bmatrix}$ enables to rewrite the Gauss sum as a product of a nonzero complex number and the Gauss sum

$$\gamma(G \otimes \mathbb{Z}^m, q \otimes A_L + (b_q \otimes A_{J,L})(-, c)) = \gamma(G \otimes W, q \otimes g + (\widehat{b}_q \otimes \widehat{g}_{\mathbb{Q}})([\omega]),$$

where g denotes the symmetric bilinear map on $W = \mathbb{Z}^m$ determined the $m \times m$ linking matrix A_L and $\omega \in G \otimes W^\sharp$ is a lift of $\sum_j c_j \otimes [\mathcal{L}_j] \in G \otimes$

$W^\sharp/W = G \otimes \text{Tors } H_1(M)$. Now we apply the formula (2.9) to obtain the identity

$$q \otimes g + (b_q \otimes g_{\mathbb{Q}})(-, \omega) = \varphi_{f\otimes g, v\otimes w - 2t} \circ j_f,$$

where (V, f, v) is a bilinear lattice equipped with an integral Wu class v for f such that $(G_f, \varphi_{f,v}) = (G, q)$([1]), where w is a Wu class for g and where $t \in V^\sharp \otimes W^\sharp$ is a lift of $\omega \in G \otimes W^\sharp$.

Finally we apply Th. 3.4, condition (4): $\gamma(G\otimes W, \varphi_{f\otimes g, v\otimes w-2t}\circ j_f) \neq 0$ if and only if $\psi(\omega) = \theta \in G_f \otimes G_g$ is the characteristic element for $\lambda_f \otimes \lambda_g = b_q \otimes \lambda_M$. This is the desired result. $\square$

Corollary 2.1. *If $\tau(M, \mathcal{L}; q, c) \neq 0$ then $\sum_j c_j \otimes [\mathcal{L}_j]$ lies in $G\otimes$Tors $H_1(M)$ and has order at most 2. In particular, if $\tau(M, \mathcal{L}, q, c) \neq 0$ and at least one of the two groups G or Tors $H_1(M)$ has odd order, then $\sum_j c_j \otimes [\mathcal{L}_j] = 0$.*

Proof. The characteristic element θ lies by definition in $G\otimes$Tors $H_1(M)$ and satisfies $2\ \theta = 0$ (see the end of §2.9). The first statement follows. The second statement is a consequence of the first one. $\square$

Corollary 2.2. *The following assertions are equivalent:*

(1) $\tau(M, q) \neq 0$;
(2) *The characteristic element θ_{b_q, λ_M} is zero;*
(3) (G, b_q) *and* (Tors $H_1(M), \lambda_M)$ *have no common orthogonal cyclic summand of even order.*

Proof. Apply Th. 2.1 with $\mathcal{L} = \varnothing$. Then $\sum_j c_j \otimes \mathcal{L}_j = 0$. This gives (1) $\Leftrightarrow$ (2). The equivalence (2) $\Leftrightarrow$ (3) follows from the definition of the characteristic element. $\square$

3. Classification results: topology

In this paragraph, we investigate the sensitivity of the invariant $\tau(M, \theta)$ to the topology of $(M, \mathcal{L})$.

There is a well defined notion of linking numbers of cycles whenever they represent torsion elements in homology. A *framing* of a smooth 1-cycle Z in M is a framing on each of its components. We denote by Z' the parallel copy of Z in M. The framing allows to define a number $q_M^{\text{fr}}(Z) \in \mathbb{Q}$ by the formula

$$q_M^{\text{fr}}(Z) = \frac{1}{2}\text{lk}_M(Z, Z') \ \in \mathbb{Q}.$$

If Z does not represent a torsion element in $H_1(M)$, then $q_M^{\text{fr}}(Z)$ is undefined.

Theorem 3.1. *Let $(M, \mathcal{L})$ and $(N, \mathcal{J})$ be two closed oriented connected 3-manifolds equipped with oriented and framed n-component links $\mathcal{L}$ and $\mathcal{J}$. The following two assertions are equivalent:*

[1] Such a triple exists by Th. 2.4.

I. (i) *There is an isomorphism*

$$(H_1(M), \lambda_M, [\mathcal{L}_1], \ldots, [\mathcal{L}_n]) \simeq (H_1(N), \lambda_N, [\mathcal{J}_1], \ldots, [\mathcal{J}_n])$$

of pointed linking groups;

(ii) *The rational linking and framing numbers are equal:*

$$\mathrm{lk}_M(\mathcal{L}_i, \mathcal{L}_j) = \mathrm{lk}_N(\mathcal{J}_i, \mathcal{J}_j) \text{ and } q_M^{\mathrm{fr}}(\mathcal{L}_i) = q_N^{\mathrm{fr}}(\mathcal{J}_i) \text{ for all } 1 \leqslant i < j \leqslant n.$$

II. $\tau(M, \mathcal{L}, q, c) = \tau(N, \mathcal{J}; q, c)$ *for any quadratic function* $q : G \to \mathbb{Q}/\mathbb{Z}$ *equipped with* $c \in G^n$.

The proof is dealt with in the next section. We first consider two simples examples of applications of Theorem 3.1.

Example 3.1. Consider the two pairs of links in S^3 representing two oriented knots $(M, \mathcal{L})$ and $(N, \mathcal{J})$ respectively (see Fig. 3.1). Since the surgery links are the same, $M = N = (S^1 \times S^2) \sharp L(5,1)$ (the connected sum of $S^1 \times S^2$ and a lens space). Set an arbitrary framing $\star \in \mathbb{Z}$ for the component $J_1 \subset S^3$ (in red in the figure; it should be the same for both components labelled J_1).

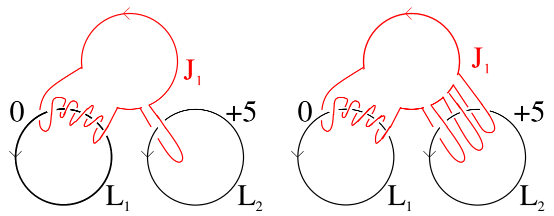


Figure 3.1. Two pairs of links representing two oriented knots in $(S^1 \times S^2) \sharp L(5,1)$.

We have

$$H_1(M) = \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}, \quad \lambda_M(x, y) = \frac{xy}{5} \bmod 1.$$

Under the identification above, $[\mathcal{L}] = (5,1)$ and $[\mathcal{J}] = (5,3)$ respectively. The extended linking matrices are $\left[\begin{array}{cc|c} 0 & 0 & 2 \\ 0 & 5 & 1 \\ \hline 2 & 1 & \star \end{array}\right]$ and $\left[\begin{array}{cc|c} 0 & 0 & 2 \\ 0 & 5 & 3 \\ \hline 2 & 3 & \star \end{array}\right]$ respectively.

Let (G, q, c) be a pointed quadratic form. The characteristic element for (b_q, λ_M) is zero since Tors $H_1(M)$ has odd order. Hence $c \otimes [\mathcal{L}]$ is characteristic if and only if $c \otimes [\mathcal{L}]$ is zero. Consider the case when G is cyclic of order n. Let k be the 5-valuation of n. Let $n' = \begin{cases} n & \text{if } k = 0, 1; \\ n/5 & \text{otherwise.} \end{cases}$. Then $c \otimes [\mathcal{L}] = 0$ if and only if $c = 0 \bmod n'$ (if and only if $c \otimes [\mathcal{J}] = 0$). Let $n = 25$ and $c = 5 \bmod 25$. Using the definition (1.1) of τ and the observation above on the characteristic element, we find that

$$\tau(M, \mathcal{L}; q, c) \neq \tau(M, \mathcal{J}; q, c).$$

Thus there is no isomorphism $\phi : H_1(M) \to H_1(M)$ such that $\phi([\mathcal{L}]) = [\mathcal{J}]$ and $\phi^*(\lambda_M) = \lambda_M$. ■

EXAMPLE 3.2. Consider again the two oriented and framed knots $\mathcal{L}$ and $\mathcal{J}$ in $S^1 \times S^2$ presented in Fig. 0.2. Here $M = N = S^1 \times S^2$. Let (G, q, c) be a pointed quadratic form. It is not hard to see that $c \otimes [\mathcal{L}]$ is characteristic if and only if $c = 0$ in G. We conclude immediately that

$$\tau(S^1 \times S^2, \mathcal{L}; q, c) = \tau(S^1 \times S^2, \mathcal{J}; q, c).$$

It follows that there exists an isomorphism $\phi : H_1(M) \to H_1(M)$ such that $\phi([\mathcal{L}]) = [\mathcal{J}]$ and $\phi^*(\lambda_M) = \lambda_M$. (This isomorphism is actually induced by a diffeomorphism of the pair $(S^1 \times S^2, \mathcal{L})$, as described in the Example 0.1.) ■

COROLLARY 3.1. *Let $(M, \mathcal{L})$ and $(N, \mathcal{J})$ be two closed oriented connected 3-manifolds with framed oriented n-component links $\mathcal{L}$ and $\mathcal{J}$. Assume that none of the components of $\mathcal{L}$ represents a torsion element in $H_1(M)$. Then the following assertions are equivalent:*

I. *There is an isomorphism*

$$(H_1(M), \lambda_M, [\mathcal{L}_1], \ldots, [\mathcal{L}_n]) \simeq (H_1(N), \lambda_N, [\mathcal{J}_1], \ldots, [\mathcal{J}_n])$$

of pointed linking groups.

II. *$\tau(M, \mathcal{L}, q, c) = \tau(N, \mathcal{J}; q, c)$ for any quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$ equipped with $c \in G^n$.*

4. The proof of the classification theorem

Both implications[2] will be derived from the formula of [**8**, Th. 3] which we recall and slight adapt to our notation. To state this formula, there are a number of choices to make (although the final result does not depend on the particular choices made). Choose a spin structure s on M, inducing a homogeneous quadratic refinement $q^s : \text{Tors } H_1(M) \to \mathbb{Q}/\mathbb{Z}$ of the linking pairing λ_M on M. Choose a lattice pairing (V, f, v) equipped with an integral Wu class such that $(V^\sharp/V, \varphi_{f,v}) = (G, q)$.

Assume first that $\sum_j c_j \otimes \mathcal{L}_j \in G \otimes \text{Tors } H_1(M)$. Choose a framed 1-cycle $\eta = \sum_j \xi_j \otimes \mathcal{L}_j$ by lifting the coefficients $c_j \in G = V^\sharp/V$ to $\xi_j \in V^\sharp$. Then $[\eta] \in V^\sharp \otimes \text{Tors } H_1(M)$. Evaluating η against the Wu class $v \in V$ yields a framed integral 1-cycle $\eta_v = \sum_j f_{\mathbb{Q}}(v, \xi_j)\mathcal{L}_j$. Note that this cycle represents a torsion element. There is an invariant of framed 1-cycles δ_s defined by the following conditions (see [**8**, §2.3, Lemma 14]):

(1) δ_s is a $\mathbb{Z}$-homomorphism and takes values in $\{0, 1/2\} = \frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$;
(2) δ_s depends on the spin structure s on M;
(3) δ_s vanishes exactly on framed 1-cycles for which the spin structure s and the framing are compatible.

In the case when the framed cycle σ represents a torsion element, we have

$$\delta_s(\sigma) = q^{\text{fr}}(\sigma) - q^s([\sigma]).$$

[2] This is not strictly necessary for the direct implication (I) $\Longrightarrow$ (II); however the idea of using characteristic elements is a key ingredient in the converse.

In particular, for $\sigma = \eta_v$, we have

$$\delta_s(\eta_v) = \sum_j f_{\mathbb{Q}}(v, \xi_j)\delta_s(\mathcal{L}_j).$$

Hence

$$\delta_s(\eta_v) = \sum_j f_{\mathbb{Q}}(v, \xi_j)\Big(q^{\mathrm{fr}}(\mathcal{L}_j) - q^s([\mathcal{L}_j])\Big) \in 1/2\mathbb{Z}/\mathbb{Z}. \tag{4.1}$$

In particular, if $v = 0 \bmod 2$, then $\delta_s(\eta_v) = 0$.

Using the fact that η also has a framing, we can slightly generalize the definition of q^{fr} above by defining

$$(f \otimes q^{\mathrm{fr}})(\eta) = \sum_j f_{\mathbb{Q}}(\xi_j, \xi_j)\ q^{\mathrm{fr}}(\mathcal{L}_j) + \sum_{j<k} f_{\mathbb{Q}}(\xi_j, \xi_k)\ \mathrm{lk}_M(\mathcal{L}_j, \mathcal{L}_k) \in \mathbb{Q}.$$

A fundamental formula [**8**, Th. 3] is the relation

(4.2)

$$\tau(M, \mathcal{L}; q, c) = e^{2\pi i\left((f\otimes q^{\mathrm{fr}})(\eta) - \delta_s(\eta_v)\right)}\ \gamma(\mathrm{Tors}\ H_1(M), q_s)^{-f_{\mathbb{Q}}(v,v)}$$
$$\gamma\Big(V \otimes \mathrm{Tors}\ H_1(M), f \otimes q_s + \Big(\widehat{f_{\mathbb{Q}}} \otimes \widehat{\lambda}_M\Big)([\eta])\Big)\ |H^1(M; G)|^{1/2}.$$

Here $f \otimes q_s$ denotes the quadratic function $V \otimes \mathrm{Tors}\ H_1(M) \to \mathbb{Q}/\mathbb{Z}$ over $f \otimes \lambda_M$ defined by

$$x \otimes y \mapsto f(x, x)\ q_s(y).$$

The map $\Big(\widehat{f_{\mathbb{Q}}} \otimes \widehat{\lambda}_M\Big)([\eta])$ is the map on $V \otimes \mathrm{Tors}\ H_1(M)$ induced by the map adjoint to the linking pairing $f \otimes \lambda_M$ at $[\eta] \in V^\sharp \times \mathrm{Tors}\ H_1(M)$. Explicitly

$$V \otimes \mathrm{Tors}\ H_1(M) \to \mathbb{Q}/\mathbb{Z},\ z \mapsto (f_{\mathbb{Q}} \otimes \lambda_M)([\eta], z).$$

Note that this map is well defined and non-trivial in general since $[\eta] \in V^\sharp \otimes \mathrm{Tors}\ H_1(M)$.

Let us prove the implication (I) $\Longrightarrow$ (II). Consider the case

$$\sum_j c_j \otimes [\mathcal{L}_j] \notin G \otimes \mathrm{Tors}\ H_1(M)$$

first. By Cor. 2.1, $\tau(M, \mathcal{L}; q, c) = 0 = \tau(N, \mathcal{J}; q, c)$, which is the desired result. Consider next the case

$$\sum_j c_j \otimes [\mathcal{L}_j] \in G \otimes \mathrm{Tors}\ H_1(M).$$

Since $\phi^*\lambda_N = \lambda_M$, choose a spin structure s' on N such that $\phi^* q_N^{s'} = q_M^s$. Thus $q_N^{s'} \simeq q_M^s$ and their associated Gauss sums are equal. Set $\eta_M = \sum_j \xi_j \otimes \mathcal{L}_j$ and $\eta_N = \sum_j \xi_j \otimes \mathcal{J}_j$. The isomorphism $\phi : H_1(M) \to H_1(N)$ induces an isomorphism $1_{V^\sharp} \otimes \phi$ sending $[\eta_M] \in V^\sharp \otimes H_1(M)$ to $[\eta_N] \in V^\sharp \otimes H_1(N)$. The isomorphism $1_{V^\sharp} \otimes \phi$ induces an isomorphism

$$f \otimes q_M^s + (\widehat{f_{\mathbb{Q}}} \otimes \widehat{\lambda}_M)([\eta_M]) \simeq f \otimes q_N^s + (\widehat{f_{\mathbb{Q}}} \otimes \widehat{\lambda}_N)([\eta_N]).$$

Hence their associated Gauss sums are equal. In view of $H_1(M) \simeq H_1(N)$, the equality $|H^1(M;G)| = |H^1(N;G)|$ is clear. Finally it follows from (iii) that

$$(f \otimes q_M^{\mathrm{fr}})(\eta_M) = (f \otimes q_N^{\mathrm{fr}})(\eta_N)$$

and it follows from (i) and the definition of s' that

$$\delta(\eta_M) = \delta(\eta_N).$$

This is the desired result.

We now prove the converse.

Step 1: we recover the homology by taking the absolute value of $\tau(M;q) = \tau(M, \varnothing; q, 0)$. By Corollary 2.2, if (G,q) and $(\mathrm{Tors}\ H_1(M))$ have no common orthogonal cyclic summand of even order, then

$$|\tau(M;q)| = |H^1(M;G)|^{1/2} = |H_1(M) \otimes G|^{1/2}.$$

By allowing G to vary, we recover all p-components of $H_1(M)$ for all odd primes p. For 2-components, for each $k \geqslant 1$ and $G_k = \mathbb{Z}/2^k\mathbb{Z}$, we compute $\tau(M;q)$. By Corollary 2.2, we know that $H_1(M)$ has G_k as an orthogonal summand if and only if $\tau(M;q) = 0$. We only need to know the number of such summands. We observe that given any finite abelian group G, the group $G \oplus G$ carries a hyperbolic linking with no cyclic orthogonal summand. Lift this linking to any quadratic form q. Then $|\tau(M;q)|^2 = |H^1(M; G \oplus G)| = |H^1(M;G)|^2$. By allowing G to vary over all 2-groups, we recover all 2-components of $H_1(M)$. Finally we recover in this fashion the isomorphism class of $H_1(M)$. (For more details on this step, see [**7**].) ∎

Step 2: For $c = 0$, the formula (1.1) simplifies to
(4.3)
$$\tau(M, \mathcal{L}; q, 0) = \tau(M, q) = \frac{\gamma(V \otimes \mathrm{Tors}\ H_1(M), f \otimes q^s)}{\gamma(\mathrm{Tors}\ H_1(M), q^s)^{f(v,v)}} \cdot |H^1(M;G)|^{1/2}.$$

Using the discriminant construction (§2.4, Th. 2.4), we may choose any bilinear even lattice $(V, f, 0)$ equipped with Wu class $v = 0 \in \mathrm{Wu}^V(f)$ such that $(G_f, \varphi_{f,0}) = (G, q)$. Then the formula simplifies further to

$$\frac{\tau(M,q)}{|H^1(M;G)|^{1/2}} = \gamma(V \otimes \mathrm{Tors}\ H_1(M), f \otimes q^s) \tag{4.4}$$

$$= \gamma(V \otimes H_1(M), \frac{1}{2} f \otimes \lambda_M). \tag{4.5}$$

For the second equality above, we regard $\frac{1}{2}f$ as the homogeneous quadratic function defined by $(\frac{1}{2}f)(x) = \frac{f(x,x)}{2} \in \mathbb{Z}$ for all $x \in V$ and we use the fact that $2\ q^s(y) = \lambda_M(y,y)$ for all $y \in H_1(M)$. We now apply the classification theorem (Th. 3.1) to the Gauss sums $\tau(M,q) = \gamma_{1/2\ f,0}(\lambda_M, 0)$ (this is the special case when the distinguished element is trivial): we recover the isomorphism class of the linking pairing λ_M. It follows from Step 1 and Corollary 3.4 that we recover the isomorphism class of the linking group $(H_1(M), \lambda_M)$. ∎

Step 3: we show how to detect whether a $\mathbb{Z}$-linear combination $\sum_{j\in J} a_j[\mathcal{L}_j]$ is torsion in $H_1(M)$ and if it is, we show how to compute its order.

Let $J \subseteq \{1,\dots,n\}$. Recall the corresponding ordered sublink $\mathcal{L}^J = \cup_{j\in J}\mathcal{L}_j$ of $\mathcal{L}$. Let $(a_j)_{j\in J} \in \mathbb{Z}^{|J|}$. We prove that τ detects whether the $\mathbb{Z}$-linear combination $\sum_{j\in J} a_j[\mathcal{L}_j]$ is torsion in $H_1(M)$. Note that it actually suffices to detect whether $\sum_{j\in J} a_j\ [\mathcal{L}_j] = 0$ in $H_1(M)$. We use the following fact from p-adic numbers. Let

$$\widehat{\mathbb{Z}} = \lim_{\rightarrow} \mathbb{Z}/n\mathbb{Z} = \{(x_k)_{k\geqslant 1} \in \prod_{k\geqslant 1} \mathbb{Z}/k\mathbb{Z},\ \text{for all } n|m,\ x_m = x_n \bmod n\}.$$

Note that the natural homomorphism

$$\mathbb{Z} \to \prod_{k\geqslant 1} \mathbb{Z}/k\mathbb{Z},\ x \mapsto (x \mod k)_{k\geqslant 1}$$

factors through a map $\mathbb{Z} \to \widehat{\mathbb{Z}}$.

Lemma 4.1. *Let H be an abelian group. The map $H \to \widehat{\mathbb{Z}} \otimes H$ is injective.*

Applying this lemma to $H = H_1(M)$ for the particular element $\sum_j a_j\ [\mathcal{L}_j]$ yields the

Corollary 4.1.
(4.6)
$$\sum_{j\in J} a_j\ [\mathcal{L}_j] = 0 \textit{ if and only if } \sum_{j\in J} (a_j \bmod k) \otimes [\mathcal{L}_j] = 0\ \in \mathbb{Z}/k\mathbb{Z},\ \textit{for all}\, k \geqslant 1.$$

Let $k \geqslant 1$. Set $C_k = \mathbb{Z}/k\mathbb{Z}$. By 1_k, we denote $(1 \bmod k) \in C_k$. Let $1_k^* \in C_k^*$ be the unique form defined by $1_k^*(1_k) = \frac{1}{k} \bmod 1$. Let $G_k = C_k \oplus C_k^*$. Define $q_k : G_k \to \mathbb{Q}/\mathbb{Z}$ by $q_k(x,\alpha) = \alpha(x)$. The quadratic form q_k is hyperbolic. In particular, (G_k, q_k) has no cyclic orthogonal summand. Hence the characteristic homomorphism vanishes: $\chi_{b_{q_k},\lambda_M} = 0$. Equivalently the characteristic element is zero. Denote by $c^J = (c_j)_{j\in J} \in G_k^{|J|}$ the color vector associated to the ordered sublink $\mathcal{L}^J$. By the vanishing condition (Th. 3.4), $\tau(M, \mathcal{L}^J; q_k, c^J) \neq 0$ if and only if $\sum_{j\in J} c_j \otimes [L_j] = 0$. This holds for any color vector $c^J = (c_j)_{j\in J}$ and for any $k \geqslant 1$. In particular, for each $k \geqslant 1$, we take $c_j = c_{j,k} = a_j\ (1_k, 1_k^*) \in G_k$. For each $k \geqslant 1$, set $c_k^J = (c_{j,k})_{j\in J} \in G_k^{|J|}$. We have $\sum_{j\in J} c_j \otimes [\mathcal{L}_j] = 0$ if and only if $\sum_{j\in J} (a_j \mod k) \otimes [\mathcal{L}_j] = 0$. By corollary 4.1, we conclude that

$$\sum_{j\in J} a_j[\mathcal{L}_j] = 0 \text{ if and only if for all } k \geqslant 1,\ \tau(M, \mathcal{L}^J; q_k, c_k^J) \neq 0.$$

In particular the invariant τ detects whether $\sum_{j\in J} a_j\ [L_j] = 0$, as claimed.

The order of $\sum_{j\in J} a_j\ [\mathcal{L}_j]$ is the smallest $n \geqslant 1$ such that $\sum_{j\in J} n a_j\ [\mathcal{L}_j] = 0$. It follows that it is the smallest $n \geqslant 1$ such that $\tau(M, \mathcal{L}^J; q_k, n\ c_k^J) \neq 0$.

As a particular case, τ detects whether a given component $\mathcal{L}_j$ represents a torsion element in $H_1(M)$ of fixed finite order. ∎

Step 4. Set $F_1(M) = H_1(M)/\mathrm{Tors}\ H_1(M)$. We show how to detect whether any $\mathbb{Z}$-linear combination $\sum_{j\in J} a_j[\mathcal{L}_j]$ projects onto a primitive element in $F_1(M)$. (Recall that a primitive element in a lattice V is an element $x \in V$ that can be completed to a $\mathbb{Z}$-basis $(x, x_2, \ldots, x_n)$ of V.)

We use the notation of the previous step.

LEMMA 4.2. *Let $k \geqslant 1$. Let $c = c^J \in G_k^{|J|}$ denote the color vector defined by $c_j = c_{j,k} = a_k\ (1_k, 1_k^*) \in G_k$ for all $j \in J$.*

(1) *The element $\sum_{j\in J} a_j[L_j]$ projects onto a nonzero element in $k\, F_1(M)$ if and only if the following two conditions are verified:*
  (1.1) *$\tau(M, \mathcal{L}; q_k, c^J) \neq 0$ for $c_j = a_k\ (1_k, 1_k^*) \in G_k$, $j \in J$.*
  (1.2) *For any positive integer k' not multiple of k, for any nonzero $c^J \in (G_{k'})^{|J|}$, $\tau(M, \mathcal{L}; q_{k'}, c^J) = 0$.*

(2) *The element $\sum_{j\in J} a_j[\mathcal{L}_j]$ projects onto a primitive element in $F_1(M)$ if and only if $\tau(M, \mathcal{L}; q_k, c^J) = 0$ for all $k \geqslant 1$.*

Step 5: we show how to recover $q^{\mathrm{fr}}(\mathcal{L}_j) \in \mathbb{Q}$ for all $1 \leqslant j \leqslant n$ such that $[\mathcal{L}_j] \in \mathrm{Tors}\ H_1(M)$. By Step 3, we know which components represents torsion elements. Henceforth we assume that there is at least one component $\mathcal{L}_j$ that represents a torsion element in $H_1(M)$. Consider the set S of all pairs (G, c) where G is a finite Abelian group and $c = (c_1, \ldots, c_n) \in G^n$ is an n-tuple of colors such that $[\theta] = c \otimes [\mathcal{L}] = \sum_j c_j \otimes [\mathcal{L}_j] = 0$. Let $(G, c) \in S$. By Theorem 2.4, choose a triple (V, f, v) over (G, q). Since $[\theta] = 0 \in G \otimes H_1(M)$, we choose a lift η such that $[\eta] = 0$. Then the formula (4.2) reduces to

(4.7)
$$\tau(M, \mathcal{L}; q, c) = e^{2\pi i\left((f\otimes q^{\mathrm{fr}})(\eta)-\delta(\eta)\right)} \cdot \frac{\gamma(V \otimes \mathrm{Tors}\ H_1(M), f \otimes q^s)}{\gamma(\mathrm{Tors}\ H_1(M), q^s)^{f(v,v)}} \cdot |H^1(M; G)|^{\frac{1}{2}}.$$

Since we know the isomorphism class of λ_M, we can freely choose a spin refinement q^s of λ_M. Hence we can compute the Gauss sums of the right hand side of (4.7). We already know the order $|H^1(M; G)|$ by Step 1. Since $[\theta] = 0$, we know that the invariant does not vanish if and only if (G, q) and $(\mathrm{Tors}\ H_1(M), \lambda_M)$ have no common cyclic orthogonal summand of even order. Let (V, f, v) be a triple over (G, q) satisfying this condition and the condition that $(G, c) \in S$. Then we recover

$$\exp 2\pi i \left((f \otimes q^{\mathrm{fr}})(\eta) - \delta(\eta) \right).$$

In particular for $v = 0$, we have $\delta(\eta) = 0$, hence we recover the term

$$\exp 2\pi i (f \otimes q^{\mathrm{fr}})(\eta).$$

We now prove our claim. Let $1 \leqslant j \leqslant n$. Let d be the order of $\mathrm{Tors}\ H_1(M)$. Set

$$\alpha = \begin{cases} 1 & \text{if } d \text{ is even;} \\ 2 & \text{if } d \text{ is odd.} \end{cases}$$

For each $N \geqslant 2$, consider the pair formed by the group $G = \mathbb{Z}/\alpha d^N\mathbb{Z}$ and the colors defined by

$$c_k = \begin{cases} \alpha d \bmod \alpha d^N & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

Clearly

$$[\theta] = c \otimes [\mathcal{L}] = c_j \otimes [\mathcal{L}_j] = 1 \otimes \alpha d\ [\mathcal{L}_j] \in \mathbb{Z}/\alpha d^N\mathbb{Z} \otimes H_1(M).$$

Thus $(G, c) \in S$ since $d\ [\mathcal{L}_j] = 0$ in Tors $H_1(M)$. Consider the bilinear lattice $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(x, y) = \alpha d^N\ x \cdot y$. Observe that f is always even (so that $v = 0$ is a Wu class) and $G_f = G$. Note that $q = \phi_{f,0}$ is a quadratic form over a cyclic group of order αd^N. Thus (G, q) and (Tors $H_1(M), \lambda_M)$ have no common cyclic orthogonal summand. We lift $\theta = (\alpha d \bmod \alpha d^N) \otimes \mathcal{L}_j$ to $\eta = \alpha d/\alpha d^N \otimes \mathcal{L}_j = 1/d^{N-1} \otimes \mathcal{L}_j$. Given our choice of $(V, f, 0)$, the argument above applies: we recover the term

$$\begin{aligned} \exp\left(2\pi i (f \otimes q^{\mathrm{fr}})(\eta)\right) &= \exp\left(2\pi i f_{\mathbb{Q}}(\xi_j, \xi_j)\ q^{\mathrm{fr}}(\mathcal{L}_j)\right) \\ &= \exp\left(2\pi i\ \tfrac{\alpha}{d^{N-2}}\ q^{\mathrm{fr}}(\mathcal{L}_j)\right). \end{aligned}$$

Hence we recover $\frac{\alpha}{d^{N-2}} q^{\mathrm{fr}}(\mathcal{L}_j) \bmod 1$ for all $N \geqslant 2$. The lemma below (well known in the context of p-adic numbers when d is prime) implies that we recover $\alpha\ q^{\mathrm{fr}}(\mathcal{L}_j) \in \mathbb{Q}$ and hence $q^{\mathrm{fr}}(\mathcal{L}_j) \in \mathbb{Q}$.

Lemma 4.3. *Let $d \geqslant 2$. Let Q_d be the inverse limit of $\mathbb{Q}/\epsilon d^N\mathbb{Z}$. The map*

$$\mathbb{Q} \to Q_d,\ r \mapsto (r \bmod d^N)_{N \geqslant 1}$$

is injective.

The following fact is also well known: a sequence $0 \leqslant r_N < d^N$ of rational numbers such that $r_N = r_{N+1} \bmod d^N$ for all $N \geqslant 1$ corresponds to a rational number $r \in \mathbb{Q}$ provided that there exists $N_0 \geqslant 0$ such that $r_N = r_{N+1}$ for all $N > N_0$. ■

Step 6. Let $p : H_1(M) \to F_1(M)$ denote the canonical projection. We show that the invariant detects the isomorphism class of the pointed (plain) lattice $(F_1(M), p([\mathcal{L}_1]), \ldots, p([\mathcal{L}_n]))$. From Step 1, we know $\rho = \operatorname{rank}(F_1(M))$. From the previous two steps, we can find in a finite number of steps the (unique) maximal subset $I \subseteq \{1, \ldots, n\}$ such that $p([\mathcal{L}_i]) \neq 0$ in $F_1(M)$ for each $i \in I$.

For all $c \in G^{|I|}$, $\tau(M, \mathcal{L}; q, c^I) = \tau(M, \mathcal{L}^I; q, c)$. By Step 4, the invariant detects whether any $\mathbb{Z}$-combination $\sum_i a_i\ p([\mathcal{L}_i])$ lies in $kF_1(M)$. Apply the classification of pointed plain lattices (Prop. 3.1) to deduce the isomorphism class of $(F_1(M), p([\mathcal{L}_1]), \ldots, p([\mathcal{L}_n]))$. This proves our claim. ■

Step 7. Let $r : H_1(M) \to$ Tors $H_1(M)$ be a retraction. We claim that the invariant τ determines the isomorphism class of the pointed linking pairing $(\lambda_M, r([\mathcal{L}]))$.

Preliminary step: let (G, q, c) be a pointed quadratic form. We prove that for any choice of a 4-tuple (V, f, v, ξ) such that $(G_f, \varphi_{f,v}, [\xi]) = (G, q, c)$,

(4.8)

$$\gamma(V \otimes \text{Tors } H_1(M), f \otimes q_{s_M} + (\widehat{f_{\mathbb{Q}}} \otimes \widehat{\lambda}_M)([\eta_M])) =$$
$$= \gamma(V \otimes \text{Tors } H_1(N), f \otimes q_{s_N} + (\widehat{f_{\mathbb{Q}}} \otimes \widehat{\lambda}_N)([\eta_N])).$$

Set

$$\theta_M = \sum_k c_k \otimes \mathcal{L}_k, \ \theta_N = \sum_k c_k \otimes \mathcal{J}_k \in G \otimes H_1(M).$$

First, notice that according to Th. 2.1, if $[\theta_M]$ is non characteristic then $[\theta_N]$ is also non characteristic and both Gauss sums in (4.8) vanish (whatever our choice for (V, f, v, ξ)). Hence (4.8) is verified.

Next, suppose that $[\theta_M]$ is characteristic. Then $[\theta_N]$ also is characteristic. There is an isomorphism $\psi : (\text{Tors } H_1(M), \lambda_M) \to (\text{Tors } H_1(N), \lambda_N)$ of linking pairings (by Step 2), hence $\text{id}_{G_f} \otimes \psi$ is also an isomorphism of linking pairings

$$(G_f \otimes \text{Tors } H_1(M), \lambda_f \otimes \lambda_M) \simeq (G_f \otimes \text{Tors } H_1(N), \lambda_f \otimes \lambda_N).$$

Since the characteristic element is preserved under linking isomorphisms (by (**??**)), this is actually an isomorphism

$$(G_f \otimes \text{Tors } H_1(M), \lambda_f \otimes \lambda_M, [\theta_M]) \simeq (G_f \otimes \text{Tors } H_1(N), \lambda_f \otimes \lambda_N, [\theta_N])$$

of pointed linking pairings. Observe that $\text{id}_{G_f} \otimes \psi$ lifts to an isomorphism $\text{id}_{V^\sharp} \otimes \psi : V^\sharp \otimes \text{Tors } H_1(M) \simeq V^\sharp \otimes \text{Tors } H_1(N)$. Thus for any lift ξ of $c \in G^n$,

$$(V^\sharp \otimes \text{Tors } H_1(M), f \otimes \lambda_M, [\eta_M]) \simeq (V^\sharp \otimes \text{Tors } H_1(N), f \otimes \lambda_N, [\eta_N]).$$

This isomorphism lifts to an isomorphism of pointed quadratic forms $(f \otimes q_{s_M}, [\eta_M]) \simeq (f \otimes q_{s_M}, [\eta_N])$. Therefore, (4.8) is again verified. This completes the preliminary step.

Since $F_1 M$ is free, there is a commutative diagram of split exact sequences

$$\begin{tikzcd}
0 \arrow[r] & V^\sharp \otimes \text{Tors } H_1(M) \arrow[r] \arrow[d] & V^\sharp \otimes H_1(M) \arrow[r] \arrow[d] & V^\sharp \otimes F_1(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \otimes \text{Tors } H_1(M) \arrow[r] & G \otimes H_1(M) \arrow[r] & G \otimes F_1(M) \arrow[r] & 0
\end{tikzcd}$$

with retractions $1_{V^\sharp} \otimes r$ and $1_G \otimes r$ respectively. If follows that $\sum_k c_k \otimes [\mathcal{L}_k] \in G \otimes \text{Tors } H_1(M)$ if and only if $\sum_k c_k \otimes [\mathcal{L}_k] = \sum_k c_k \otimes r[\mathcal{L}_k]$. Thus

$$[\eta_M] = \sum_k \xi_k \otimes [\mathcal{L}_k] \in V^\sharp \otimes \text{Tors } H_1(M) \implies [\eta_M] = \sum_k \xi_k \otimes r[\mathcal{L}_k].$$

Let $(V, f, 0)$ be an even lattice over (G, q) (cf. Th. 2.4). For any homogeneous quadratic form q, we have

$$f \otimes q = \frac{1}{2} f \otimes b_q,$$

where $\frac{1}{2}f$ denotes the homogeneous quadratic function defined on the lattice V by $(1/2\ f)(x) = \frac{f(x,x)}{2} \in \mathbb{Z}$ for all $x \in V$. In particular,

$$f \otimes q_{s_M} = \frac{1}{2} f \otimes \lambda_M .$$

Now equality (4.8) reads

$$\gamma_{h,s}\left(\lambda_M, r([\mathcal{L}_1]), \ldots, r([\mathcal{L}_n])\right) = \gamma_{h,s}\left(\lambda_N, r([\mathcal{J}_1]), \ldots, r([\mathcal{J}_n])\right), \tag{4.9}$$

where $h = \frac{1}{2}f$ and $s = \widehat{f}_{\mathbb{Q}}(\xi)|_{V^n} = (\widehat{f}_{\mathbb{Q}}(\xi_1)|_V, \ldots, \widehat{f}_{\mathbb{Q}}(\xi_n)|_V) \in (V^*)^n$. We apply Theorem 3.1: Step 1 ensures that condition (1) is satisfied and (4.9) ensures that condition (2) is satisfied. The isomorphism

$$(\text{Tors } H_1(M), \lambda_M, r([\mathcal{L}_1]), \ldots, r([\mathcal{L}_n])) \simeq (\text{Tors } H_1(N), \lambda_N, r([\mathcal{J}_1]), \ldots, r([\mathcal{J}_n])))$$

follows.

Step 8. Steps 7 and 8 imply that the hypotheses of Lemma 3.4 are satisfied. We conclude that there is an isomorphism of pointed linking groups

$$(H_1(M), \lambda_M, [\mathcal{L}_1], \ldots, [\mathcal{L}_n]) \simeq (H_1(N), \lambda_N, [\mathcal{J}_1], \ldots, [\mathcal{J}_n]) .$$

This concludes the proof. ■

5. The extension of the monoid pairing

Theorem 3.1 suggests extending the monoid pairing $\langle -, - \rangle : \mathfrak{M}^+ \times \mathfrak{Q}^0 \to \mathbb{C}$ defined in §**??**. This section is devoted to the construction of this extension. Our main result is that this extended pairing is nondegenerate.

Let $n \geqslant 0$. Let $\mathfrak{M}^+(n)$ denote the monoid of n-pointed linking groups. In other words, $\mathfrak{M}^+(n)$ consists of triples (H, λ, ℓ) where H is a finitely generated abelian group, $\lambda : \text{Tors } H \times \text{Tors } H \to \mathbb{Q}/\mathbb{Z}$ is a linking pairing and $\ell = (\ell_1, \ldots, \ell_n) \in H^n$ is an n-tuple of distinguished elements. The operation is the expected one, induced componentwise by orthogonal sum and addition. There is a natural embedding $\mathfrak{M}^+(n) \to \mathfrak{M}^+(p)$ for any $n \leqslant p$ defined by adding $p - n$ zeros on the right on the distinguished n-tuple to form a distinguished p-tuple. We define also a monoid $\mathfrak{Q}^0(n)$ that consists of pairs (q, c) where $q : G \to \mathbb{Q}/\mathbb{Z}$ is a nondegenerate homogeneous quadratic function on a finite abelian group G and c is a distinguished element in G^n. Clearly, $\mathfrak{Q}^0 = \mathfrak{Q}^0(0)$ embeds in $\mathfrak{Q}^0(n)$ in the usual way for any $n > 0$. An ordered link $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$ in a closed connected oriented 3-manifold M induces an element

$$[\mathcal{L}] = ([\mathcal{L}_1], \ldots, [\mathcal{L}_n]) \in H_1(M)^n$$

and therefore determines an element $(H_1(M), \lambda_M, [\mathcal{L}]) \in \mathfrak{M}^+(n)$.

LEMMA 5.1. *Let $(H, \lambda, \ell) \in \mathfrak{M}^+(n)$. There exists a closed oriented 3-manifold M equipped with an oriented and ordered link $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_n \subset M$ such that $(H_1(M), \lambda_M, [\mathcal{L}]) = (H, \lambda, \ell)$.*

PROOF. By [**33**, Th. 6.1], any (nondegenerate) linking pairing can be realized as the linking pairing of a closed oriented 3-manifold. Actually, the 3-manifold can be chosen as a rational homology 3-sphere M' ($b_1(M') = 0$).

Let m be the rank of H. One needs to modify M' to another 3-manifold M such that $b_1(M) = m$ so that $H_1(M) = H$. The simplest way to do this is to make connected sums with $S^1 \times S^2$: $M = M' \sharp\sharp_{j=1}^m S^1 \times S^2$. Then $b_1(M \sharp\sharp_{j=1}^m S^1 \times S^2) = b_1(M) + m\ b_1(S^1 \times S^2) = 0 + m\ 1 = m$. It remains to choose an ordered oriented link $\mathcal{L} \subset M'$ such that its components represent prescribed homology classes $\ell_1, \ldots, \ell_n$. Since the dimension of each components is one, we can achieve this component by component. Since the codimension of each component in M is two, we can ensure that the components are pairwise disjoint. $\square$

Let $n \geqslant 0$. We define a pairing $\mathfrak{M}^+(n) \times \mathfrak{Q}^0(n) \to \mathbb{C}$ by

$$\langle H, \lambda, \ell; q, c\rangle = \tau(M, \mathcal{L}; q, c), \tag{5.1}$$

where M is any closed oriented 3-manifold equipped with a link $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_n \subset M$ provided by the lemma above. For $n = 0$ (no distinguished element), this coincides with the previous definition (see §**??**). For $n \geqslant 1$ (when there is at least one distinguished element), the definition is unambiguous *only if we fix the framing* $\mathrm{lk}_M(\mathcal{L}_j, \mathcal{L}'_j)$ of torsion components of $\mathcal{L}$ in M. We require that for each $1 \leqslant j \leqslant n$,

$$\mathrm{lk}_M(\mathcal{L}_j, \mathcal{L}'_j) = \frac{+1}{N_j} \in \mathbb{Q}$$

where N_j is the order of the homology class $[\mathcal{L}_j]$ in $H_1(M)$. We refer to this framing as the *reference framing* of $\mathcal{L}$. For non torsion components, the framing is undefined. Theorem 3.1 ensures that the pairing $\langle -; -\rangle$ is well defined by (5.1).

THEOREM 5.1. *The pairing*

$$\mathfrak{M}^+(n) \times \mathfrak{Q}^0(n) \to \mathbb{C}, \quad \Big((H, \lambda, \ell), (q, c)\Big) \mapsto \langle \lambda, \ell; q, c\rangle$$

is bilinear, left and right nondegenerate.

Bilinearity is meant with respect to the operations

$$(H, \lambda, \ell) \oplus (H', \lambda', \ell') = (H \oplus H', \lambda \oplus \lambda', \ell \oplus \ell')$$

and

$$(q, c) \oplus (q', c') = (q \oplus q', c \oplus c').$$

Clearly the pairing above generalizes the pairing introduced in §**??**:

$$\langle H, \lambda, \varnothing; q, c\rangle = \langle H, \lambda; q, \varnothing\rangle = \langle H, \lambda; 0\rangle = \langle H, \lambda, \ell; q, 0\rangle.$$

PROOF. Bilinearity follows from the definition. Th. 3.1 asserts that $\langle -, -\rangle$ is left nondegenerate. It remains to prove that it is also right nondegenerate.

Step 1: we recover G by taking the absolute value of $\tau(M, \mathcal{L}; q, c)$. This step is completely symmetric to the first step of the proof of Th. 3.1. We have

$$|\langle \lambda, m, \ell; q, \mathbf{c}\rangle| = |H^1(M; G)|^{1/2} = |G \otimes \mathrm{Tors}\ H_1(M)|^{1/2},$$

if G and Tors $H_1(M)$ have no common cyclic orthogonal summand of even order (and is zero otherwise). According to the previous lemma, for any pointed linking pairing $(\lambda, \ell) \in \mathfrak{M}(n)$, there is a closed oriented rational homology 3-sphere M equipped with an oriented link $\mathcal{L}$ such that $(\lambda, 0, \ell) = (\lambda_M, 0, [\mathcal{L}])$. Endow $\mathcal{L}$ with the reference framing. By appropriate choices of $(\mathrm{Tors}\, H_1(M), \lambda_M)$, we recover all p-components of G, hence the isomorphism class of G itself.

Step 2: we establish a formula for $\tau(M, \mathcal{L}; q, c)$ in a particular case (first proved in [**12**, Cor. 4]). Let $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n \subset M$ be an oriented link whose components are all homologically trivial: $[\mathcal{L}_j] = 0$ in $H_1(M)$ for $1 \leqslant j \leqslant n$. We have $[\eta] = \sum_k \xi_k \otimes [\mathcal{L}_k] = 0$ for any lift of $[\theta_M] = \sum_k c_k \otimes [\mathcal{L}_k] = 0$. Furthermore,

$$\begin{array}{rcl} \delta_s(\eta_v) & = & \sum_j f_{\mathbb{Q}}(v, \xi_j)\left(q^{\mathrm{fr}}(\mathcal{L}_j) - q^s([\mathcal{L}_j])\right) \\ & = & \sum_j f_{\mathbb{Q}}(v, \xi_j)\left(q^{\mathrm{fr}}(\mathcal{L}_j) - 0\right) \\ & = & \sum_j f_{\mathbb{Q}}(v, \xi_j) q^{\mathrm{fr}}(\mathcal{L}_j). \end{array}$$

Since each component is homologically trivial, all framing and linking numbers are integers. It follows that

$$(q \otimes \mathrm{lk}_M)(\theta_M) = (\varphi_{f,v} \otimes \mathrm{lk}_M)\left(\sum_j c_j \otimes \mathcal{L}_j\right) = (f \otimes q^{\mathrm{fr}})(\eta) - \delta_s(\eta_v) \bmod 1.$$

It follows from the formula (4.2) that

$$\tau(M, \mathcal{L}; q, c) = \exp\Big(2\pi i (q \otimes \mathrm{lk}_M)(\theta)\Big)\ \tau(M, q). \tag{5.2}$$

In particular, if M is a $\mathbb{Z}$-homology 3-sphere, $\tau(M, q) = 1$. Hence

$$\tau(M, \mathcal{L}; q, c) = \exp\Big(2\pi i (q \otimes \mathrm{lk}_M)(\theta)\Big). \tag{5.3}$$

for an empty (or zero-framed algebraically split) link, we have

$$\langle \lambda, m, \varnothing; q, c\rangle = \langle \lambda, m; q\rangle.$$

We have noted earlier that the pairing $\langle -, -\rangle : \mathfrak{M}^+ \times \mathfrak{Q}^0 \to \mathbb{C}$ is right nondegenerate. Hence the (isomorphism class of the) homogeneous quadratic form (G, q) is determined. In particular, the Gauss sum $\gamma(G, q)$ is recovered.

Step 3: consider the case when M is a $\mathbb{Z}$-homology 3-sphere with an oriented link $\mathcal{L} \subset M$. Then it can be shown ([**8**, corollary 5]) that

$$\langle \lambda_M, 0, [\mathcal{L}]; q, c\rangle = \tau(M, \mathcal{L}; q, c) = \exp\Big(2\pi i (q \otimes A_{\mathcal{L}})(c)\Big),$$

where $A_{\mathcal{L}}$ denotes the $n \times n$ symmetric integral linking matrix of $\mathcal{L}$ in M. Hence by varying the zero-framed link $\mathcal{L}$ in M, we can realize any symmetric integral $n \times n$ matrix with zeros on the diagonal. It follows that $q(c_j)$, $b_q(c_j, c_k) \in \mathbb{Q}/\mathbb{Z}$, $1 \leqslant j, k \leqslant n$, are all determined.

Step 4: given any pointed linking pairing (λ, ℓ), we realize it as the pointed linking pairing $(\lambda_M, [\mathcal{L}])$ associated to a closed oriented 3-manifold M equipped with a zero-framed oriented link $\mathcal{L}$. The pair $(M, \mathcal{L})$ itself can be realized as a pair (L, J) of disjoint links in S^3 where L is a framed m-component link (on which the surgery is performed) and J is an oriented framed n-component link (giving rise to $\mathcal{L}$ once the surgery on L is performed). Such a pair determines a linking matrix $A_{L\cup J}$. This symmetric integral matrix decomposes as

$$A_{L\cup J} = \begin{bmatrix} A_L & A_{L,J} \\ A_{J,L} & A_J \end{bmatrix}$$

where A_L is the linking $m \times m$ matrix of L in S^3, A_J is the linking $n \times n$ matrix of J in S^3 and $A_{J,L} = A^{\mathrm{t}}_{L,J}$ is the $n \times m$ matrix of the linking numbers of the components of L with the components of J in S^3. Hence, for $x \in G^m$ and $c \in G^n$,

$$(q \otimes A_{L\cup J})(x, c) = (q \otimes A_L)(x) + (b_q \otimes A_{J,L})(x, c) + (q \otimes A_J)(c),$$

where $q \otimes A_{L\cup J}$ is regarded as a quadratic function on $G \otimes \mathbb{Z}^{m+n} = G^{m+n} = G^m \times G^n$, $q \otimes A_L$ is regarded as a quadratic function on $G \otimes \mathbb{Z}^m = G^m$, $q \otimes A_J$ is regarded as a quadratic function on $G \otimes \mathbb{Z}^n = G^n$ and $b_q \otimes A_{J,L}$ is regarded as a bilinear pairing $(G \otimes \mathbb{Z}^m) \times (G \otimes \mathbb{Z}^n) = G^m \times G^n \to \mathbb{Q}/\mathbb{Z}$. It follows from the previous step that we recover the term

$$(q \otimes A_J)(c) = \sum_k q(c_k)\ \mathrm{lk}_{S^3}(J_k, J'_k) + \sum_{k<l} b_q(c_k, c_l)\ \mathrm{lk}_{S^3}(J_k, J_l).$$

Step 5: with the notation previously introduced, we have

$$\frac{\tau(M, \mathcal{L}; q, c)}{\gamma(G, q)^{-\mathrm{sign}(L)}|G|^{-m/2}} = \sum_{\mathbf{x}\in G^m} \exp(2\pi i (q \otimes A_{L\cup J})(\mathbf{x}, \mathbf{c})).$$

By Steps 1 and 3, we know the factors $\gamma(G, q)^{-\mathrm{sign}(L)}$ and $|G|^{-m/2}$. Therefore, we recover the Gauss sum

$$\sum_{\mathbf{x}\in G^m} \exp(2\pi i (q\otimes A_{L\cup J})(\mathbf{x}, \mathbf{c})) = \sum_{\mathbf{x}\in G^m} \exp\Big(2\pi i\Big((q\otimes A_L)(x) + (b_q\otimes A_{J,L})(x, c) + (q\otimes A_J)(c)\Big)\Big).$$

According to Step 4, we know the term $(q \otimes A_L)(c)$. Hence we recover the Gauss sum

$$\sum_{\mathbf{x}\in G^m} \exp\Big(2\pi i\Big((q \otimes A_L)(x) + (b_q \otimes A_{J,L})(x, c)\Big)\Big).$$

Since we know its absolue value $|G \otimes \mathrm{Tors}\ H_1(M)|^{\frac{1}{2}}$ from Step 1, we recover the Gauss sum

$$\gamma(G \otimes \mathbb{Z}^m, q \otimes A_L + (b_q \otimes A_{J,L})(-, c)) = \gamma(G \otimes W, q \otimes g + (\widehat{b}_q \otimes \widehat{g}_{\mathbb{Q}})([\omega]),$$

where g denotes the symmetric bilinear map on $W = \mathbb{Z}^m$ determined the $m \times m$ linking matrix A_L and $\omega \in G \otimes W^\sharp$ is a lift of $\theta = \sum_j c_j \otimes [L_j] \in G \otimes W^\sharp/W = G \otimes \mathrm{Tors}\ H_1(M)$. It follows that we recover all Gauss sums $\gamma_{g,s}(q, c)$ for all bilinear pairings (W, g). Therefore, applying the classification result for homogeneous quadratic functions (Corollary 3.2) yields the desired result. $\square$

CHAPTER 8

Abelian topological quantum field theory

Let $M = (M, \Sigma_-, \Sigma_+)$ be a connected compact oriented 3-cobordism. In other words, M is a connected compact oriented 3-manifold such that

$$\partial M = \Sigma_+ \coprod -\Sigma_-.$$

The surfaces Σ_+ and Σ_-, called the *bases* of the cobordism, are closed and oriented. We also write

$$\Sigma_+ = \partial_+ M,\ \Sigma_- = \partial_- M.$$

If each base is connected, we say that the cobordism is *elementary*. Note that a given orientation restricts to an orientation on each of the connected component. The opposite orientation is denoted by a minus sign. Each connected component Σ of the base carries a nondegenerate symplectic pairing

$$H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z},\ ([a],[b]) \mapsto [a] \bullet [b]$$

in the first homology with integral coefficients, namely the *intersection pairing*. If Σ is not connected (if the cobordism is not elementary), the pairing may be degenerate. Recall that $H_1(-\Sigma) = -H_1(\Sigma)$ is the same space as $H_1(\Sigma)$ but carries the symplectic pairing *opposite* to that of $H_1(\Sigma)$.

We regard two cobordisms as equivalent if they are equivalent by an orientation preserving homeomorphism that is the identity on the boundary. As is well-known, 3-cobordisms form a category Cob where objects are closed oriented surfaces and morphisms are oriented 3-cobordisms and the composition is provided by the gluing along a common base. The composition will be denoted by $\circ$. For our purpose, this category has the right notion of morphisms but too many objects (too many noncanonical objects). In order to reduce the number of objects (to make them more canonical), we need to enrich somewhat this category.

Once we have defined the right category, we extend the invariant defined in the previous chapters to cobordisms. There are actually two equivalent constructions.

The first one is based on the following idea: glue to each base of a cobordism (M, Σ_-, Σ_+) a finite union of standard handlebodies. This yields a closed 3-manifold $\tilde{M}$ with a pair of distinguished framed oriented links L^+, L^- in $-\Sigma_- \cup \Sigma_+$ (the links are the images of the meridians of the handlebodies respectively). Let g^- (resp. g^+) be the total genus of Σ_- (resp. Σ_+). Color the links with some elements $x^-, x^+ \in G^{g_-} \times G^{g_+}$. According to the previous chapter, to a pair $(\tilde{M}, L^- \cup L^+)$ is associated a topological invariant $\tau(\tilde{M}, L^- \cup L^+, (g^-, g^+)) \in \mathbb{C}$. By varying the colors and using a normalization factor, we obtain a linear operator $\tau(M)$ associated to the cobordism M.

The second construction uses only linking invariants associated to the cobordism M itself. It relies ultimately on the reciprocity.

Finally the invariant has special functorial properties: it is a topological quantum field theory (TQFT). It can be approximately regarded as a functor from an appropriate category of cobordisms to the category of finite-dimensional $\mathbb{C}$-vector spaces.

1. The parametrized cobordism category

Let g be a natural number. For each g, we fix an oriented handlebody H_g of genus g as follows. The *standard* handlebody H_0 of genus 0 is the unit closed 3-ball $D^3 \subset \mathbb{R}^3$. For $g \geqslant 1$, consider a union U_g of g circles $C_1, \dots, C_g$ of radius 1 in the plane $z = 0$ of $\mathbb{R}^3$ centered on the x-axis such that $C_j \cap C_{j+1}$ is a singleton for $1 \leqslant j \leqslant g-1$. The *standard* handlebody H_g of genus g is a closed tubular neighborhood of U_g in $\mathbb{R}^3$ symmetric with respect to the planes $z = 0$ and $y = 0$. All the standard handlebodies are orientable as submanifolds of $\mathbb{R}^3$: we choose the orientation so that the outward normal vector is last. The map $\mathrm{mir} : (x, y, z) \mapsto (x, y, -z)$ is an orientation reversing homeomorphism that restricts to H_g. It is the *standard* orientation reversing homeomorphism of H_g. We also fix a closed oriented surface Σ_g of genus g by $\Sigma_g = \partial H_g$. The j-th simple closed curve $\star \times S^1 \subseteq \Sigma_g$ (resp. $S^1 \times \star \subseteq \Sigma_g$) is the j-th *meridian* (resp. the j-th longitude) of Σ_g. A closed surface Σ is *standard* if Σ is empty or if there is $g \in \mathbb{N}$ such that $\Sigma = \Sigma_g$.

DEFINITION 1.1. A *standard 3-cobordism* is a 3-cobordism (M, Σ_-, Σ_+) whose bases are finite disjoint unions of standard surfaces.

The definition of a standard cobordism is rigid. It is completely dependent of our geometrical model of a closed surface of genus g. The idea is to make all the gluings and all the computations using standard cobordisms and standard surfaces.

Two standard cobordisms (M, Σ_-, Σ_+) and $(N, \Sigma'_-, \Sigma'_+)$ such that $\Sigma_+ = \Sigma'_-$ can be composed by gluing along the common basis: $N \circ M = N \cup M$. Composition is an associative operation.

We need to define an equivalence relation on cobordisms. A first attempt could be to define a weak equivalence between two standard cobordism as an orientation preserving homeomorphism that sends the bottom (resp. top) base to the bottom (resp. top) base. If two standard cobordisms are weakly equivalent then their bottom and top bases respectively coincide. We certainly want this property to hold. As any homeomorphism restricts to a homeomorphism of the boundary, the only difference between this definition and a general homeomorphism is to distinguish between top and bottom bases.

DEFINITION 1.2. An *equivalence* between two standard cobordisms is an orientation preserving homeomorphism that induces the identity on the bases.

Denote equivalence by $\sim$.

LEMMA 1.1. *Let M, M', N, N' be standard cobordisms. Suppose that $N \circ M$ and $N' \circ M'$ are well-defined. If $M \sim M'$ and $N \sim N'$ then $N \circ M \sim N' \circ M'$.*

PROOF. Trivial. □

LEMMA 1.2. *Equivalence classes of standard cobordisms form a small category with finite coproducts (disjoint unions).*

PROOF. Cobordisms are morphisms between two standard surfaces (possibly empty). Formally speaking, an object is a finite sequence of elements in $\{\star, 0, 1, 2, \ldots\}$. We set $\Sigma_\star = \varnothing$. For instance, a morphism between g_- (one term sequence) and g_+ (one term sequence) is represented by an elementary standard cobordism $(M, \Sigma_{g_-}, \Sigma_{g_+})$. In general, a morphism between $(g_1^-, \ldots, g_r^-)$ and $(g_1^+, \ldots, g_s^+)$ is a standard cobordism with bottom base (resp. top base) a surface with connected closed components of genus $g_1^-, \ldots, g_r^-$ (resp. of genus $g_1^+, \ldots, g_s^+$) respectively. The notion of equivalence enables to have an identity cobordism for each object g: it is empty if the object is $\star$ or it is the cylinder $\Sigma_g \times [0, 1]$ otherwise. □

The category of equivalence classes of standard cobordisms is denoted Cob^0.

DEFINITION 1.3. A *parametrized 3-cobordism* is a 3-cobordism (M, Σ_-, Σ_+) equipped with two orientation preserving homeomorphisms $f_- : \Sigma_{g_-} \to \Sigma_-$ and $f_+ : \Sigma_{g_+} \to \Sigma_+$ respectively.

EXAMPLE 1.1. The standard handlebody H_g of genus g can be regarded as a parametrized 3-cobordism $(H_g, \varnothing, \Sigma_g)$ parametrized by the identity $f_+ = \mathrm{id}_{\Sigma_g}$. More generally, any standard cobordism provides an example of a trivially parametrized cobordism with identity parametrizations.

DEFINITION 1.4. Let (M, Σ_-, Σ_0) and (N, Σ_1, Σ_+) be two parametrized cobordisms such that there is an orientation preserving homeomorphism sending Σ_0 to Σ_1. Let g be the genus of Σ_0. Denote by $f_0 : \Sigma_g \to \Sigma_0 \subset M$ and $f_1 : \Sigma_g \to \Sigma_1 \subset N$ the parametrizations of Σ_0 and Σ_1 respectively. The *composition* $N \circ M$ of N and M is defined as

$$N \circ M = N \coprod_{f_1} \Sigma_g \times [-1, 1] \coprod_{f_0} M.$$

Here the identifications are given by $(s, -1) = f_0(s)$ for all $s \in \Sigma_g$ and $(s, 1) = f_1(s)$ for all $s \in \Sigma_g$.

Loosely speaking, we glue a cylinder (over the basis Σ_g) to the disjoint union $N \coprod M$ via the respective parametrizations. See Fig. 1.1.

DEFINITION 1.5. Let (M, Σ_+, Σ_-) and $(N, \Sigma'_+, \Sigma'_-)$ be two parametrized cobordisms. Let g_+ (resp. g_-) denote the genus of the surface Σ_+ (resp. g_-). Let f_-, f_+, f'_-, f'_+ be the parametrizations of $\Sigma_-, \Sigma_+, \Sigma'_-, \Sigma'_+$ respectively. An *equivalence* between M and N is an orientation preserving homeomorphism $F : M \to N$ inducing homeomorphisms on the bases such that $F|_{\Sigma_-} \circ f_- = f'_-$ and $F|_{\Sigma_+} \circ f_+ = f'_+$.

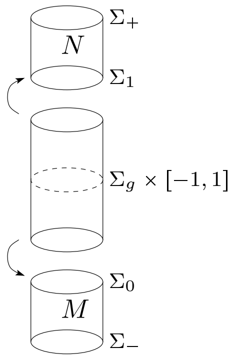


FIGURE 1.1. Gluing two parametrized cobordisms.

LEMMA 1.3. *Let M, M', N, N' be parametrized cobordisms. Suppose that $N \circ M$ and $N' \circ M'$ are well-defined. If $M \sim M'$ and $N \sim N'$ then $N \circ M \sim N' \circ M'$.*

PROOF. Let F_N (resp. F_M) be an equivalence between N and N' (resp. between M and M'). Let g be the genus of the bottom base of N which coincides with the genus of the top base of M. Define a map

$$\tilde{F} : N \coprod \Sigma \times [-1,1] \coprod M \to N' \coprod \Sigma_g \times [-1,1] \coprod M'$$

by

$$\tilde{F}(x) = \begin{cases} F_N(x) & \text{if } x \in N \\ x & \text{if } x \in \Sigma \times [-1,1] \\ F_N(x) & \text{if } x \in M. \end{cases}$$

This map induces an orientation preservation homeomorphism $F : N \circ M \to N' \circ M'$ which commutes with the parametrizations, hence is an equivalence between $N \circ M$ and $N' \circ M'$. □

LEMMA 1.4. *Equivalence classes of parametrized cobordisms form a small category with finite coproducts (disjoint unions).*

PROOF. An object is a finite sequence of elements in $\{\star, 0, 1, 2, \ldots\}$. We set $\Sigma_\star = \varnothing$. For a finite sequence $g = (g_1, \ldots, g_r)$, let Σ_g denote the disjoint union of the standard surfaces $\Sigma_{g_1}, \ldots, \Sigma_{g_r}$. A morphism between $(g_1^-, \ldots, g_r^-)$ and $(g_1^+, \ldots, g_s^+)$ is represented by a triple (M, f_-, f_+) where M is an oriented cobordism (M, Σ_-, Σ_+), $f_- : \Sigma_{g_-} \to \Sigma_-$ and $f_+ : \Sigma_{g_+} \to \Sigma_+$ are orientation preserving homeomorphisms. The identity morphism is represented by a cylinder with identity parametrizations on the bases. The other axioms are easily verified. □

The category of equivalence classes of parametrized cobordisms is denoted $\mathrm{Cob}^{\mathrm{par}}$.

Any parametrized cobordism (M, Σ_-, Σ_+) yields a standard cobordism $(\widetilde{M}, \Sigma_{g_-}, \Sigma_{g_+})$ as follows. We set $g_\pm$ to be the genus of $\Sigma_\pm$. (If $\Sigma_\pm$ is empty, then we choose

$\Sigma_{g\pm}$ to be empty.) We define

$$\tilde{M} = (\Sigma_{g-} \times [0,1]) \coprod_{f_-} M \coprod_{f_+} (\Sigma_{g+} \times [0,1]).$$

The identification are given by $(s,1) = f_-(s)$ for all $s \in \Sigma_{g-}$ and $(s,0) = f_+(s)$ for all $s \in \Sigma_{g+}$. Clearly $\tilde{M}$ is a standard cobordism $(\tilde{M}, \Sigma_{g_-}, \Sigma_+)$ obtained by gluing the cylinders over Σ_{g-} (resp. Σ_{g+}) to Σ_- (resp. Σ_+) by means of the parametrization f_- (resp. f_+) along $\Sigma_{g-} \times \{1\}$ (resp. along $\Sigma_{g+} \times \{0\}$).

Lemma 1.5. *If M and N are two equivalent parametrized cobordisms then $\tilde{M}$ and $\tilde{N}$ are two equivalent standard cobordisms.*

Proof. Suppose there is an equivalence $F : (M, \Sigma_-, \Sigma_0) \to (N, \Sigma'_-, \Sigma'_+)$ between two parametrized cobordisms. It follows that the genus of the bases coincide: $g_- = g'_-$ and $g_+ = g'_+$. Define a disjoint union of maps

$$\tilde{F} : (\Sigma_{g+} \times [0,1]) \coprod M \coprod (\Sigma_{g-} \times [0,1]) \to (\Sigma_{g'_+} \times [0,1]) \coprod N \coprod (\Sigma_{g'_-} \times [0,1])$$

by

$$\tilde{F}(x) = \begin{cases} x & \text{if } x \in \Sigma_{g+} \times [0,1] \\ F(x) & \text{if } x \in M \\ x & \text{if } x \in \Sigma_{g-} \times [0,1]. \end{cases}$$

Since F commute with parametrizations, the maps glue together to induce a map $(\tilde{M}, \Sigma_{g_-}, \Sigma_{g+}) \to (\tilde{N}, \Sigma_{g'_-}, \Sigma'_{g+})$ between standard cobordisms. The map is easily seen to be an orientation preserving homeomorphism. By construction, it preserves pointwise the bases. It is therefore an equivalence. □

Proposition 1.1. *The assignment $M \mapsto \tilde{M}$ induces a covariant full functor* $\mathrm{Cob}^{\mathrm{par}} \to \mathrm{Cob}^0$.

Proof. Lemma 1.5 implies that the assignment is well defined at the level of equivalence classes. Let $F : \mathrm{Cob}^{\mathrm{par}} \to \mathrm{Cob}^0$ denote the corresponding assignment. The identity morphism of the object $g = (g_1, \ldots, g_r)$ in $\mathrm{Cob}^{\mathrm{par}}$ is represented by the cylinder $C = \Sigma_g \times [0,1]$ with the identity as parametrization of the bases. It follows from the definition that $\tilde{C} = C$ in Cob^0. Thus F sends the identity morphism of g in $\mathrm{Cob}^{\mathrm{par}}$ to the identity morphism of g in Cob^0. The identity $F(N \circ M) = F(N) \circ F(M)$ follows from Fig. 1.2. By Example 1.1, any standard cobordism is realized as the image of a trivially parametrized cobordism. Hence F is a full functor. □

2. The Lagrangian cobordism category

Let A be a symplectic lattice. As is customary, we denote by $-A$ the same underlying lattice A with the symplectic pairing opposite to that of A. In particular, if Σ is a closed surface, then $H_1(-\Sigma) = -H_1(\Sigma)$. Any orientation preserving homeomorphism $\Sigma \to \Sigma$ induces a symplectomorphism $H_1(\Sigma) \to H_1(\Sigma)$.

The following definition should be seen as a motivation.

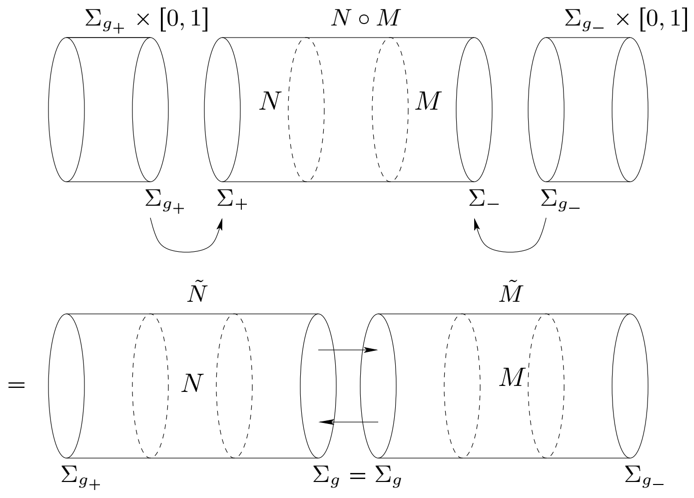


FIGURE 1.2. The composition $\tilde{N} \circ \tilde{M}$.

DEFINITION 2.1. A *Lagrangian cobordism* is an oriented compact 3-cobordism $M, \Sigma_-, \Sigma_+)$ endowed with

(1) Lagrangians $A^- \subseteq H_1(\Sigma_-)$ and $A^+ \subseteq H_1(\Sigma_+)$.
(2) A Lagrangian $\Lambda_M \subset H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$ such that Λ_M is transverse to the Lagrangian $A^- \oplus A^+$ in $H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$.

REMARK 2.1. The condition (2) is equivalent to $\Lambda \cap A^- = \Lambda \cap A^+ = 0$. This follows from Lemma 1.4.

REMARK 2.2. The Lagrangian Λ_M associated to the cobordism M is decomposable. This follows from Lemma 1.2.

EXAMPLE 2.1 (Meridional and longitudinal Lagrangians). For any standard surface Σ_g, there are two distinguished and transverse Lagrangians, namely the Lagrangian generated in 1-homology by the meridians of Σ_g (the *standard meridional Lagrangian*) and the Lagrangian generated in 1-homology by the longitudes of Σ_g (the *standard longitudinal Lagrangian*).

EXAMPLE 2.2 (The trivial Lagrangian cylinder). Let $M = \Sigma \times [0,1]$ be the oriented cylinder over a standard surface $\Sigma = \Sigma_g$. We endow $\Sigma \times \{0\}$ (resp. $\Sigma \times \{1\}$ with the longitudinal Lagrangian A^+ (resp. A^-) in $H_1(\Sigma \times \{1\})$ (resp. $H_1(\Sigma \times \{0\})$) generated in 1-homology by the longitudes $l_1, \ldots, l_g$ of $\Sigma \times \{1\}$ (resp. $H_1(\Sigma \times \{0\})$). We endow M with the Lagrangian Λ_M generated in $H_1(\partial M) = -H_1(\Sigma \times \{0\}) \oplus H_1(\Sigma \times \{1\})$ in 1-homology by the meridians of $\Sigma \times \{1\}$ and $\Sigma \times \{0\}$. Clearly Λ_M is transverse to $A^- \oplus A^+$. The cobordism acquires a structure of Lagrangian cobordism called the trivial Lagrangian cylinder.

EXAMPLE 2.3 (Lagrangian cobordism associated to a parametrization). Let $f : \Sigma_g \to \Sigma$ be an orientation preserving homeomorphism (a parametrization). We associate to f a Lagrangian cobordism $C(f)$ as follows. As a cobordism, $C(f)$ is obtained by gluing two cylinders $\Sigma_g \times [0,1]$ and $\Sigma \times [0,1]$

via f:

$$C(f) = \Sigma \times [0,1] \coprod_f \Sigma_g \times [0,1]$$

with the identification $(s, 1) = (f(s), 0)$ for all $s \in \Sigma$. Hence $C(f)$ is an oriented cobordism between Σ_g and Σ. To the bottom base we associate the standard longitudinal Lagrangian $A^- \subseteq H_1(\Sigma_g)$. To the top base we associated the Lagrangian $f(A^-) \subseteq H_1(\Sigma)$. To the cobordism $C(f)$, we associate $\Lambda = \mathrm{Graph}(f_*)$. Clearly Λ is transverse to $A^- \ominus f(A^-)$. The case when $f = \mathrm{id}_{\Sigma_g}$ yields the diagonal Lagrangian for Λ and the trivial Lagrangian cylinder.

DEFINITION 2.2. The *composition* of two Lagrangian cobordisms (M, Σ_-, Σ_0) and (N, Σ_0, Σ_+) is defined as the underlying composition of the two cobordisms $N \circ M = N \cup M$ endowed with the Lagrangians of the bottom base of M and the top base of N and the Lagrangian $\Lambda_{N \circ M} = \Lambda_N \circ \Lambda_M$. (See Chap. 2, Lemma 1.1, for the composition of Lagrangians.)

LEMMA 2.1. *The composition of two Lagrangian cobordism is a Lagrangian cobordism.*

PROOF. This follows from Lemma 1.6. $\square$

DEFINITION 2.3. Two Lagrangian cobordisms are equivalent if there exists a cobordism equivalence (an orientation preserving homeomorphism that restricts to the identity on the boundary) between them sending Lagrangian onto Lagrangian.

Equivalence classes of Lagrangian cobordisms form a category $\mathrm{Cob}^{\mathrm{lag}}$ with trivial Lagrangian cylinders being the identity morphisms. The assignment

$$(M, \Sigma_-, \Sigma_+) \mapsto (\Lambda, H_1(\Sigma_-), H_1(\Sigma_+))$$

is a functor $\mathrm{Cob}^{\mathrm{lag}} \to \mathrm{Lag}^{-1}_{\mathrm{trans}}(\mathbb{Z})$. As before the category $\mathrm{Cob}^{\mathrm{lag}}$ has too many objects. We now modify this category.

DEFINITION 2.4. A *Lagrangian decorated cobordism* is an oriented 3-cobordism (M, Σ_-, Σ_+) endowed with

(1) A pair of isotopy classes of oriented framed links $L^- \subseteq \Sigma_-$ and $L^+ \subseteq \Sigma_+$ such that the subgroups A^- and A^+ generated in 1-homology by the components of L_- and L_+ respectively are Lagrangians in $H_1(\Sigma_-)$ and $H_1(\Sigma_+)$ respectively.
(2) A Lagrangian $\Lambda \subset H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$ such that Λ is transverse to the Lagrangian $A^- \oplus A^+$ in $H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$.

EXAMPLE 2.4 (The trivial Lagrangian decorated cobordism). Let $M = \Sigma_g \times [0,1]$ be the trivial Lagrangian cobordism. If we provide M with the two underlying sets of longitudes in $\Sigma_g \times \{0\}$ and $\Sigma_g \times \{1\}$, then M is called the trivial Lagrangian decorated cobordism.

EXAMPLE 2.5 (Lagrangian decorated cobordism associated to a parametrized cobordism). Any parametrized cobordism (M, Σ_-, Σ_+) yields a Lagrangian decorated cobordism. The cobordism is topologically the same. Let L^+ be the oriented framed link formed by the images of the longitudes of Σ_{g+} by the parametrization $f_+ : \Sigma_{g+} \to \Sigma_+$. Similarly let L^- be the oriented framed link formed by the images of the longitudes of Σ_{g-} by the parametrization $f_- : \Sigma_{g-} \to \Sigma_-$. Let Λ_M be the Lagrangian in $H_1(\partial M)$ generated in 1-homology by the images of the meridians of Σ_{g+} and Σ_{g-} under the parametrizations f_+ and f_- respectively. Clearly Λ_M and $A^- \oplus A^+$ are transverse. This provides M with the structure of a Lagrangian decorated cobordism.

We now define the composition of two Lagrangian decorated cobordisms using standard surfaces.

Definition 2.5. Let (M, Σ_-, Σ_0) and (N, Σ_1, Σ_+) be two Lagrangian decorated cobordisms such that there is an orientation preserving homeomorphism $\Sigma_0 \to \Sigma_1$. Let g be the genus of Σ_0. Let L^0 (resp. L^1) be an oriented framed link inside Σ_0 (resp. Σ_1). Let $1 \leqslant j \leqslant g$. Denote by l_j the j-th longitude of Σ_g. Choose orientation preserving homeomorphisms $f_0 : \Sigma_g \to \Sigma_0$ and $f_1 : \Sigma_g \to \Sigma_1$ such that $f_0(l_i) = L_i^0$ and $f_1(l_i) = L_i^1$, $i = 1, \ldots, g$. The *composition* $N \circ M$ of M and N is defined by

$$N \circ M = N \coprod_{f_1} \Sigma_g \times [-1, 1] \coprod_{f_0} M$$

with the identifications $(s, -1) = f_0(s)$ and $(s, 1) = f_1(s)$ for all $s \in \Sigma_g$.

Remark 2.3. Each longitude m_j, $1 \leqslant j \leqslant g$, has a natural parallel in Σ: at each point $p \in m_j$, there is a well defined positive normal of length 1. The framing of the links L^0 and L^0 is inherited from the framing of the longitudes m_j, $1 \leqslant j \leqslant g$, in Σ_g.

Remark 2.4. The choice of the homeomorphisms f_0 and f_1 is not unique in general. One should complete the definition of f_0 and f_1 by defining for instance on meridians and extending completely the definition of each homeomorphism to Σ_g. However, we shall soon see that a different choice will eventually lead to an equivalent Lagrangian decorated cobordism.

Lemma 2.2. *The composition of two Lagrangian decorated cobordisms has a natural structure of Lagrangian decorated cobordism.*

Proof. Two Lagrangian decorated cobordisms (M, Σ_-, Σ_0) and (N, Σ_1, Σ_+) are composable if and only if there is an orientation preserving homeomorphism $\Sigma_0 \to \Sigma_1$ sending the link L^0 to the link L^1 (up to isotopy in Σ_1). The link associated to the bottom base (resp. top base) of $N \circ M$ is the link L^- (resp. L^+) associated to Σ_- (resp. to Σ_+). The links L^- and L^+ generate Lagrangians $A^-(M)$ and $A^+(N)$ in $H_1(\Sigma_-)$ and $H_1(\Sigma_+)$ respectively.

The only point consists in *defining* the Lagrangian $\Lambda_{N \circ M}$ in $H_1(N \circ M)$ associated to the Lagrangian decorated cobordism $N \circ M$ so that it be transverse to $A^-(M) \oplus A^+(M)$. Let Λ_M and Λ_N be the Lagrangians associated to M and N respectively. Our gluing depends on the intermediate links in Σ_0 and Σ_1, so we cannot define $\Lambda_{N \circ M} = \Lambda_N \circ \Lambda_M$ as in the case of Lagrangian cobordisms.

The cobordism $N \circ M$ can be written as the composition of cobordisms (in the sense of morphisms of Cob)

$$N \circ M = N \circ C(f_1) \circ C(f_0^{-1}) \circ M$$

where $C(f_1)$ and $C(f_0^{-1})$ are Lagrangian cobordisms associated to f_1 and f_2 respectively. (See Example 2.3.) Now regard M and M as Lagrangian cobordisms as well. We have expressed $N \circ M$ as the composition of four Lagrangian cobordisms, hence $N \circ M$ is a Lagrangian cobordism. Therefore (in the usual sense of composition of Lagrangians, by Lemma 2.1)

$$\Lambda_{N \circ M} = \Lambda_N \circ \mathrm{Graph}(f_{1*}) \circ \mathrm{Graph}(f_{0*}^{-1}) \circ \Lambda_M,$$

where f_{0*} and f_{1*} denote the symplectomorphisms induced in 1-homology by f_0 and f_1 respectively, is a Lagrangian transverse to $A^-(M)$ and $A^+(M)$.

We are therefore forced to define the composition $\Lambda_{N \circ M}$ by the formula above which proves the result. $\square$

Definition 2.6. Two decorated Lagrangian cobordisms M and N are *equivalent* if there is an orientation preserving homeomorphism $f : M \to N$ sending the top base (resp. the bottom base) to the top base (resp. the top base), sending the oriented framed link in the top base (resp. in the bottom base) to the oriented framed link in the top base (resp. in the bottom base) and sending the Lagrangian Λ_M to the Lagrangian Λ_N.

The composition is associative and the trivial Lagrangian decorated cobordisms represent the identity morphisms.

Definition 2.7. The category formed by equivalence classes of Lagrangian decorated cobordisms is denoted $\mathrm{Cob}_{\mathrm{fr}}^{\mathrm{lag}}$.

Example 2.5 shows that a Lagrangian decorated cobordism $\tilde{M}$ is naturally associated to a parametrized cobordism M.

Proposition 2.1. *The assignment $M \mapsto \tilde{M}$ induces a full covariant functor* $\mathrm{Cob}^{\mathrm{par}} \to \mathrm{Cob}_{\mathrm{fr}}^{\mathrm{lag}}$.

Proof. Let (M, Σ_-, Σ_+) be a decorated Lagrangian cobordism with transverse Lagrangians $\Lambda_M \subseteq H_1(\partial M)$ and $A^- \oplus A^+ \subseteq H_1(\partial M)$ respectively. We show that this cobordism is represented by a parametrized cobordism. First the symplectic group acts transitively on pairs of transverse Lagrangians.

we need to find parametrizations $f_- : \Sigma_{g_-} \to \Sigma_-$ and $f_+ : \Sigma_{g_+} \to \Sigma_+$ such that the images of meridians $\square$

Remark 2.5. We could introduce a more general category $\widetilde{\mathrm{Cob}}_{\mathrm{fr}}^{\mathrm{lag}}$ by removing the condition of transversality for the Lagrangians. For all practical purposes, the category $\mathrm{Cob}_{\mathrm{fr}}^{\mathrm{lag}}$ will suffice.

In the rest of this paragraph, we make a number of elementary observations about the nature of Lagrangians.

Consider a connected oriented 3-manifold M with non-empty boundary ∂M. Such a manifold gives rise to a Lagrangian Ker $i_* \subset H_1(\partial M)$ where $i_* : H_1(\partial M) \to H_1(M)$ denotes the inclusion homomorphism. See for instance [**4**, Th. 10.4]. Such a Lagrangian depends only on the topology of M and will be called *standard* in the sequel.

In general, there are many other Lagrangians in $H_1(\partial M)$; if we regard M as some Lagrangian decorated cobordism, the standard Lagrangian generally differ from the Lagrangian Λ_M associated to the cobordism. One way to see that is to remark that Ker i_* does not have to be decomposable.

EXAMPLE 2.6. Let $M = T^2 \times [0,1]$, the cylinder over the 2-torus. The boundary of M consists of two copies

$$T_- = T \times 0,\ T_+ = T \times 1$$

(with opposite orientations) of the 2-torus. The standard Lagrangian Ker i_* is the Lagrangian generated by pairs $(-x, x) \in -H_1(T_-) \oplus H_1(T_+)$. It is *not decomposable*. Denote by

$$l_\pm, m_\pm \in H_1(T_\pm)$$

the homological classes represented by the longitude $S^1 \times \star$ and the meridian $\star \times S^1$ of $T_\pm = S^1 \times S^1$ respectively. They clearly generate $H_1(T_\pm)$. Any pair of primitive elements of $H_1(T_\pm)$ forms a symplectic basis of $H_1(T_\pm)$ if and and only if it is uniquely represented by a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ with respect to the basis $(l_\pm, m_\pm)$. In particular, there is a one-to-one correspondence between Lagrangians in $H_1(T_\pm)$ and primitive elements in $H_1(T_\pm)$. Let now $\Lambda \in H_1(\partial M) = -H_1(T_-) \oplus H_1(T_+)$ be a Lagrangian. It is generated by two independent primitive elements. If one of these two elements can be taken to be in $H_1(T_\pm)$ then (since it is primitive) it generates a Lagrangian $\Lambda_\pm \in H_1(T_\pm)$. If both elements can be taken in $H_1(T_-)$ and $H_1(T_+)$ respectively, then Λ is a decomposable Lagrangian and $\Lambda = \Lambda_- \oplus \Lambda_+$. Conversely, any decomposable Lagrangian $\Lambda \subset -H_1(T_-) \oplus H_1(T_+)$ is generated by two primitive elements in $H_1(T_-)$ and $H_1(T_+)$ respectively. For instance,

$$\Lambda_{11} = \langle l_- \rangle \oplus \langle l_+ \rangle,\ \Lambda_{12} = \langle l_- \rangle \oplus \langle m_+ \rangle,\ \Lambda_{21} = \langle m_- \rangle \oplus \langle l_+ \rangle,\ \Lambda_{22} = \langle m_- \rangle \oplus \langle m_+ \rangle$$

are four distinct decomposable Lagrangians of $H_1(\partial M) = H_1(-T_-) \oplus H_1(T_+)$.

It is not hard to extend the observation of the previous example.

LEMMA 2.3. *Let (M, Σ_-, Σ_+) be a 3-cobordism with ∂M consisting of exactly two connected components, Σ_- and Σ_+. A Lagrangian $\Lambda \subset H_1(\partial M)$ is decomposable (with respect to the decomposition $H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$) if and only if Λ is generated by elements represented by simple closed oriented curves.*

In the example above for $M = T \times [0,1]$, the standard Lagrangian Λ_M is generated by pairs $(-x, x) \in -H_1(T_-) \oplus H_1(T_+)$, $x \in H_1(T)$. None of these pairs (except the trivial one) can be represented by *one* single simple closed oriented curve.

PROOF. The representation of generators of Λ by simple closed oriented curves is a sufficient condition: each simple closed curve must lie in one single connected component of ∂M and in particular will induce a well-defined homological class in $H_1(\Sigma_-)$ or in $H_1(\Sigma_+)$. Since Λ is Lagrangian, there are exactly $g_\pm$ simple closed curves in $\Sigma_\pm$ where $g_\pm$ denotes the genus of $\Sigma_\pm$. The result follows.

Conversely, if Λ is decomposable then $\Lambda_\pm = \Lambda \cap H_1(\Sigma_\pm)$ is a Lagrangian in $H_1(\Sigma_\pm)$. Such a Lagrangian is generated by a system of g independent primitive elements in $H_1(\Sigma_\pm)$. An element in $H_1(\Sigma_\pm)$ is primitive if and only if it is represented by a simple closed oriented nonseparating curve (see e.g., [**42**]). Hence the result. □

3. Linking pairings and Lagrangians

3.1. The linking pairing of a 3-manifold. Let M be any connected compact oriented 3-manifold with boundary ∂M. Poincaré-Lefschetz duality and the (torsion) universal coefficient theorem lead to the following sequence of isomorphisms

$$\begin{aligned}\text{Tors } H_1(M) \simeq \text{Tors } H^2(M, \partial M) &\simeq \text{Tors Ext}(H_1(M, \partial M), \mathbb{Z}) \\ &\simeq \text{Hom}(\text{Tors } H_1(M, \partial M), \mathbb{Q}/\mathbb{Z}).\end{aligned}$$

There is therefore a nonsingular bilinear pairing

$$a_M : \text{Tors } H_1(M) \times \text{Tors } H_1(M, \partial M) \to \mathbb{Q}/\mathbb{Z}.$$

A geometrical definition of this pairing is as follows. First define linking numbers for cycles. Let x be an integral 1-cycle in M and let y be a relative 1-cycle in $(M, \partial M)$ representing homology classes in Tors $H_1(M)$ and Tors $H_1(M, \partial M)$ respectively. We may assume that x and y are in general position. There exists $n \in \mathbb{Z}$ and a 2-chain C in M such that $n\ x = \partial C$. We may further assume that C and y intersect transversally, i.e. in a finite number of points away from the boundary. Define

$$\text{lk}_M(x, y) = \frac{C \cdot y}{n} \in \mathbb{Q}.$$

EXAMPLE 3.1. Let $M = S^1 \times D^2$ be the solid torus. Let $\star$ denote an arbitrary point on S^1. Let $x = \star \times \partial D^2$ be a meridian and $y = S^1 \times 0$ be a longitude. Clearly x bounds a disc $C = \star \times D^2$ and y is a boundary modulo $\partial M = S^1 \times \partial D^2$ (the torus surface). Since C and y intersect in exactly one point, it follows that for a suitable choice of orientations, $\text{lk}(x, y) = +1$.

At the level of homology, define

$$a_M([x], [y]) = \text{lk}(x, y) \bmod 1.$$

Here we denote by a small dot the algebraic intersection number. The *linking pairing* λ_M associated to M is defined by

$$\lambda_M = a_M \circ (j_* \times \text{id}),$$

where $j_* : \text{Tors } H_1(M) \to \text{Tors } H_1(M, \partial M)$ denotes the natural projection homomorphism. In particular, if j_* is an isomorphism or when M has no nonempty boundary, the linking pairing λ_M is nonsingular. In general when M has a nonempty boundary, the linking pairing is degenerate.

3.2. Lagrangians. We consider a slight variation of the previous setting. Suppose that M is equipped with an isotropic subgroup $\Lambda \subset H_1(\partial M)$ (isotropic with respect to the intersection pairing on $H_1(\partial M)$). Let x, y be two 1-cycles in general position in M such that their homology classes lie in $\mathrm{Tors}\,(H_1(M)/i_*(\Lambda))$. Unravelling the definition, we see that there exist $n \in \mathbb{Z}$, a 1-cycle z whose homology class lies in Λ and a 2-chain C in M such that $n\ x = i_* z + \partial C$. Assuming that C is in general position with respect to y and denoting by a dot algebraic intersection, we define the linking number

$$\mathrm{lk}_\Lambda(x, y) = \frac{C \cdot y}{n} \in \mathbb{Q}.$$

LEMMA 3.1. lk_Λ *is well-defined, symmetric and bilinear.*

REMARK 3.1. The linking number takes values in $\mathbb{Z}$ if and only if one of the 2-cycles is a boundary.

PROOF. If $n\ x = i_* z' + \partial C'$ is another decomposition, then $\partial(C - C') = i_*(z' - z)$ represents an element in $i_*(\Lambda)$. In particular, $C - C'$ is a relative 2-cycle, i.e. represents an element in $H_2(M, \partial M)$. Let $p \in \mathbb{Z}$ such that $p\ [y] = i_*(w)$ in $H_1(M)$ for some $w \in \Lambda$. Thus the algebraic intersection $(C - C') \cdot p\ y$ is computed using the homological intersection product $\bullet$: $H_2(M, \partial M) \times H_1(M) \to \mathbb{Z}$. Let $a = [C - C'] \in H_2(M, \partial M)$. Since this product takes value in $\mathbb{Z}$, no torsion occurs. Hence

$$(C - C') \cdot y = \frac{1}{p}\big(a \cdot p\ y\big) = \frac{1}{p}\big(a \bullet_M i_*(w)\big) = \frac{1}{p}(\partial a \bullet_{\partial M} w) = 0.$$

The third equality is a well-known property of intersection products with respect to the long exact sequence associated to $(M, \partial M)$. In the last equality, the product is the intersection product on $H^1(\partial M)$. Since both ∂a and w lie in the same isotropic Λ, their product vanishes. This proves that lk_Λ is well defined.

Since lk_Λ is defined for 1-cycles, it is a bilinear pairing. To see that the pairing is symmetric, let C' be a 2-chain in M and z' a 1-cycle whose homology class lies in $i_*(\Lambda)$ such that $p\ y = i_* z' + \partial C'$. Assume transversality, the intersection of two 2-chains C and C' is a 1-cycle in M. Hence

$$\begin{aligned} 0 = \partial(C \cdot C') = \partial C \cdot C' - C \cdot \partial C' &= (nx - i_* z) \cdot C' - C \cdot (ny - i_* z') \\ &= n\ x \cdot C' - C \cdot p\ y - i_* z \cdot C' + C \cdot i_* z' \\ &= n\ x \cdot C' - C \cdot p\ y. \end{aligned}$$

Dividing by $n\ p$ gives $0 = \mathrm{lk}_\Lambda(y, x) - \mathrm{lk}_\Lambda(x, y)$. ☐

Given the special role that Lagrangians play in 3-cobordisms (see §...), we are interested in the special case when Λ is a Lagrangian.

LEMMA 3.2. *Let M be any connected compact oriented 3-manifold with boundary ∂M. Let Λ be a Lagrangian in $H_1(\partial M)$. Let*

$$i_* : H_1(\partial M) \to H_1(M)$$

be the homomorphism induced by inclusion. The linking pairing on M induces a linking pairing

$$\lambda_\Lambda : \mathrm{Tors}\,\Big(H_1(M)/i_*(\Lambda)\Big) \times \mathrm{Tors}\,\Big(H_1(M)/i_*(\Lambda)\Big) \to \mathbb{Q}/\mathbb{Z}$$

defined by

$$\lambda_\Lambda([a],[b]) = \mathrm{lk}_\Lambda(a,b) \bmod 1.$$

REMARK 3.2. The case when ∂M is empty, we recover the usual linking pairing. See [**25**, Chap. 4].

PROOF. See [**45**, §6]. Another proof follows from Lemma 3.3 below. $\square$

In the sequel, we set

$$G_\Lambda = H_1(M)/i_*(\Lambda), \quad T_\Lambda M = \mathrm{Tors}\; G_\Lambda.$$

According to Lemma 3.2, $T_\Lambda M$ carries a linking pairing λ_Λ. An important case arises when the Lagrangian is decomposable.

3.3. Lagrangians and gluings. Consider a disjoint union H of standard oriented solid handlebodies $H_1, \dots, H_r$ and an orientation reversing homeomorphism $f : \cup_j \partial H_j \to \partial M$. Let $\widetilde{M} = M \cup_f H$ be the closed oriented 3-manifold obtained by gluing the handlebodies to ∂M via f. Let Λ be the Lagrangian generated in $H_1(\partial M)$ by the images by f of the meridians of $\partial H_1, \dots, \partial H_r$. Let j denote the map $M \to \widetilde{M}$ induced by the inclusion $M \subset M \cup H$ and the gluing. The following observation is useful.

LEMMA 3.3. *Let x, y be two 1-cycles in general position in M such that some of their multiples lie in $i_*\Lambda$. Then*

$$\mathrm{lk}_\Lambda(x,y) = \mathrm{lk}_{\widetilde{M}}(j_*x, j_*y).$$

In particular, there is a linking pairing isomorphism

$$(\mathrm{Tors}\; H_1(\widetilde{M}), \lambda_{\widetilde{M}}) \simeq (T_\Lambda M, \lambda_\Lambda).$$

In particular, λ_Λ is nondegenerate.

EXAMPLE 3.2. Let M be the solid torus $S^1 \times D^2$. Its boundary is $\partial M = S^1 \times \partial D^2 = S^1 \times S^1$. The first integral homology of ∂M is freely generated by a meridian $m = \star \times \partial D^2$ and a longitude $l = S^1 \times \star$. Let Λ be the Lagrangian generated by the longitude $l = S^1 \times \star \subset S^1 \times \partial D^2$. Then $G_\Lambda M = H_1(M)/i_*\Lambda = 0$, the linking numbers $\mathrm{lk}_\Lambda(x,y)$ are integers and the linking pairing lk_Λ is trivial. Consider a homeomorphism $f : \partial(S^1 \times D^2) \to M$ sending the meridian $\star \times \partial D^2$ to the longitude l of M. Then $\widetilde{M} = S^3$. Hence linking numbers are usual linking numbers in S^3 (hence are integers) and the linking pairing on S^3 is trivial since the homology of S^3 is trivial.

EXAMPLE 3.3. Consider the same solid torus $M = S^1 \times D^2$. Let n be a nonzero integer and m and l the meridian and longitude as before. Let $\Lambda = \Lambda_n$ be the Lagrangian in $H_1(\partial M)$ generated by $m + n\,l$. Then $H_1(M)/i_*\Lambda \simeq \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{lk}_\Lambda(p\;[l], q\;[l]) = \pm\frac{pq}{n} \bmod 1$. If $f : \partial(S^1 \times D^2) \to M$ is a homeomorphism sending the meridian $\star \times D^2$ to $m + n\,l$ in M, then $\widetilde{M}$ is the lens space $L(n,1)$ and we recover the cyclic linking pairing on $L(n,1)$ in this fashion.

Under the hypothesis of this paragraph, any quadratic enhancement q_Λ of λ_Λ is nondegenerate. We observe that any quadratic enhancement can be regarded as partially induced by a relative spin structure s on the 3-manifold M with boundary. We describe it as follows. With the same notation as above, endow the disjoint union of oriented solid handlebodies $H_1, \dots, H_r$ with relative spin structures $s_1, \dots, s_r$ respectively, in such a way that

$$s|_{\partial M} = \cup_i f^* s_i|_{\partial H_i}$$

Then gluing the handlebodies to M via f yields a closed 3-manifold $\widetilde{M}$ with spin structure $\widetilde{s}$. Any quadratic enhancement of λ_Λ is obtained as the quadratic form induced by $\widetilde{s}$ for some suitable choice of $s_1, \dots, s_r$.

Remark 3.3. We regard the Lagrangian Λ as a kind of algebraic remnant of the gluing. Topologically it is easier to think in terms of gluings; algebraically (specifically in relation with the Weil representation), it is easier to think in terms of Lagrangians. This motivates the following definitions.

Definition 3.1. Let $K, K' \subset M$ be two disjoint oriented framed knots and Λ a Lagrangian in $H_1(\partial M)$. If for any parametrization $f : \cup_j \partial H_j \to \partial M$ such that Λ is generated by the images by f of all the meridians of the standard handlebodies H_j, the framed knot K (thought as an annulus) bounds a disc in the closed manifold $\widetilde{M}$ (resp. the knots K and K' are separated by a 2-sphere in $\widetilde{M}$), then we say that K *Λ-unknotted* or is *Λ-trivial* (resp. we say that K and K' are *Λ-unlinked*). Given an oriented link $L \subset M$, we say that L is *λ-unlinked* if all pairs of components of L are Λ-unlinked. We say that L is *Λ-trivial* if L is *Λ-unlinked* and each individual component of L is Λ-trivial.

Example 3.4. Consider the cylinder M over the 2-torus equipped with the four closed simples curves m_-, m_+, l_-, l_+ as depicted in Fig. 2.1. Endow these curves with collars provided by the orientation of the tori, which turn them into annuli (framed curves). The curves m_+ and l_- generate in 1-homology a Lagrangian Λ. Then both m_- and l_+ are Λ-unknotted. Since

$$\mathrm{lk}_\Lambda(i_* m_-, i_* l_+) = +1$$

the oriented framed link $m_- \cup l_+$ is not Λ-unlinked.

4. Construction I: filling in

This section is devoted to a first construction of the abelian TQFT associated to a quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group. This construction relies on "closing" cobordisms by standard handlebodies. The ground cobordism category is the category $\mathrm{Cob}^{\mathrm{par}}$ of parametrized cobordisms.

Let (M, Σ_-, Σ_+) be a Lagrangian decorated cobordism. For simplicity we shall assume that Σ_- and Σ_+ are connected. By definition, M is endowed with two parametrizations $f_- : \Sigma_{g_-} \to \Sigma_-$ and $f_+ : \Sigma_{g_+} \to \Sigma_+$. Consider the closed oriented manifold

$$\widetilde{M} = H_{g_-} \cup_{f_-} M \cup_{f_+ \circ \mathrm{mir}} -H_{g_+}$$

obtained by gluing two standard handlebodies to M via the parametrization maps. Let L^- (resp. L^+) be the image in Σ_- (resp. in Σ_+) of the standard longitudes of ∂H_{g_-} (resp. ∂H_{g_+}). Hence L^- (resp. L^+) is a g_--component (resp. g_+-component) oriented link that can be colored correspondingly by an element $c^- \in G^{g_-}$ (resp. $c^+ \in G^{g_+}$). Furthermore, since a standard longitude has a preferred standard parallel, each link comes with a framing. Let A^- (resp. A^+) denote the abelian group freely generated by L^- (resp. L^+). Let $\mathcal{T}(\Sigma_-) = \mathbb{C}[G \otimes A^-]$ and $\mathcal{T}(\Sigma_+) = \mathbb{C}[G \otimes A^+]$. Define a $\mathbb{C}$-linear map $\tau(M) : \mathcal{T}(\Sigma_-) \to \mathcal{T}(\Sigma_+)$ in matrix form by $\tau(M) = (\tau_{c^-,c^+})_{c^- \in G^{g_-}, c^+ \in G^{g_+}}$ with

$$\tau_{c^-,c^+} = |G|^{-g_+/2}\ \tau(\tilde{M}, L^- \cup L^+; q, (c^-, c^+)).$$

THEOREM 4.1. *The assignment* $\tau : \mathrm{Cob}^{\mathrm{par}} \to \mathrm{Vect}_{\mathbb{C}}, (M, \Sigma_-, \Sigma_+) \mapsto \tau(M)$ *defines a TQFT in dimension 3.*

This is a simplified case of Turaev's construction [**61**, Chapter IV]. For a proof, see [**9**, Theorem 2].

5. The construction II: counting cycles

First we fix some auxiliary algebraic data. We need a fixed homogeneous nondegenerate quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group. We shall use the discriminant construction in the form of a presentation of (G, q) given by a triple (V, f, v) where $f : V \times V \to \mathbb{Z}$ is a (nondegenerate) bilinear lattice equipped with an integral Wu class $v \in \mathrm{Wu}^V(f)$. We note that there are many choices for the presentation (V, f, v) and for the lift of coefficients according to a given presentation of (G, q) as a discriminant quadratic function. However, the outcome of our construction will be independent of all these choices.

Consider a 3-cobordism (M, Σ_-, Σ_+) in $\mathrm{Cob}^{\mathrm{fr}}_{\mathrm{Lag}}$. Recall the free abelian groups $A^- = A^-, A^+$ generated by the framed 1-cycles L_i^-, $(1 \leqslant i \leqslant g_-)$ and L_i^+, $(1 \leqslant i \leqslant g_+)$ respectively. We shall use the same notation, A^- and A^+, to denote the Lagrangian they generate in $H_1(\Sigma_-)$ and $H_1(\Sigma_+)$ respectively. Furthermore, $A^- \oplus A^+$ is a (decomposable) Lagrangian in $H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$. Define two *state modules* by

$$\mathcal{T}(\Sigma_\pm) = \mathbb{C}[G \otimes A^\pm].$$

These are simply vector spaces of formal combinations over $\mathbb{C}$ of certain framed 1-cycles in M (with coefficients in G) of dimensions $|G|^{g_-}$ and $|G|^{g_+}$ respectively. (The algebra structure on $\mathcal{T}(\Sigma_\pm)$ will not be used.)

Consider the two nondegenerate linking pairings $b_q : G \times G \to \mathbb{Q}/\mathbb{Z}$ and $\lambda_M : T_\Lambda M \times T_\Lambda M \to \mathbb{Q}/\mathbb{Z}$. According to Corollary **??**, there is a characteristic element

$$\theta = \theta_{b_q, \lambda_M} \in G \otimes T_\Lambda M \subset G \otimes G_\Lambda M$$

of order at most 2.

There are natural maps $i_*^{\pm} : G \otimes A^{\pm} \to G \otimes G_\Lambda$ defined as the composition

$$G \otimes A^{\pm} \xrightarrow{\subset} G \otimes Z_1(\Sigma_{\pm}) \longrightarrow G \otimes Z_1(M) \longrightarrow G \otimes \frac{H_1(M)}{i_*\Lambda} = G \otimes G_\Lambda$$

where $Z_1(M)$ denotes the group of 1-cycles in M and the middle map is induced by the inclusion homomorphism. For each $x \in G \otimes A^-$, let

$$\mathcal{H}(x) = \{y \in G \otimes A^+ \mid i_*^+([y]) - i_*^-([x]) = \theta \in G \otimes G_\Lambda M\}.$$

An informal (and probably more inspiring) way of defining the set $\mathcal{H}(x)$ is to declare that a 1-cycle $y \in G \otimes A^+$ lies in $\mathcal{H}(x)$ if and only if when viewed inside M, the difference of the cycles x and y lies in the class defined by the characteristic element $\theta \in G \otimes T_\Lambda M$. Since G is finite, the set $\mathcal{H}(x)$ is finite. Note that $\mathcal{H}(x)$ can be empty.

Example 5.1. In the case $T_\Lambda M = 0$, then $\lambda_\Lambda = 0$ and $\theta = 0$. Thus

$$\mathcal{H}(x) = \{y \in G \otimes A^+ \mid i_*^+([y]) - i_*^-([x]) = 0 \in G \otimes G_\Lambda M\}.$$

Example 5.2. In the case $G_\Lambda M = 0$, then for all $y \in G \otimes A^+$, the equality $i_*^+([y]) - i_*^-([x]) = 0$ in $G \otimes G_\Lambda M = 0$ is satisfied. Hence $\mathcal{H}(x) = G \otimes A^+$.

The 1-cycle $\kappa_{xy} = i_*^+(y) - i_*^-(x)$ (with coefficients in G) inherits a framing from the original framings of x and y. This cycle lifts to a framed oriented 1-cycle $\widetilde{\kappa}_{xy}$ with coefficients in $V^\sharp$ by lifting coefficients. Similarly the characteristic element $\theta \in G \otimes T_\Lambda M$ lifts to an element $\widetilde{\theta} \in V^\sharp \otimes T_\Lambda M$.

Note that there exists a lift $\widetilde{\kappa}_{xy}$ such that $[\widetilde{\kappa}_{xy}] = \widetilde{\theta} \in V^\sharp \otimes T_\Lambda M$ if and only if $[\kappa_{xy}] = \theta \in G \otimes T_\Lambda M$ if and only if $y \in \mathcal{H}(x)$.

Let $\widetilde{\kappa}_v$ denote the framed 1-cycle obtained by evaluating (coefficients of) $\widetilde{\kappa}_{xy}$ against the integral Wu class $v \in \mathrm{Wu}^V(f) \subset V$: $\widetilde{\kappa}_v = (\widehat{f}_{\mathbb{Q}} \otimes \mathrm{id})(v \otimes \mathrm{id})(\widetilde{\kappa}_{xy})$. We shall need to use the invariant $\delta_s(\widetilde{\kappa}_v) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ defined by (4.1). (As it was defined, this invariant depends on the choice of a spin structure, but the spin structure was used only to select a quadratic enhancement of the linking pairing.) We denote this quadratic form by q_Λ. We denote the resulting invariant by $\delta_\Lambda(\widetilde{\kappa}_v) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

(5.1)
$$\begin{aligned} C_\Lambda(M) &= \gamma(T_\Lambda M, q_\Lambda)^{-f_{\mathbb{Q}}(v,v)} \\ &\quad \gamma\Big(V \otimes T_\Lambda M, f \otimes q_\Lambda + \big(\widehat{f}_{\mathbb{Q}} \otimes \widehat{\lambda}_\Lambda\big)(\widetilde{\theta})\Big)\ |G \otimes G_\Lambda|^{1/2} |G|^{-g_+/2}. \end{aligned}$$

The number $C_\Lambda(M) \in \mathbb{C}$ is independent of the lift $\tilde{\theta}$ for θ. It is nonzero because of Theorem 2.1.

We define a linear operator $\tau(M) = \tau(M, q) : \mathcal{T}(\Sigma_-) \to \mathcal{T}_+(\Sigma_+)$ by setting

$$\tau(M)x = C_\Lambda(M) \sum_{y \in \mathcal{H}(x)} \exp\Big(2\pi i\ \Big((f \otimes q^{\mathrm{fr}})(\widetilde{\kappa}_{xy}) - \delta_\Lambda(\widetilde{\kappa}_v)\Big)\Big)\ y \tag{5.2}$$

for any $x \in G \otimes A^-$ and extending by $\mathbb{C}$-linearity.

In an informal way (forgetting about the choice of the Lagrangian), we may say that $\tau(M)x$ computes a weighted sum of cycles in Σ_+ almost homologous to x when viewed inside M.

The major result of the paper [**9**] is the following theorem.

THEOREM 5.1. *The assignment* $(M, \Sigma_-, \Sigma_+) \mapsto \tau(M)$ *defines a* TQFT *in dimension* 3.

The assignment actually takes a cobordism (a morphism in the category $\mathrm{Cob}^{\mathrm{fr}}_{\mathrm{Dec}}$) to a unitary linear operator (cobordism invariant operator). In particular, $\tau(M)$ depends on a Lagrangian $\Lambda \in H_1(\partial M)$. It is part of the statement of Th. 5.1 that if M and N are equivalent, then $\tau(M) = \tau(N)$.

A consequence of the classification theorem (Th. 3.1).

Let us record a particular case of Theorem 5.1.

COROLLARY 5.1. *Suppose that G or $T_\Lambda M$ has odd order. Then*

$$C_\Lambda(M) = \gamma(T_\Lambda M, q_\Lambda)^{-f_{\mathbb{Q}}(v,v)}\gamma\Bigg(V \otimes T_\Lambda M, f \otimes q_\Lambda\Bigg)\ |G \otimes G_\Lambda|^{1/2}|G|^{-g_+/2}. \tag{5.3}$$

and for any $x \in G \otimes A^-$,

$$\tau(M)x = C_\Lambda(M) \sum_{y \in \mathcal{H}(x)} \exp(2\pi i(q \otimes \mathrm{lk}_\Lambda(\kappa_{xy})))\ y. \tag{5.4}$$

PROOF. Either hypothesis implies that the characteristic element θ is zero in homology. Then each cycle κ_{xy} representing θ can be written as a linear combination, with coefficients in G, of boundaries, say $\kappa_{xy} = \sum_k c_k \otimes L_k$. Lift coefficients $c_k \in G$ to coefficients $\xi_k \in V^\sharp$ and get a cycle $\widetilde{\kappa}_{xy}$ with coefficients in $V^\sharp$. Recall that (G, q) is the discriminant quadratic form derived from (V, f, v). It follows that

$$\begin{array}{rcl}
(q \otimes \mathrm{lk}_\Lambda)(\kappa_{xy}) & = & \sum_k q(c_k)\mathrm{lk}_\Lambda(L_k, L'_k) + \sum_{j<k} b_q(c_j, c_k)\mathrm{lk}_M(L_j, L_k) \\
& = & (f \otimes q^{\mathrm{fr}})(\sum_k \xi_k \otimes L_k) - \sum_k f_{\mathbb{Q}}(\xi_k, v)\frac{1}{2}\mathrm{lk}_\Lambda(L_k, L'_k) \\
& = & (f \otimes q^{\mathrm{fr}})(\widetilde{\kappa}_{xy}) - \sum_k f_{\mathbb{Q}}(\xi_k, v)\left(\frac{1}{2}\mathrm{lk}_\Lambda(L_k, L'_k) - 0\right) \\
& = & (f \otimes q^{\mathrm{fr}})(\widetilde{\kappa}_{xy}) - \sum_k f_{\mathbb{Q}}(\xi_k, v)\left(\frac{1}{2}\mathrm{lk}_\Lambda(L_k, L'_k) - q_\Lambda([L_k])\right) \\
& = & (f \otimes q^{\mathrm{fr}})(\widetilde{\kappa}_{xy}) - \delta_\Lambda(\widetilde{\kappa}_v).
\end{array}$$

Since in homology we can take $\tilde{\theta} = 0 \in V^\sharp \otimes T_\Lambda M$, $C_\Lambda(M)$ is as stated in (5.3) and the result stated follows. ☐

COROLLARY 5.2. *If $T_\Lambda M = 0$ then*

$$\tau(M)x = |G \otimes G_\Lambda|^{\frac{1}{2}}|G|^{-g_+/2} \sum_{\substack{y \in G \otimes A^+ \\ i^+_*([y]) = i^-_*([x])}} \exp(2\pi i(q \otimes \mathrm{lk}_\Lambda)(\kappa_{xy}))\ y. \tag{5.5}$$

A complete proof of Th. 5.1 was given in [**9**] for $v = 0$. The same proof carries over in the general case. Here we wish to make a few comments. The crucial point in the proof of Theorem 5.1 lies in the exact behavior of

τ under the gluing of 3-cobordisms. To the best of the author's knowledge, this is done in an indirect way: the 3-cobordisms are viewed as boundaries of 4-cobordisms and the composition of the 3-cobordisms is computed as the boundary of the composition of 4-cobordisms; then Wall's corrective formula for the signature of the composition is used to compute the anomaly[1]. Wall's formula involves the Leray-Maslov index. However, the Leray-Maslov index is an invariant of a triple of Lagrangians (with respect to an antisymmetric bilinear pairing). We deduce the following result:

THEOREM 5.2. *The assignment* $(M, \Sigma_-, \Sigma_+) \mapsto \tau(M)$ *defines a* TQFT *in dimension* $4k-1$ *for any* $k \geqslant 1$.

PROOF. Each closed $(4k-2)$-manifold is naturally equipped with its intersection pairing $H_{2k-1}(\Sigma) \times H_{2k-1}(\Sigma) \to \mathbb{Z}$ which is antisymmetric. In addition, it is equipped with the isotopy class of framed $(2k-1)$-cycles that generate a Lagrangian. A cobordism M between Σ_- and Σ_+ is equipped with a Lagrangian $\Lambda \subset H_{2k-1}(\partial M) = H_{2k-1}(-\Sigma_-) \oplus H_{2k-1}(\Sigma_+)$. The definitions of linking number lk_Λ, linking pairing λ_Λ and quadratic enhancement q_Λ are the same. Then formula (5.2) makes sense for a $(4k-1)$-cobordism M with boundary $\partial M = -\Sigma_- \coprod \Sigma_+$. All the axioms for a TQFT are easily seen to hold except maybe the gluing axiom. Let $N \circ M$ be the composition of two cobordisms (M, Σ_-, Σ) and (N, Σ, Σ_+). Let A_-, A, A_+ the respective Lagrangians in $H_{2k-1}(\Sigma_-)$, $H_{2k-1}(\Sigma)$ and $H_{2k-1}(\Sigma_+)$. Consider the standard Lagrangians $\Lambda_M = \mathrm{Ker}(i_* : H_{2k-1}(\partial M) \to H_{2k-1}(M))$ and $\Lambda_N = \mathrm{Ker}(i_* : H_{2k-1}(\partial N) \to H_{2k-1}(N))$ respectively. The subspaces

$$\lambda_- = (\Lambda_M)_* A_- = \{y \in H_{2k-1}(\Sigma) \mid (x, y) \in \Lambda_M \text{ for some } x \in A_-\},$$
$$\lambda_+ = (\Lambda_N)^* A_+ = \{y \in H_{2k-1}(\Sigma) \mid (y, z) \in \Lambda_N \text{ for some } z \in A_+\}$$

are Lagrangians in $H_{2k-1}(\Sigma)$. As in the case of 3-cobordisms, one finds that

$$\tau(N \circ M) = \gamma(G, q)^{-\mu(\lambda_-, A, \lambda_+)} \tau(N) \circ \tau(M),$$

where $\mu(\lambda_-, A, \lambda_+) \in \mathbb{Z}$ denotes the Leray-Maslov index (see for [**64**] and [**61**, IV,4]) of the three Lagrangians λ_-, A, λ_+ in $H_{2k-1}(\Sigma)$. $\square$

REMARK 5.1. A careful reader may notice that not only the cobordism invariant map $\tau(M) : \mathcal{T}(\Sigma_-) \to \mathcal{T}(\Sigma_+)$ but the state modules $\mathcal{T}(\Sigma_\pm)$ themselves depend on the extra structure on $\Sigma_\pm$, namely the oriented framed links $L^\pm$. The key dependency is that of the cobordism invariant operator. If we think of the link $L \subset H_1(\Sigma)$ as playing the rôle of a *base* of a fixed Lagrangian, we can identify the module of states $\mathcal{T}(\Sigma)$ to a fixed vector space (thought of as a *color module*). Suppose first that Σ has genus g and let A be the Lagrangian generated in homology by the components of L. Consider the canonical isomorphism

$$\mathrm{can}_L : \mathbb{C}[G^g] \to \mathbb{C}[G \otimes A] = \mathcal{T}(\Sigma), \quad (x_1, \ldots, x_g) \mapsto x_1 \otimes L_1 + \cdots + x_g \otimes L_g.$$

Define the *color module* of Σ to be $T(\Sigma) = \mathbb{C}[G^g]$. Suppose next that Σ consists of several connected components $\Sigma_1, \ldots, \Sigma_r$. We define $T(\Sigma)$ to be

[1] The exact computation is not needed, for instance, if one is interested only in the projective representation of the mapping class groups; in this case, Wall's formula is not necessary.

the (non-ordered) tensor product of all the color modules of the components: $T(\Sigma) = \bigoplus_{j=1}^{r} T(\Sigma_j)$. The isomorphism can $T(\Sigma) \to \mathcal{T}(\Sigma)$ is defined to be the (non-ordered) tensor products of the isomorphisms corresponding to the components. Then we may define the cobordism invariant operator as a map

$$T(\Sigma_-) \to T(\Sigma_+),\ \tau'(M) = \mathrm{can}_{L^+}^{-1} \circ \tau(M) \circ \mathrm{can}_{L^-}.$$

In particular, a cylinder (with extra structures at its bases) on a surface Σ of genus g gives rise to an operator $T(\Sigma) \to T(\Sigma)$. This is especially relevant in the next section when we derive representations of the mapping class group of surfaces from this TQFT.

CHAPTER 9

The return of the Weil representation

According to the general theory, any TQFT in dimension 3 yields a projective representation of the mapping class group of surfaces. We shall outline the procedure in our setting and proceed to the explicit computation of the representation. Then we state in a more precise form the identification with the Weil representation.

1. The mapping class group and parametrized cylinders

Let Σ be an oriented connected compact surface of genus g without boundary. Let $\mathcal{M}(\Sigma)$ denote the *mapping class group* of Σ, that is the group that consists of isotopy classes of orientation preserving homeomorphisms of Σ. We begin with a tautological representation of $\mathcal{M}(\Sigma)$.

Definition 1.1. A *parametrized cylinder* C_φ over Σ is an oriented cylinder $\Sigma \times [0,1]$ equipped with a homeomorphism $\varphi : \Sigma \times 0 \to \Sigma \times 1$.

Remark 1.1. A parametrized cylinder C_φ over Σ is equivalently defined as the oriented cylinder $\Sigma \times [0,1]$ equipped with a homeomorphism $\Sigma \to \Sigma \times 0$, parameterizing the bottom base. Thus a parametrized cylinder is a particular parametrized cobordism $(\Sigma \times [0,1], \Sigma \times \{1\}, \Sigma \times \{0\})$ where the top base is parametrized by the identity and the bottom base is parametrized by a fixed homeomorphism.

Definition 1.2. An *equivalence* between parametrized cylinders is an orientation preserving homeomorphism $\Phi : C_\phi \to C_\psi$ such that

$$\Phi|_{\Sigma \times \{1\}} = \mathrm{id}_{\Sigma \times \{1\}}, \Phi|_{\Sigma \times \{0\}} \circ \phi(x) = (\psi(x), 0) \quad \text{for all } x \in \Sigma. \tag{1.1}$$

Remark 1.2. Two parametrized cylinders are equivalent if and only if they are equivalent as parametrized cobordisms.

Denote by $\mathrm{Cyl}(\Sigma)$ the set of parametrized cylinders up to equivalence.

Lemma 1.1. *The map $\varphi \mapsto C_\varphi$ from the group of homeomorphisms to the set of parametrized cylinders induces a map*

$$\mathfrak{M}(\Sigma) \to \mathrm{Cyl}(\Sigma),\ [\varphi] \mapsto [C_\varphi].$$

Proof. Let $(x,t) \mapsto \varphi_t(x)$ be an isotopy between two homeomorphisms φ_0 and φ_1 of Σ. We need to show that there is an equivalence between C_{φ_0} and C_{φ_1}. The map

$$\Phi : \Sigma \times [0,1] \to \Sigma \times [0,1],\ (x,t) \mapsto (\varphi_t \varphi_0^{-1}, t)$$

is clearly a level-preserving homeomorphism and commutes with parametrizations. □

LEMMA 1.2. *The set* $\mathrm{Cyl}(\Sigma)$ *is equipped with a product defined as follows: for* $[C_\varphi], [C_\psi] \in \mathrm{Cyl}(\Sigma)$*, let*

$$C_\varphi \circ C_\psi = \big(C_\varphi \coprod C_\psi\big)/\sim,$$

where for any $x \in \Sigma$, $C_\varphi \ni (\varphi(x), 0) \sim (x, 1) \in C_\psi$. *In other words,* $C_\varphi \cdot C_\psi$ *is the cylinder* $\Sigma \times [0,1]$ *obtained by identifying the top base of a cylinder with the bottom base of a cylinder via* φ *and compressing the result in the* t*-coordinate. Define a product in* $\mathrm{Cyl}(\Sigma)$ *by*

$$[C_\varphi] \circ [C_\psi] = [C_\varphi \circ C_\psi].$$

In other words, the top base of C_ψ *is identified via* φ *with the bottom base of* C_φ. *With this product,* $\mathrm{Cyl}(\Sigma)$ *becomes a group so that the natural map*

$$\mathfrak{M}(\Sigma) \to \mathrm{Cyl}(\Sigma)$$

is a group isomorphism.

PROOF. The product at the level of $\mathrm{Cyl}(\Sigma)$ is the composition of (equivalence classes of) parametrized cobordisms. Thus the map $[\varphi] \mapsto [C_\varphi]$ is multiplicative. There is obviously a map from the set of parametrized cylinders over Σ to the mapping class group $\mathfrak{M}(\Sigma)$ that sends C_φ to $[\varphi]$. We claim that this map induces a map $\mathrm{Cyl}(\Sigma) \to \mathfrak{M}(\Sigma)$. Let $\Phi : C_\phi \to C_\psi$ be an equivalence between two parametrized cylinders. Let $\varphi_t(x) = \Phi|_{\Sigma\times\{t\}}$ for all $x \in \Sigma$. Then $\varphi_1(x) = \mathrm{id}_\Sigma$ and $\varphi_0(x) \circ \varphi = \psi$. Therefore $f_t = \varphi_t \circ \varphi$ defines an isotopy between ψ $(t = 0)$ and φ $(t = 1)$. Hence $[\varphi] = [\psi]$. So we have just defined an inverse map $\mathrm{Cyl}(\Sigma) \to \mathfrak{M}(\Sigma), [C_\varphi] \mapsto [\varphi]$. $\mathrm{Cyl}(\Sigma)$ inherits its group structure from that of $\mathfrak{M}(\Sigma)$. □

REMARK 1.3. This implies that any obvious generalization of $\mathrm{Cyl}(\Sigma)$, for instance *homology cylinders*, will contain the mapping class groups.

DEFINITION 1.3. A *geometric symplectic basis* for $H_1(\Sigma)$ is a system of $2g$ oriented simple closed curves $(m_1, l_1, \ldots, m_g, l_g)$ on Σ such that

(1) the complement $\Sigma - (m_1 \cup \ldots \cup m_g)$ is connected;
(2) the system $([m_1], [l_1], \ldots, [m_g], [l_g])$ of their 1-homology classes is a symplectic basis for the intersection pairing $\bullet : H_1(\Sigma)\times H_1(\Sigma) \to \mathbb{Z}$.

LEMMA 1.3. *For any system* $(m_1, \ldots, m_r)$ *of oriented simple closed curves whose complement is connected, there exists a geometric symplectic basis extending it.*

PROOF. See [**17**, §1.3]. □

We call the curves $m_1, \ldots, m_g$ (resp. $l_1, \ldots, l_g$) *meridians* (resp. *longitudes*).

REMARK 1.4. We could have as well defined meridians and longitudes as images by some parametrization of meridians and longitudes of a *standard* surface (Cf. §1).

The surface Σ shall always be equipped with a geometric symplectic basis. See Fig. 1.1.

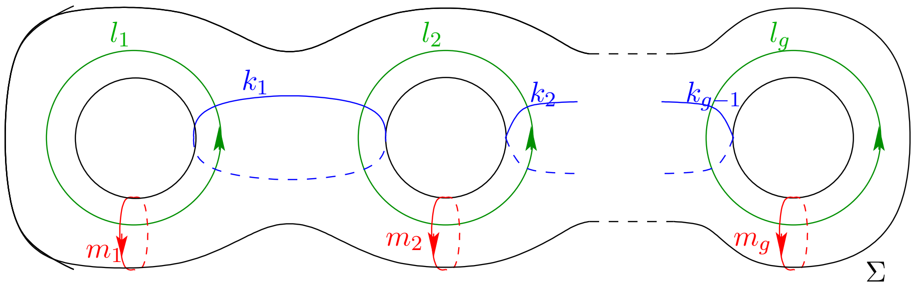


FIGURE 1.1. The standard oriented surface Σ of genus g equipped with $2g$ oriented simple closed curves and $g-1$ unoriented simple closed curves. The geometric symplectic basis of $H_1(\Sigma)$ is $(m_1, l_1, \ldots, m_g, l_g)$.

2. Lagrangian cylinders and Seifert pairings

DEFINITION 2.1. A *Lagrangian cylinder* C_Λ over an oriented compact surface Σ is an oriented cylinder $C = \Sigma \times [0,1]$ equipped with Lagrangians $A^- \subseteq H_1(\Sigma_-)$, $A^+ \subseteq H_1(\Sigma_+)$ and $\Lambda \subseteq H_1(\partial C)$ such that $\Lambda \oplus A^- \oplus A^+ = H_1(\partial C)$.

A Lagrangian cylinder is a special case of a Lagrangian decorated cobordism (and hence, a Lagrangian cobordism). As a consequence, a parametrized cylinder induces a Lagrangian cylinder (cf. Prop. 2.1) as we recall now. Let C_φ be a parametrized cylinder. As a parametrized cobordism, C_φ comes equipped with two sets of oriented links L^- and L^+ respectively: L^- consists of the image of the longitudes $l_1^-, \ldots, l_g^-$ in Σ_- by $\varphi : \Sigma_- \to \Sigma_-$ while L^+ consists of the longitudes $l_1^+, \ldots, l_g^+$ in Σ_+. The links L^- and L^+ are *framed* by the framing given by a small positive collar of L^- and L^+ in Σ_- and Σ_+. The links L^- and L^+ generate Lagrangians A_- and A_+ in $H_1(\Sigma_-) = H_1(\Sigma)$ and in $H_1(\Sigma_+)$ respectively. The Lagrangian $\Lambda = \Lambda_\varphi$ is generated by the image by φ of the meridians in Σ_- and by the meridians in Σ_+.

The observation of this paragraph is that a Lagrangian cylinder over a surface gives rise to a Seifert pairing on the homology of the surface.

Let C_Λ be a Lagrangian cylinder over an oriented compact surface Σ. Let $\Sigma_+ = \Sigma \times \{1\}$ and $\Sigma_- = \Sigma \times \{0\}$. We regard Σ as embedded in $\Sigma \times [0,1]$ via the inclusion $\Sigma \to \Sigma \times \{1/2\} \subset \Sigma \times [0,1]$. The cylinder structure provides Σ with a natural bicollar. For a 1-cycle $x \in \Sigma = \Sigma \times \{1/2\}$, we denote by x^+ the 1-cycle corresponding to $x \times \{1\}$ and by x^- the 1-cycle corresponding to $x \times \{0\}$. Denote as usual the inclusion homomorphisms by $i^\pm : H_1(\Sigma_\pm) \to H_1(\Sigma \times [0,1])$. We keep the same notation for the inclusion homomorphisms followed by the projection map $H_1(\Sigma \times [0,1]) \to G_\Lambda = H_1(\Sigma \times [0,1])/i_*\Lambda$. Let x, y be two 1-cycles in Σ. We define a map $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ by

$$\beta([x],[y]) = \mathrm{lk}_\Lambda(i_*^- x^-, i_*^+ y^+).$$

LEMMA 2.1. *The pairing $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ is a Seifert pairing with respect to the intersection pairing on Σ.*

EXAMPLE 2.1 (The Hopf Seifert pairing). Let (M, Σ_-, Σ_+) be the trivial cylinder over the 2-torus $\Sigma = -\Sigma_- = \Sigma_+ = S^1 \times S^1$. Recall that $\partial M = -\Sigma_- \cup \Sigma_+$.

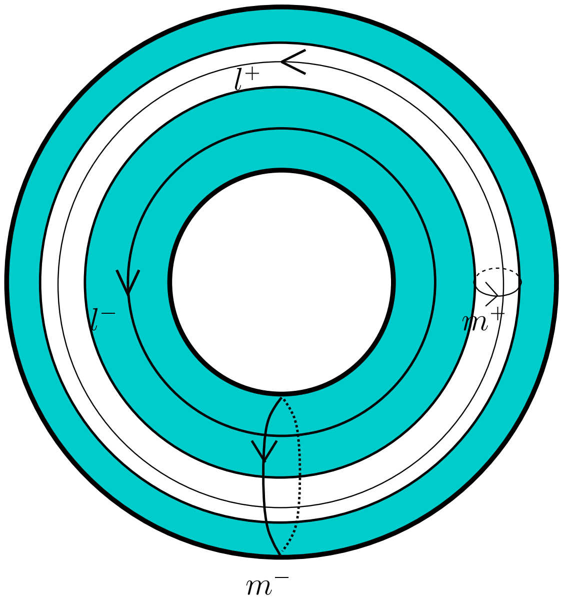


FIGURE 2.1. The cylinder over the torus equipped with a geometric symplectic basis on the components Σ_- and Σ_+.

Choose two systems (m^-, l^-) and (m^+, l^+) of oriented closed curves forming geometric symplectic basis for Σ_- and Σ_+ respectively. In particular,

$$m^- \bullet_{\Sigma_-} l^- = +1 = -m^- \bullet_{-\Sigma_-} l^- = -m^- \bullet_{\partial M} l^-.$$

Let Λ be the Lagrangian in $H_1(\partial M)$ generated by $[m^+]$ and $[l^-]$. Then

$$\mathrm{lk}_\Lambda(i_*^- m^-, i_*^+ l^+) = m^+ \bullet_{\Sigma_+} l^+ = +1.$$

Glue two copies of a standard solid torus $H = S^1 \times D^2$ of genus g to $\Sigma \times [0,1]$ as follows. We glue the first solid torus via a homeomorphism $-\partial H \to \Sigma \times \{1\}$ sending the meridian of ∂H to m^+. We glue the second solid torus via a homeomorphism $\partial H \to \Sigma \times \{0\}$ sending the meridian $\star \times \partial D^2$ of ∂H to l^-. The resulting closed 3-manifold is S^3. We call this gluing the *Hopf gluing*. By Lemma 3.3, for any disjoint 1-cycles x, y in $\Sigma \times [0,1]$,

$$\mathrm{lk}_\Lambda(x, y) = \mathrm{lk}_{S^3}(i_* x, i_* y),$$

In particular, $\beta(m, l) = \mathrm{lk}_\Lambda(i_*^- m^-, i_*^+ l^+)$ is the linking number of a positive Hopf link in S^3. For this reason we call the Lagrangian Λ above the *Hopf Lagrangian* and the corresponding Seifert form the *Hopf Seifert form*.

These considerations extend obviously to the case of an oriented closed connected surface Σ of arbitrary genus.

DEFINITION 2.2. Let $\Sigma \times [0,1]$ be the cylinder over the standard surface of genus $g \leqslant 1$. Let H_g denote the standard oriented handlebody of genus g. Let $f_+ : \Sigma_{g+} \to \Sigma \times \{1\}$ be the identity and let $f_- : \Sigma_{g-} \to \Sigma \times \{0\}$ be an orientation preserving homeomorphism sending the j-th meridian of $\Sigma_{g-} = \partial H_g$ onto the j-th longitude of $\Sigma \times \{0\}$. The *Hopf gluing* is defined as the gluing that consists in gluing two handlebodies to the cylinder $\Sigma \times [0,1]$ via the parametrization f_- and f_+ on the bases:

$$S^3 = H_g \cup_{f_-} \Sigma \times [0,1] \cup_{f_+ \circ \mathrm{mir}} -H_g.$$

The Lagrangian $\Lambda \subseteq H_1(\partial(\Sigma \times [0,1]))$ generated by the meridians of $\Sigma \times \{1\}$ and the longitudes of $\Sigma \times \{0\}$ is called the *Hopf Lagrangian*. The *Hopf Seifert form* is defined similarly.

3. The modular representation

According to the TQFT τ defined in the previous chapter, the cylinder $M(h)$ gives rise to a cobordism map $\tau(h) : \mathcal{T}(\Sigma_-) \to \mathcal{T}(\Sigma_+)$. Note that $\tau(h) \circ \tau(h^{-1})$ is a multiple of $\tau(\mathrm{id}_\Sigma)$, hence a multiple of the identity, it follows that $\tau(h)$ is invertible.

The isomorphism $\tau(h)$ depends on the quadratic form q: when we need to emphasize this, we write $\tau_q(h)$. It follows from Lemma 1.2 that the cobordism $M(h)$ depends only on the mapping class $[h] \in \mathcal{M}(\Sigma)$.

Any oriented closed surface can be endowed with a geometric symplectic basis. Since cylinders form a very particular class of cobordisms, we can be more specific about our choices here. We choose two *geometric* symplectic bases $(m_1^-, l_1^-, \ldots, m_g^-, l_g^-)$ and $(m_1^+, l_1^+, \ldots, m_g^+, l_g^+)$ for $H_1(\Sigma_-)$ and $H_1(\Sigma_+)$ respectively, as follows. Since any cobordism $M(h)$ is a cylinder over Σ, the natural inclusion map

$$i^\pm : \pm\Sigma_\pm \to \Sigma \times \{\tfrac{1}{2}\} \subset \Sigma \times [0,1] = M(h)$$

is a positive embedding which induces an isomorphism

$$i_*^\pm : H_1(\pm\Sigma_\pm) \xrightarrow{\simeq} H_1(M(h)).$$

We require the geometric symplectic bases to verify

$$i_*^-(l_j^-) = i_*^+(l_j^+),\ i_*^-(m_j^-) = i_*^+(m_j^+).$$

In particular, let

$$m_1 = i_*^+([m_1^+]), \ldots, m_g = i_*^+(m_g^+), l_1 = i_*^+(l_1^+), \ldots, l_g = i_*^+(l_g^+).$$

The set $[m_1], [l_1], \ldots, [m_g], [l_g]$ is a $\mathbb{Z}$-basis for $H_1(M(h))$.

DEFINITION 3.1. The Lagrangian generated in $H_1(M(h))$ by $l_1, \ldots, l_g$ is the *longitudinal* Lagrangian and is denoted Υ. The *state module* $T(\Sigma)$ associated to a standard oriented closed surface Σ is the group algebra $\mathbb{C}[G \otimes \Upsilon]$. More generally, the state module associated to an oriented closed surface endowed with a Lagrangian $A \subseteq H_1(\Sigma)$ is $T(\Sigma) = \mathbb{C}[G \otimes A]$.

For a standard surface, the state module $T(\Sigma)$ plays the rôle of a reference state module. If $f : \Sigma_g \to \Sigma$ is a parametrization of Σ sending Υ to Λ...

By definition $\tau(h)$ depends on the choice of geometrical Lagrangians A^- and A^+ in Σ_- and Σ_+. In the particular setting of parametrized cylinders, A^- (resp. A^+) is the lattice generated by the oriented framed link $h(l_1^-), \ldots, h(l_g^-)$ (resp. by $l_1^+, \ldots, l_g^+$). Note that A^- and A^+ both identify to Υ in $H_1(M(h))$ via the maps i_*^- and i_*^+ respectively. With this identification in mind, we can set $T(\Sigma_-) = T(\Sigma_+) = \mathbb{C}[G \otimes \Upsilon]$.

The following result is a consequence of Theorem 5.1.

Lemma 3.1 (Modular representation). *The map*

$$\mathcal{M}(\Sigma) \to \operatorname{Aut}(T(\Sigma)),\ [h] \mapsto \tau_q(h)$$

defines a projective representation of $\mathcal{M}(\Sigma)$.

The fact that the representation is projective and not just linear is a consequence of the non trivial anomaly in the TQFT.

Let us describe more explicitly the modular representation τ_q above in terms of the formulas (5.1) and (5.2) for the TQFT given in the previous section.

By construction, $i_*(\Lambda)$ is the subgroup generated by $i_*^- h_*([m_1^-]), \ldots, i_*^- h_*([m_g^-])$ and $i_*^+([m_1^+]), \ldots, i_*^+([m_g^+])$. Writing in the basis of $H_1(M)$

$$i_*^- h_*([m_j^-]) = \sum_k \left(a_{jk}\ [l_k] + b_{jk}\ [m_k]\right),\ k = 1, \ldots, g,$$

we find that $G_\Lambda = H_1(M(h))/i_*\Lambda$ is the abelian group generated by $[l_1], \ldots, [l_g]$ with relations $\sum_k a_{jk}\ [l_k]$. In particular, G_Λ has rank at most g.

Therefore, $T_\Lambda M(h) = \text{Tors Coker}(a_{jk})_{1 \leqslant j,k \leqslant g}$ depends only on the matrix $(a_{jk})_{1 \leqslant j,k \leqslant g}$. According to the previous section, this matrix has a simple interpretation:

$$a_{jk} = \operatorname{lk}_\Lambda\left(i_*^- h_*(m_j^-), i_*^+(l_k^+)\right) \in \mathbb{Z}.$$

On the other hand, linking numbers inside $M(h)$ between images of longitudes are rational in general.

The pair $(G_\Lambda, \lambda_\Lambda)$ is the discriminant linking group associated to the matrix $(a_{jk})_{1 \leqslant j,k \leqslant g}$. The characteristic element $\theta \in G \otimes G_\Lambda$ is the characteristic element associated to λ_Λ and λ_q. Any choice $x = (x_1, \ldots, x_g) \in G^g$ of colors determines an oriented framed 1-cycle $\sum_j x_j \otimes (i^- \circ h)(l_j^-)$ in $M(h)$ with coefficients in G. Similarly any choice $y = (y_1, \ldots, y_g) \in G^g$ of colors determines an oriented framed 1-cycle $\sum_j y_j \otimes i^+(l_j^+)$ in $M(f)$ with coefficients in G. Given $x = (x_1, \ldots, x_g) \in G^g$, we set

$$\mathcal{H}(x) = \Big\{y = (y_1, \ldots, y_g) \in G^g \mid \underbrace{i_*^+\Big(\sum_j y_j \otimes [l_j^+]\Big) - i_*^-\Big(\sum_j x_j \otimes h_*([l^-])\Big)}_{[\kappa_{xy}]} = \theta\Big\}.$$

Choose a lattice presentation (V, f, v) for (G, q) and a quadratic enhancement (G, q_Λ) of (G, λ_Λ). Then we may lift θ to an element $\widetilde{\theta} \in V^\sharp \otimes T_\Lambda M(h)$ as before. More generally, we may lift 1-cycles with coefficients in G to

1-cycles with coefficients in $V^\sharp$. In particular, for any choices of colors x, y, we obtain a 1-cycle $\widetilde{\kappa}_{xy}$ with coefficients in $V^\sharp$.

We define a normalization coefficient $C(h)$ using (5.1):

$$C(h) = C_\Lambda(M(h)). \tag{3.1}$$

Given $x \in G^g$ and $y \in \mathcal{H}(x)$, we define the *phase weight* associated to y by

$$\Omega(x,y) = \exp\left(2\pi i\left(\left(f \otimes q^{\mathrm{fr}}\right)(\widetilde{\kappa}_{xy})\right) - \delta_v(\widetilde{\kappa}_v)\right).$$

Let $e_x,\ x \in G^g$ be the standard basis of $\mathbb{C}[G^g]$. Then

$$\tau(f)e_x = C(h) \cdot \sum_{y \in \mathcal{H}(x)} \Omega(x,y)\ e_y. \tag{3.2}$$

As noted above, it is a consequence of Th. 5.1 that the map

$$\mathcal{M}(\Sigma) \to \mathrm{Aut}(\mathbb{C}[G^g]),\ [h] \mapsto \tau(h) \tag{3.3}$$

is a projective representation of $\mathcal{M}(\Sigma)$.

PROPOSITION 3.1. *The representation $[h] \mapsto \tau(h)$ is unitary and factors through the symplectic linear representation*

$$\mathcal{M}(\Sigma_g) \to \mathrm{Sp}(H_1(\Sigma)),\ [h] \mapsto h_*$$

induced by homology.

PROOF. We first show that $\tau(h)$ only depends on $h_* \in \mathrm{Sp}(H_1(\Sigma))$. □

(Since $\tau(M) = 1$ if M is an integral homology 3-sphere, the representation $[h] \mapsto \tau(h)$ factors through the symplectic representation.)

The following theorem is our main goal. It asserts that the Abelian TQFT representation based on a finite quadratic form q is essentially the Weil representation associated to q. Together with our description of the Abelian TQFT representation (3.2), it provides a new description of the Weil representation.

We first define the Weil representation in the appropriate setting. The group G is endowed with its quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$, which turns it into a quadratic group and in particular, into a linking group. The homology group $H_1(\Sigma)$ is endowed with its intersection form $\bullet$, which turns it into a symplectic group. In particular, the group $G \otimes H_1(\Sigma)$ becomes a symplectic group with the symplectic form $b_q \otimes \bullet$.

We endow the standard surface Σ with the Hopf Lagrangian Λ and the corresponding Hopf Seifert pairing $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ (see §2), defined by

$$\beta([x],[y]) = \mathrm{lk}_\Lambda(i_*^- x^-, i_*^+ y^+) \tag{3.4}$$

where as usual $i_*^{\pm} : H_1(\Sigma_\pm) \to H_1(\Sigma \times [0,1])/i_*\Lambda$ denotes the inclusion homomorphisms. This Seifert pairing β induces a Seifert pairing

$$b_q \otimes \beta : H_1(\Sigma; G) \times H_1(\Sigma; G) \to \mathbb{Q}/\mathbb{Z}$$

for the symplectic form $b_q \otimes \bullet$ which we still call the *Hopf Seifert pairing*.

One needs to select a Lagrangian in $G \otimes H_1(\Sigma) = H_1(\Sigma; G)$. We choose the Lagrangian L_0 in $H_1(\Sigma)$, which induces the Lagrangian $G \otimes L_0$ in $H_1(\Sigma; G)$.

Define a character $\chi : H_1(\Sigma; G) \to \mathbb{C}^\times$ by $\chi = \exp\Big(2\pi i \big(b_q \otimes \beta\big)\Big)$.

Consider the Weil representation

$$\rho : \mathrm{Sp}(H_1(\Sigma)) \to \mathrm{U}(L^2 G \otimes L_0) = \mathrm{U}(\mathbb{C}[G \otimes L_0])$$

associated to Lagrangian L_0 and character χ (see §...).

Theorem 3.1 ("**Weil = TQFT**"). *The* TQFT *representation* $[h] \mapsto \tau(h)$ *factors through the Weil representation* $\rho : \mathrm{Sp}(H_1(\Sigma)) \to \mathrm{U}(\mathbb{C}[G \otimes L_0])$. *In other words, the diagram*

$$\begin{tikzcd}
\mathcal{M}(\Sigma) \arrow[r, "\tau"] \arrow[d] & \mathrm{U}(\mathbb{C}[G^g]) \\
\mathrm{Sp}(H_1(\Sigma)) \arrow[r] \arrow[ur, dotted, "\rho"] & \mathrm{ASp}(H_1(\Sigma; G)) \arrow[u]
\end{tikzcd}$$

is commutative.

4. Abelian skein theory

In this section we give a topological ("skein") interpretation of the Heisenberg algebra and the Schrödinger representation of the Heisengerg algebra. These results are interesting by themselves because they provide state modules with extra structures. They give a completely skein-theoretic approach to the construction of Abelian TQFTs and they are building blocks used in the proof of Theorem 3.1. For this, we develop an appropriate calculus, called skein calculus, pioneered by J. H. Przytycki and V.G. Turaev.

4.1. Heisenberg skein modules. We fix as before a quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group. Denote by $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ the associated linking pairing. Let

$$U = \{t_g, t_g^{-1} \mid g \in G\} \text{ and } V = \{t_{g,h}, t_{g,h}^{-1} \mid \{g, h\} \subset G\}$$

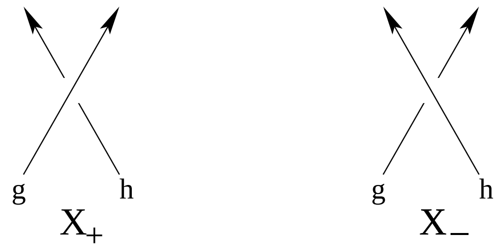
be two independent sets of free and commuting variables. The set U has cardinality $2|G|$. The set V is indexed by all nonempty subsets of $|G|$ of cardinality less or equal to 2. Let $V^* = V - \{t_{gg}, t_{gg}^{-1} \mid g \in G\}$ the subset of V corresponding to indices by all unordered pairs of distinct elements in G. We have $|V^*| = |G| \cdot (|G| - 1)$. Finally we set $S_G = U \cup V^*$. We shall drop the subscript G if the group G is understood from the context.

Let M be an oriented compact 3-manifold. A *colored* link in M is a link L together with a map

$$\mathrm{col} : \pi_0(L) \to G.$$

If $\pi_0(L) = \{L_1, \ldots, L_n\}$, we usually denote $\mathrm{col}(L_j) = g_j$, $1 \leqslant j \leqslant n$. A framed link is conveniently thought of as an annulus. We use the blackboard convention for the framing, meaning that the annulus determining the framing lies in the plane of the figure.

Consider the free module $\mathscr{L}(M)$ over $\mathbb{C}[U, V]$ generated by the set of all isotopy classes of oriented framed colored links in M, including the empty link denoted $\varnothing$. Figure 4.1 below represents two oriented framed colored links $X_+(g, h)$ and $X_-(g, h)$ which are identical except in a small embedded ball in M where they look exactly as shown, where one arc is part of a component colored by an element $g \in G$ and the other arc is part of a component colored by $h \in G$.

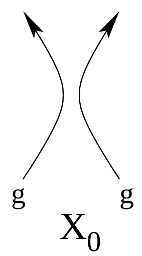


Consider now the elements

$$X_+(g, h) - t_{gh} X_-(g, h),\ X_-(g, h) - t_{gh}^{-1} X_+(g, h) \tag{4.1}$$

where $X_+(g, h)$ and $X_-(g, h)$ are two oriented framed colored links which are identical except in a small embedded ball D in M where they look exactly as shown in the figure, where one arc is part of a component colored by an element $g \in G$ and the other arc is part of a component colored by $h \in G$. The possibility $g = h$ is accepted, whether the arcs belong to distinct components or not.

The next figure (Fig. 4.1 below) represents an arbitrary oriented framed link $X_0(g, g)$ in M, where the interior of a small embedded ball D^3 is specified as shown.



We consider also the elements

$$X_+(g, g) - t_g X_0(g, g),\ X_-(g, g) - t_g^{-1} X_0(g, g) \tag{4.2}$$

where $X_+(g, g)$ and $X_0(g, g)$ are two oriented framed colored links which are identical except in a small embedded ball in M where they look exactly as shown in the figure. (Note that the number of components of $X_\pm(g, g)$ is the number of components of $X_0(g, g)$ plus or minus one.)

Let X be an arbitrary element in $\mathscr{L}(M)$. Denote by XO the element in $\mathscr{L}(M)$ that consists of X and an extra annulus that bounds in M a disc disjoint from X (the trivial framed unknot). Consider finally all the elements

$$X - XO. \tag{4.3}$$

Let $\mathscr{S}(M)$ be the submodule spanned by all elements of the three kinds enumerated above, respectively by (4.1), (4.2) and (4.3). The corresponding relations are called *skein relations*. It is easy to see that the skein relations preserve the last two Reidemeister moves, hence preserve the isotopy class of the framed link.

Definition 4.1. The *Heisenberg skein module* $\mathscr{A}(M)$ is the quotient

$$\mathscr{L}(M)/\mathscr{S}(M)$$

Elements of $\mathscr{A}(M)$ are called *skeins*. The skein represented by an oriented framed link L will be denoted by $\langle L\rangle$. If we need to emphasize the color, we include it in the notation. For instance, if K is an oriented framed knot, then $\langle K, g\rangle$ denotes the skein represented by K colored with $g \in G$. Similarly, if we need to emphasize that the Heisenberg skein module depends on G, we denote it by $\mathscr{A}_G(M)$.

Lemma 4.1. *The following relations hold in $\mathscr{A}(M)$:*

(1) *The commutativity relations: $t_g t_h = t_h t_g$, $t_{gh} = t_{hg}$ and $t_g t_{hk} = t_{hk} t_g$ for all $g, h, k \in G$.*
(2) *The doubling relations: $t_g^2 = t_{gg}$ for all $g \in G$.*

Proof. (1) The order in which the skein relations are processed is irrelevant. (2) The following relations hold in $\mathscr{A}(M)$:

$$X_+(g,g) = t_g X_0(g), \quad X_+(g,g) = t_{gg} X_-(g,g) = t_{gg} t_g^{-1} X_0(g,g).$$

Therefore

$$t_g X_0(g,g) = t_{gg} t_g^{-1} X_0(g,g)$$

for any skein. The result follows. $\square$

Example 4.1. Consider the 3-manifold $M = S^3$ (or an integral homology 3-sphere). Let $L = L_1 \cup \cdots \cup L_n$ be an oriented framed link colored with $g_1, \ldots, g_n$. Then

$$\langle L\rangle = \prod_{1\leqslant i<j\leqslant n} t_{g_i,g_j}^{\mathrm{lk}(L_i,L_j)} \cdot \prod_{1\leqslant i\leqslant n} t_{g_i}^{\mathrm{lk}(L_i,L_i')} \langle\varnothing\rangle$$

where L_i' denotes the component parallel to L_i determined by the framing. It follows that $\mathscr{A}(S^3) \simeq \mathbb{C}[U, V^*]$.

Example 4.2. Consider the case when G is the trivial group. We can set $t_0 = t$ and $t_{00} = u$. According to the lemma, $t^2 = u$. The relations of the first kind are $X_+ = tX_-$, $X_- = t^{-1}X_+$. The relations of the second kind are $X_+ = uX_0$, $X_- = u^{-1}X_0$. The Heisenberg module $\mathscr{A}(M)$ is a $\mathbb{C}[t, t^{-1}]$-module.

EXAMPLE 4.3. Consider the oriented framed and colored two component link L in the solid torus (oriented handlebody of genus one) $S^1 \times D^2$ as pictured in Fig. 4.1. Denote as usual by $m = \star \times \partial D^2$ a meridian of $S^1 \times D^2$ and by $l = S^1 \times \star$ a longitude of $S^1 \times D^2$. Color the meridian by an element $g \in G$ and the longitude by an element $h \in G$.

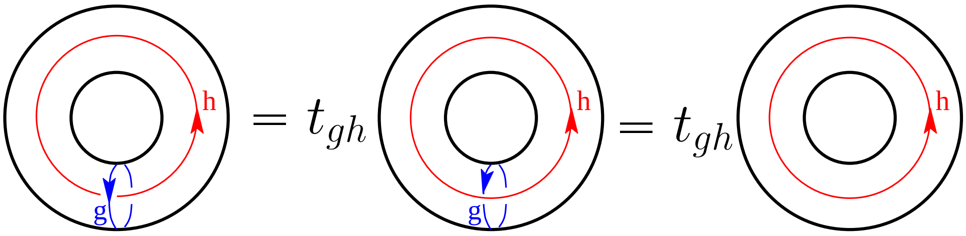


FIGURE 4.1. A skein relation in $S^1 \times D^2$.

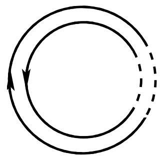
Fig. 4.1 shows that in the Heisenberg skein module $\mathscr{A}(S^1 \times D^2)$, the following relation holds:

$$\langle L \rangle = t_{gh} \langle l; h \rangle.$$

Note that both a relation of the first type (4.1) and a relation of the third type (4.3) were used.

EXERCISE 4.1. In general, nontrivial identities are produced on a link diagram by a combination of Reidemeister moves and skein moves.

Consider the skein $\langle L \rangle$ represented by two parallel framed knots with opposite orientations in an arbitrary 3-manifold M and colored with the same color as represented here opposite. Prove that $\langle L \rangle = \langle \varnothing \rangle$.



In the sequel we adopt the normalization $\langle \varnothing \rangle = 1$.

Consider the 3-manifold $M = \Sigma \times [0, 1]$ where Σ is a closed oriented surface. Provide M with the product orientation as usual. The product of two elements $L, L' \in \mathscr{L}(\Sigma \times [0, 1])$ is defined by uniformly compressing L in $\Sigma \times [0, 1/2]$, respectively L' in $\Sigma \times [1/2, 1]$, and juxtaposing in $\Sigma \times [0, 1] = \Sigma \times ([0, 1/2] \cup [1/2, 1])$. The result $L \cdot L'$ is clearly an oriented framed colored link in $\Sigma \times [0, 1]$.

DEFINITION 4.2. The product

$$(L, L') \mapsto L \cdot L' \tag{4.4}$$

induces a product on $\mathscr{A}(\Sigma \times [0, 1])$, called the *skein product*.

The skein product turns $\mathscr{A}(\Sigma \times [0, 1])$ into an associative algebra with the empty link being the unit. The n-th power of a skein $\langle L \rangle$ is easily seen to be represented by n parallel copies of L. By Exercise 4.1 and our normalization, we have $\langle L \rangle^{-n} = \langle -L \rangle$ where $-L$ denotes L with the reversed orientation. By definition, $\langle L \rangle^0 = \langle \varnothing \rangle = 1$.

EXAMPLE 4.4. If K is an oriented framed knot in $\Sigma \times [0,1]$ then $\langle K; g\rangle \cdot \langle K; h\rangle = \langle K; h\rangle \cdot \langle K; g\rangle$ for any colors $g, h \in G$.

EXAMPLE 4.5. Consider the skein $\langle L\rangle$ in the cylinder $T \times [0,1]$ over the torus T^2 represented by the two component oriented framed colored link L as pictured in Fig. 4.2. Denote as usual by m the meridian (here colored by

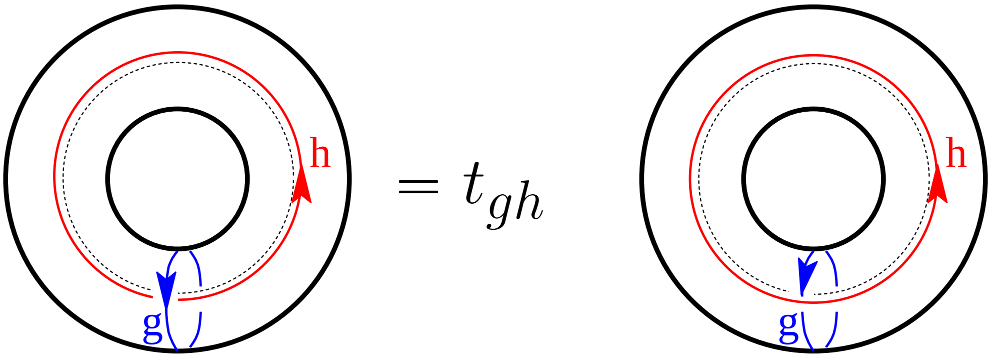


FIGURE 4.2. A skein relation in $T \times [0,1]$.

$g \in G$) and by l (here colored by h) the longitude of $T = \partial S^1 \times D^2$. Then Fig. 4.2 shows that the following relations hold in the Heisenberg skein algebra $\mathscr{A}(T \times [0,1])$:

$$\langle L\rangle = \langle m; g\rangle \cdot \langle l; h\rangle = t_{gh}\, \langle l; h\rangle \cdot \langle m; g\rangle.$$

This example shows that $\mathscr{A}(T \times [0,1])$ is not commutative.

EXAMPLE 4.6. Consider the skein $\langle L\rangle$ in the cylinder $T \times [0,1]$ over the torus T represented by the two component oriented framed colored link L as pictured in the left side of Fig. 4.3. This is the same oriented framed link as in the previous example except that the two components are colored by the same element $g \in G$. An application of the second skein relation is pictured in Fig. 4.3. Observe that the integral homology of the link in

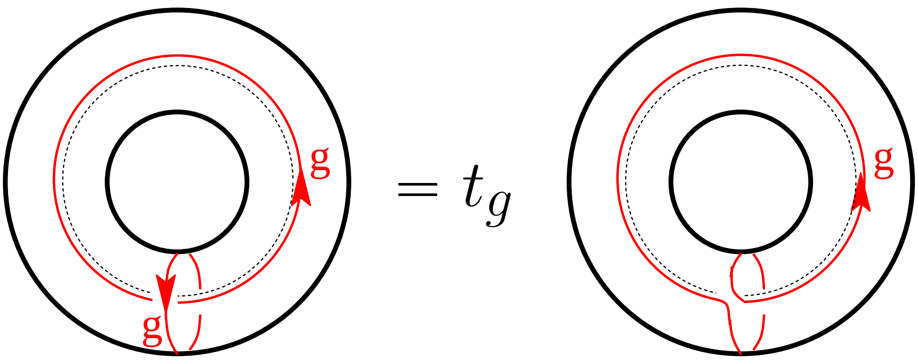


FIGURE 4.3. Another skein relation in $T \times [0,1]$.

$T \times [0,1]$ is unchanged after the skein relation.

4.2. A skein multivariable polynomial. We now proceed to generalize Examples 4.3 and 4.5. Let Σ be a compact oriented connected surface. Embed $\Sigma \times [0,1]$ into S^3 by the Hopf gluing (see Example 2.1). Denote by Λ the corresponding Hopf Lagrangian in $H_1(\partial\Sigma \times [0,1])$.

LEMMA 4.2. *Let L be a framed oriented colored link in $\Sigma \times [0,1]$. There is an oriented framed colored link $L_0 \subseteq \Sigma \times \{1/2\}$ and a Laurent polynomial $P_L \in \mathbb{Z}[S_G]$ such that*

$$\langle L\rangle = P_L \cdot \langle L_0\rangle \in \mathscr{A}(\Sigma \times [0,1]) \tag{4.5}$$

with

$$P_L = \prod_{\substack{\text{pairs} \\ \jmath,\ell\in\pi_0(L) \\ \jmath\neq\ell}} t_{\mathrm{col}(\jmath),\mathrm{col}(\ell)}^{\mathrm{lk}_\Lambda(\jmath,\ell)} \cdot \prod_{\ell\in\pi_0(L)} t_{\mathrm{col}(\ell)}^{\mathrm{lk}_\Lambda(\ell,\ell')} \tag{4.6}$$

with the following properties:

(1) *The oriented links L and L_0 represent the same homology class: $[L] = [L_0]$ in $H_1(\Sigma; G)$.*
(2) *The oriented framed link L_0 is Λ-trivial.*
(3) *The decomposition is canonical in the sense that the Laurent polynomial P_L and the* skein *$\langle L_0\rangle$ are unique.*

Here $\mathrm{lk}_\Lambda(\ell,\ell')$ denotes the Λ-linking number of ℓ and its distinguished parallel ℓ', i.e., the Λ-framing number of ℓ.

REMARK 4.1. The link L_0 itself (even up to isotopy) may not be unique: for instance, the relation of the third type does not affect properties (1), (2) and (3) (but changes the framed isotopy class).

PROOF. Using the skein relations and by induction on the number of crossings of an appropriate generic projection, one easily obtains a decomposition satisfying the properties (1), (2) and (3) with P_L as stated. The skein relations in $\Sigma\times[0,1]$ do not modify the framing nor the linking numbers inside S^3. Since P_L only depends on the framing and the linking numbers inside S^3, P_L is an invariant of the skein $\langle L\rangle$. This proves the uniqueness property. $\square$

COROLLARY 4.1. *The Laurent polynomial P_L associated to a link L is an invariant of the skein $\langle L\rangle$. In particular, it is an invariant of framed isotopy of L.*

DEFINITION 4.3. The Laurent polynomial P_L associated to the link L is called the *linking number skein polynomial* of L.

EXAMPLE 4.7. For the skein $\langle L\rangle$ of Example 4.5 (see Fig. 4.1), we have $P_L = t_{gh}$. One verifies that the link represented on the right hand side of Fig. 4.1 is Λ-trivial.

We record the behaviour of the linking number skein polynomial under the skein product.

LEMMA 4.3. *Let J and L be two oriented framed colored links in $\Sigma \times [0,1]$. Then*

$$P_{J \cdot L} = \prod_{(\jmath,\ell) \in \pi_0(J) \times \pi_0(L)} t_{\mathrm{col}(\jmath),\mathrm{col}(l)}^{\mathrm{lk}_\Lambda(\jmath,\ell)} \, P_J P_L.$$

PROOF. There are three Hopf Lagrangians: let Λ_0 (resp. Λ_1) be associated to the Hopf gluing applied to the first copy of $\Sigma \times [0,1]$ which contains J (resp. applied to the second copy of $\Sigma \times [0,1]$ which contains L). Finally let Λ be associated to the Hopf gluing applied to $\Sigma \times [0,1] \simeq \Sigma \times [0,1] \cup \Sigma \times [0,1]$. Abusing notations and denoting by the same letter a component in possibly three distinct manifolds, we have $\mathrm{lk}_{\Lambda_0}(\jmath,\ell) = \mathrm{lk}_\Lambda(\jmath,\ell)$ for any pair $\jmath,\ell$ of components of J. (if $\jmath = \ell$, the framing number $\mathrm{lk}_\Lambda(\ell,\ell')$ is meant.) Similarly, $\mathrm{lk}_{\Lambda_1}(\jmath,\ell) = \mathrm{lk}_\Lambda(\jmath,\ell)$ for any pair $\jmath,\ell$ of components of L. Since $\pi_0(JL)$ is the disjoint union of $\pi_0(J)$ and $\pi_0(L)$, we compute by means of Lemma 4.2, the skein polynomial P_{JL} and the product of skein polynomials $P_J P_L$. Comparing the two yields the desired formula. $\square$

4.3. Heisenberg skein modules and homology. We fix a standard surface of genus g with its geometric symplectic basis $(m_1, l_1, \ldots, m_g, l_g)$.

PROPOSITION 4.1. *The Heisenberg skein algebra $\mathscr{A}(\Sigma \times [0,1])$ is a free $\mathbb{C}[S_G]$-module whose basis consists of all elements of the form*

$$\prod_{j=1}^{g} \prod_{l=1}^{r_j} \langle m_j; x_{j,l} \rangle \cdot \prod_{j=1}^{g} \prod_{l=1}^{s_j} \langle l_j; y_{j,l} \rangle \tag{4.7}$$

where the indices $r_1, \ldots, r_g, s_1, \ldots, s_g$ lie in $\mathbb{Z}$ and the colors $x_{j,l}$ and $y_{j,l}$ lie in G.

The Heisenberg skein module $\mathscr{A}(H_g)$ of the genus g oriented handlebody is a free $\mathbb{C}[S_G]$-module with basis

$$\prod_{j=1}^{g} \prod_{l=1}^{s_j} \langle l_j; y_{j,l} \rangle,$$

where each longitude is colored with an arbitrary element of G.

PROOF. Let $\langle L \rangle$ be a skein in $\mathscr{A}(\Sigma \times [0,1])$. By Lemma 4.2, $\langle L \rangle = P_L \cdot \langle L_0 \rangle$ where L_0 is a Λ-trivial oriented framed link. It remains to show that $\langle L_0 \rangle$ is equivalent to a skein of the form stated in the Proposition. View Σ as the connected sum of g tori. Using isotopy and skein calculus (second relation), we can realize L_0 in such a way that every component lives in no more than one torus. So each component is a torus knot $T_{a,b}$, $a, b \in \mathbb{Z} \times \mathbb{Z}$ (see for instance [**54**, 2C]). Using skein calculus, we see by induction on (a,b) that $\langle T_{a,b} \rangle = \langle m \rangle^a \cdot \langle l \rangle^b$ (The case $a = b = 1$ is provided by Example 4.6). The first statement follows.

For the second statement, view the solid handlebody H as containing the cylinder over the closed oriented surface Σ so that ∂H identifies with one of the bases, say $\Sigma \times 0$, of the cylinder over Σ. Let $\langle L \rangle$ be a skein in $\mathscr{A}(H)$. By isotoping L is necessary, we may assume that L lies in $\Sigma \times [0,1] \subset H$. By the previous argument, $\langle L \rangle$ is proportional to an element of the form (4.7).

Now in $\mathscr{A}(H)$, each meridian is a trivial knot so $\langle m\rangle = \langle\varnothing\rangle$. This gives the desired result. $\square$

Remark 4.2. There is a slight abuse of notation in the second statement of Prop. 4.1. Indeed, $\mathscr{A}(H)$ has not been given yet any natural product structure. What the second statement really means is that any skein in H can be geometrically represented by a disjoint union of parallel longitudes (with the standard orientation) arbitrarily colored.

Let K be an oriented framed knot in M. The orientation and the framing of K determine an oriented knot K'. Extend the framing of K to a framing for K'. Let $g, h \in G$ be arbitrary colors for K and K' respectively. Consider the element

$$\langle (K, g) \cup (K', h)\rangle - \langle K, g + h\rangle \tag{4.8}$$

in $\mathscr{A}(M)$. For any arbitrary oriented framed link $L \in \mathscr{S}(M)$, consider the elements

$$t_g\langle L\rangle - \exp(2\pi i q(g))\langle L\rangle, \quad t_{g,h}\langle L\rangle = \exp(2\pi i b_q(g, h))\langle L\rangle \tag{4.9}$$

in $\mathscr{A}(M)$. Let $\mathscr{I}(M)$ be the submodule of $\mathscr{A}(M)$ spanned by all elements of the form (4.8) and (4.9) respectively.

Definition 4.4. The *reduced Heisenberg skein module* $\widehat{\mathscr{A}}(M)$ is the quotient $\mathscr{A}(M)/\mathscr{I}(M)$. If we need to emphasize that the reduced Heisenberg skein module depends on the quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$, we denote it $\widehat{\mathscr{A}_G}(M)$.

In other words, in the reduced skein module, (1) we replace two parallel knots by one of them and add their original colors; (2) we evaluate $t_g = \exp(2\pi i q(g))$ and $t_{g,h} = \exp(2\pi i b_q(g, h))$. The first relation allows in particular to replace n parallels of a link by the link with n times the original color and conversely (for any $n \in \mathbb{Z}$). In particular, a knot colored by $0 \in G$ represents $\langle\varnothing\rangle = 1$.

Remark 4.3. The skein relations do not affect the 1-homology of the link with coefficients in G. There is a natural $\mathbb{C}$-linear epimorphism

$$\mathrm{c} : \widehat{\mathscr{A}_G}(M) \to \mathbb{C}[H_1(M; G)]$$

which consists in assigning to a skein $\langle L\rangle$ its homology class

$$[L] = \sum_{\ell\in\pi_0(L)} \mathrm{col}(\ell) \otimes [\ell] \in H_1(M; G)$$

and extending by $\mathbb{C}$-linearity.

In the case $M = \Sigma \times [0, 1]$, the product defined above induces a product on $\widehat{\mathscr{A}}(M)$ which turns it into an associative algebra with unit. In particular, in $\widehat{\mathscr{A}}(\Sigma \times [0, 1])$,

$$\langle K; g\rangle \cdot \langle K; h\rangle = \langle K; g + h\rangle$$

for any oriented framed knot K and colors $g, h \in G$.

PROPOSITION 4.2. *Let L be an oriented framed colored link in $\Sigma \times [0,1]$. Let $\widehat{\theta}_L = \sum_{\ell \in \pi_0(L)} \mathrm{col}(\ell) \otimes \ell$ the framed 1-cycle determined by L in $\Sigma \times [0,1]$. Denote by $[L] = \theta_L$ its homology class in $H_1(\Sigma; G)$. Then*

$$\langle L \rangle = e^{2\pi i (q \otimes \mathrm{lk}_\Lambda)(\widehat{\theta}_L)} \cdot \langle L_0 \rangle \text{ in } \widehat{\mathscr{A}}(\Sigma \times [0,1]) \tag{4.10}$$

where L_0 is an oriented framed colored link such that

- $[L_0] = \theta_L = [L]$ *in* $H_1(\Sigma; G)$;
- L_0 *is* Λ*-trivial.*

Furthermore, the decomposition (4.10) *is unique.*

PROOF. We apply Lemma 4.2 to L in $\mathscr{A}(\Sigma \times [0,1])$. Now we observe that

$$P_L\Big(t_g = \exp(2\pi i q(g)), t_{gh} = \exp(2\pi i b(g,h)),\ g, h \in G\Big) = e^{2\pi i (q \otimes \mathrm{lk}_\Lambda)(\widehat{\theta}_L)}.$$

This gives the desired formula. $\square$

Let H_g be an oriented handlebody of genus g, so that $\partial H_g = \Sigma_g$. Recall the geometric symplectic basis $(m_1, l_1, \ldots, m_g, l_g)$ for the surface Σ. Then $(l_1, \ldots, l_g)$ is a geometric basis for the first homology of H_g.

COROLLARY 4.2. *The map defined by*

$$\langle L \rangle \mapsto \delta_{[L]}$$

defines a linear isomorphism $\widehat{\mathscr{A}}(\Sigma \times [0,1]) \to L^2(H_1(\Sigma; G))$. In particular, the Heisenberg skein algebra $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ is a vector space over $\mathbb{C}$ with basis

$$\langle m_1 \rangle \cdots \langle m_g \rangle \cdots \langle l_1 \rangle \cdots \langle l_g \rangle,$$

where the geometric elements of the symplectic basis are colored with arbitrary elements of G.

PROOF. We apply the extra relations (4.8) (which removes powers of elementary skeins) and (4.9) (which gives complex coefficients) to the bases respectively obtained in Prop. 4.1. We obtain the elements described in the statement above. Therefore they form a set $\mathcal{S}$ of generators for $\widehat{\mathscr{A}}(\Sigma \times [0,1])$. Consider the linear epimorphism $\widehat{\mathscr{A}}(\Sigma \times [0,1]) \to \mathbb{C}[H_1(\Sigma; G)]$. A basis for $\mathbb{C}[H_1(\Sigma; G)]$ consists of all elements in $H_1(\Sigma; G)$. The image of $\mathcal{S}$ under the epimorphism is precisely that basis. Thus $\mathcal{S}$ is also a basis for $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ and the map is a linear isomorphism. $\square$

COROLLARY 4.3. *Let Λ be the Lagrangian in $H_1(\Sigma)$ generated by the longitudes. The map defined by*

$$\langle L \rangle \mapsto \delta_{[L]}$$

defines a linear isomorphism $\widehat{\mathscr{A}}(H_g) \to L^2(G \otimes \Lambda)$. In particular, the Heisenberg skein module $\widehat{\mathscr{A}}(H_g)$ of the genus g oriented handlebody is a vector space over $\mathbb{C}$ with basis

$$\langle l_1 \rangle \cdots \langle l_g \rangle,$$

where each longitude is colored with an arbitrary element of G.

Proof. The proof is similar to that of Cor. 4.2. Alternatively apply Cor. 4.2 to Σ being the disc with g punctures. $\square$

In other words, an element of the basis of $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ is an arbitrary product of the skeins of the symplectic basis. Similarly, an element of the basis of $\widehat{\mathscr{A}}(H_g)$ is an arbitrary product of the skeins of the basis of the first homology of H_g. Whether a given skein appears in the product is determined by the color. In particular $\dim \widehat{\mathscr{A}}(\Sigma \times [0,1]) = |G|^{2g}$ and $\dim \widehat{\mathscr{A}}(H_g) = |G|^g$.

Definition 4.5. The product structure on $\widehat{\mathscr{A}}(H)$ is defined by as being induced by that of $L^2(G \otimes \Lambda)$ on the generators given in Cor. 4.3. This product turns $\widehat{\mathscr{A}}(A)$ into an algebra isomorphic to $L^2(G \otimes \Lambda)$.

In particular, the algebra $\widehat{\mathscr{A}}(H)$ is commutative. The product has a simple geometric meaning.

Proposition 4.3. *Let L and L' be two oriented framed links in H. If L and L' are topologically disjoint in H or if L' is parallel to L then*

$$\langle L \cup L' \rangle = \langle L \rangle \cdot \langle L' \rangle. \tag{4.11}$$

In particular, this justifies our previous notation in Corollaries 4.2 and 4.3 (see Remark 4.2) for $\langle l_1 \rangle \cdots \langle l_g \rangle = \langle l_1 \cup \cdots \cup l_g \rangle$.

Proof. By means of Prop. 4.2, any link is skein equivalent (up to a unit complex number) to a disjoint union of oriented framed colored longitudes. Therefore it suffices to verify the identity for an oriented framed colored link that is a disjoint union of oriented framed colored longitudes $l_1 \ldots, l_g$. Let $[l_i] \in H_1(H;G)$ denote the 1-homology class of the colored i-th longitude. Then

$$\mathrm{c}(\langle L \rangle) = \delta_{[L]} = \delta_{[l_1]+\cdots+[l_g]} = \delta_{[l_1]} \star \cdots \star \delta_{[l_g]}.$$

This justifies our previous notation: $\langle L \rangle = \langle l_1 \rangle \cdots \langle l_g \rangle = \langle l_1 \cup \cdots \cup l_g \rangle$. If l_i and l'_i denote the same longitude l with different colors $x, y \in G$, then

$$\delta_{\langle l_i \rangle \cdot \langle l'_i \rangle} = \delta_{[l_i]} \star \delta_{[l'_i]} = \delta_{[l_i]+[l'_i]} = \delta_{(x+y)\otimes[l]},$$

where $[l]$ denotes the integral 1-homology of l. Hence

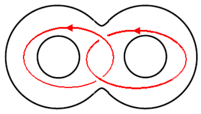
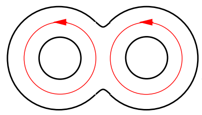
$$\langle l; x \rangle \cdot \langle l; y \rangle = \langle l; x + y \rangle.$$

Since for a longitude l' parallel to l (determined by the framing of l), $\langle l'; y \rangle = \langle l; y \rangle$, the definition of the reduced Heisenberg module implies that

$$\langle l; x \rangle \cdot \langle l'; y \rangle = \langle l; x + y \rangle = \langle (l; x) \cup (l'; y) \rangle.$$

The result follows. $\square$

Remark 4.4. If the links are not topologically disjoint or not parallel, then the formula (4.11) does not hold. For instance, consider the Hopf link $L \cup L'$ inside the handlebody of genus 2 depicted below. Denote by $x, y \in G$ the respective colors of the components. Then $\langle L\rangle \cdot \langle L'\rangle = \langle l_1\rangle \cdot \langle l_2\rangle = \langle l_1 \cup l_2\rangle$. But

 $= \exp(2\pi i b_q(x,y)) \cdot$ .

Hence $\langle L \cup L'\rangle = \exp(2\pi i b_q(x,y)) \cdot \langle l_1 \cup l_2\rangle$. So if $b_q(x,y) \neq 0 \pmod 1$, then $\langle L \cup L'\rangle \neq \langle L\rangle \cdot \langle L'\rangle$.

There is a natural question: do the linear isomorphism $\widehat{\mathscr{A}}(\Sigma \times [0,1]) \to L^2(H_1(\Sigma; G))$ of Cor. 4.3 extend to an algebra isomorphism ? It turns out that the algebra structure on $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ defined geometrically above is not compatible with this linear isomorphism.

We now turn to the study of the natural module epimorphism

$$\langle L\rangle \mapsto [L],\ \widehat{\mathscr{A}_G}(\Sigma) \to \mathbb{C}[H_1(\Sigma; G)]$$

defined in Remark 4.3. Can it be extended to an *algebra* morphism ? The answer (which is "almost") is given in the next section (see Th. 4.1).

4.4. Relation to the Heisenberg group algebras. Our goal is to compare the Heisenberg skein algebra to the group algebra of a suitable Heisenberg group. Let $\mathbb{Z}[G]$ be the group algebra of the group G, which we identify as usual to the algebra of $\mathbb{Z}$-valued functions over G with the canonical basis given by all maps δ_g ($g \in G$) defined by $\delta_g(h) = 1$ if $g = h$ and $\delta_g(h) = 0$ if $g \neq h$.

LEMMA 4.4. *The assignment, from the set of all isotopy classes of oriented framed colored links to* 1*-homology with coefficients in* L^2G*, defined by*

$$L \mapsto \sum_{\ell \in \pi_0(L)} \delta_{\mathrm{col}(\ell)} \otimes [\ell]$$

induces a map $\varkappa : \mathscr{A}(\Sigma \times [0,1]) \to H_1(\Sigma; L^2G)$ *that verifies*

$$\varkappa(\langle L\rangle\ \langle L'\rangle) = \varkappa(\langle L\rangle) + \varkappa(\langle L'\rangle) = \varkappa(\langle L\rangle + \langle L'\rangle). \tag{4.12}$$

PROOF. It is not hard to verify that the map is invariant under the skein relations. For instance, let us consider the second skein relation. This relation does not change the integral 1-homology class of the whole link but increases (or decreases) by one the number of components. Suppose that L and L' are related by a skein relation of the second kind. Suppose for instance that the skein relation involves two components ℓ, ℓ' of L (both labelled by g) and that after the skein move, the two components merge into one component ℓ'' (labelled by g) of L'. Then $[\ell''] = [\ell] + [\ell']$ in $H_1(\Sigma \times [0,1])$ and all other components are unaffected. Hence the result follows. To verify the identity (4.12), it suffices to observe that $\pi_0(L \cdot L')$ is the disjoint union of $\pi_0(L)$ and $\pi_0(L')$. Finally, we extend the map $\varkappa$ by taking the action of the Laurent polynomial algebra $\mathbb{C}[S]$ on $H_1(\Sigma; L^2G)$ to be the trivial action. □

We define a set-theoretic map

$$\Phi : \mathscr{A}(\Sigma \times [0,1]) \to \mathbb{C}[H_1(\Sigma; L^2G) \times \mathbb{Z}[S]]$$

as follows. First, we define a map with values in $H_1(\Sigma; L^2G) \times \mathbb{Z}[S]$ on generators of $\mathscr{A}(\Sigma \times [0,1])$ by the formulas

$$t^n_{g,h}\, \langle L\rangle \mapsto (\varkappa(L), t^n_{g,h}\ P_L), \quad t^n_g\, \langle L\rangle \mapsto (\varkappa(L), t^n_g\ P_L)$$

for any skein $\langle L\rangle$, any pair of distinct elements $g, h \in G$ and any integer $n \in \mathbb{Z}$. Secondly, we extend the map by $\mathbb{C}$-linearity.

Lemma 4.5. *The $\mathbb{C}$-linear map $\Phi : \mathscr{A}(\Sigma \times [0,1]) \to \mathbb{C}\big[H_1(\Sigma; L^2G) \times \mathbb{Z}[S]\big]$ is injective.*

Hence Φ induces a $\mathbb{C}$-linear isomorphism on its image $\mathrm{Im}(\Phi)$.

Proof. Define a map $\Psi : \mathrm{Im}(\Phi) \to \mathscr{A}(\Sigma \times [0,1])$ as follows. Let $(\sum_\ell \delta_{g_\ell} \otimes [\ell], P) \in \mathrm{Im}(\Phi)$. Set $n = \{s \in G \mid \deg_s P \neq 0\} \in \mathbb{N}$. Lift $\sum_\ell \delta_{g_\ell} \otimes [\ell]$ to some oriented colored n-component link L in $\Sigma \times [0,1]$ such that $\varkappa(L) = \sum_\ell \delta_{g_l} \otimes [\ell]$. Then we use the monomial P to modify L in such a way that the linking and framing numbers of the components match the corresponding partial degrees of P. Namely two components ℓ and ℓ' are linked algebraically $\deg_{\mathrm{col}(\ell),\mathrm{col}(\ell')} P$ times and a one-component ℓ is framed algebraically $\deg_{\mathrm{col}(\ell)} P$ times. Using skein relations and the definition of Φ, one observes that the skein $\langle L\rangle$ obtained in this fashion is independent of the original choice of L. Thus the map Ψ is well-defined. It is easily seen that $\Psi \circ \Phi = \mathrm{Id}_{\mathscr{A}(\Sigma\times[0,1])}$. □

The map Φ is not multiplicative but the cocycle is easily computed: $\varkappa(\langle L\rangle \cdot \langle L'\rangle)$ is given by Lemma 4.4 and $P_{L\cdot L'}$ is given by Lemma 4.3. This motivates the following definition. Let $\mathscr{H}(H_1(\Sigma; L^2G))$ be the set of all pairs $(\varkappa(L), P_L)$ in $\mathrm{Im}(\Phi)$. Endow $\mathscr{H}(H_1(\Sigma; L^2G))$ with the product defined by

$$(\varkappa(L), P_L)\cdot(\varkappa(L'), P_{L'}) = \Big(\varkappa(L)+\varkappa(L'), \prod_{(\ell,\ell')\in\pi_0(L)\times\pi_0(L')} t^{\mathrm{lk}_\Lambda(\ell,\ell')}_{\mathrm{col}(\ell),\mathrm{col}(\ell')}\ P_L P_{L'}\Big).$$

This turns $\mathscr{H}(H_1(\Sigma; L^2G))$ into a Heisenberg group over the group $H_1(\Sigma; L^2G)$. The (multiplicatively written) value group is $\mathbb{Z}[S]$ and any Lagrangian Λ in $H_1(\Sigma)$ induces a Lagrangian $L^2G \otimes \Lambda$ in $H_1(\Sigma; L^2G)$. The corresponding Seifert form is given by

$$\beta(\varkappa(L), \varkappa(L')) = \prod_{\ell,\ell'} t^{\beta_\Lambda([\ell],[\ell'])}_{\mathrm{col}(\ell),\mathrm{col}(\ell')}$$

where β_Λ is the usual Hopf Seifert form associated to the Hopf Lagrangian Λ. The product above turns $\mathrm{Im}(\Phi) = \mathbb{C}[\mathscr{H}(H_1(\Sigma; L^2G)]$ into a $\mathbb{C}$-algebra. Therefore we have proved

Proposition 4.4. *With the product above, the map*

$$\Phi : \mathscr{A}(\Sigma \times [0,1]) \to \mathbb{C}[\mathscr{H}(H_1(\Sigma; L^2G))]$$

is a $\mathbb{C}$-algebra isomorphism.

Consider the particular case when G is trivial. Then $\mathscr{H}(H_1(\Sigma; L^2G))$ is the Heisenberg group formed by the integral 1-homology group $H_1(\Sigma)$ (with multiplicative notation for the value group). The Seifert form $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}[t, t^{-1}]$ is explicitly given by

$$\beta([L],[L']) = t^{\beta_\Lambda([L],[L'])}$$

where β_Λ is the usual Hopf Seifert form associated to the Hopf Lagrangian Λ. The algebra $\mathrm{Im}(\Phi)$ is isomorphic to the group algebra $\mathbb{C}[\mathscr{H}(H_1(\Sigma)]$. Thus we have proved

Corollary 4.4 (Trivial color case). *For G trivial, the map*

$$\Phi : \mathscr{A}(\Sigma \times [0,1]) \to \mathbb{C}[\mathscr{H}(H_1(\Sigma)]$$

is an algebra isomorphism.

This result was proved by R. Gelca and A. Uribe in [**24**, Theorem 5.6]. We now turn to the quotient (reduced) Heisenberg skein algebra. For this, we need to define three ingredients which are all dictated by the definition of the reduced Heisenbrg skein algebra.

First, there is a natural surjective map

$$\mathfrak{t} : L^2G \to G, \quad f \mapsto \sum_{g \in G} f(g)g.$$

Clearly $\mathfrak{t}(f \star g) = \mathfrak{t}(f+g) = \mathfrak{t}(f) + \mathfrak{t}(g)$ for any $f, g \in L^2G$. Alternatively $\mathfrak{t}$ is defined by setting $\mathfrak{t}(\delta_g) = g$ and then extending to a $\mathbb{Z}$-map. The map $\mathfrak{t}$ induces a surjective additive map $H_1(\Sigma; L^2G) \to H_1(\Sigma; G)$, still denoted $\mathfrak{t}$.

The second ingredient is a specialization map. This is the algebra epimorphism $\mathrm{ev} : \mathbb{Z}[S] \mapsto \mathbb{C}$ defined by

$$\mathrm{ev}(P) = P\Big(t_g = \exp(2\pi i q(g)), t_{gh} = \exp(2\pi i b(g,h)),\ g, h \in G\Big).$$

The third ingredient is a quotient of the group algebra of the finite Heisenberg group $\mathscr{H}(H_1(\Sigma; G))$. Recall that $Z = 0 \times \mathbb{Q}/\mathbb{Z}$ is the center of $\mathscr{H}(H_1(\Sigma; G))$.

Let I_q be the two-sided ideal of $\mathbb{C}[\mathscr{H}(H_1(\Sigma; G))]$ generated by all combinations

$$(0,t) \cdot h - \chi(t)h, \quad t \in \mathbb{Q}/\mathbb{Z},\ h \in \mathscr{H}(H_1(\Sigma; G)).$$

Definition 4.6. The *reduced group algebra of the Heisenberg group $H_1(\Sigma; G)$* is defined by

$$V_q[\mathscr{H}(H_1(\Sigma; G))] = \mathbb{C}[\mathscr{H}(H_1(\Sigma; G))]/I_q.$$

Theorem 4.1. *There is a $\mathbb{C}$-algebra isomorphism*

$$\Psi : \widehat{\mathscr{A}_G}(\Sigma \times [0,1]) \to V_q[\mathscr{H}(H_1(\Sigma; G))]$$

such that the diagram of algebras

$$\begin{tikzcd}
\mathscr{A}_G(\Sigma \times [0,1]) \arrow[r, "\Phi"] \arrow[d] & \mathbb{C}[\mathscr{H}(H_1(\Sigma; L^2G)] \arrow[d] \\
\widehat{\mathscr{A}_G}(\Sigma \times [0,1]) \arrow[r, "\Psi"] & V_q[\mathscr{H}(H_1(\Sigma; G))]
\end{tikzcd}$$

is commutative. The left vectical arrow is the natural projection and the right vertical arrow is the map induced by $\mathfrak{t} \times \mathrm{ev}$.

PROOF. Define an assignment

$$\langle L\rangle \mapsto \Big([L], \mathrm{ev}(P_L)\Big)$$

and extend it by $\mathbb{C}$-linearity to a map $\Psi : \widehat{\mathscr{A}_G}(\Sigma\times[0,1]) \to V_q[\mathscr{H}(H_1(\Sigma; G))]$. It follows from definitions that it is an algebra map making the diagram commute. It remains to see that Ψ is a linear isomorphism. For this, we use Cor. 4.2. The details are left to the reader. $\square$

For each Lagrangian Λ of $H_1(\Sigma)$, there is a Schrödinger representation $\pi_\Lambda : \mathscr{H}(H_1(\Sigma; G)) \to \mathrm{U}(L^2(G\otimes\Lambda))$.

LEMMA 4.6. *There is a unique* $\mathbb{C}$*-linear extension*

$$\tilde{\pi}_\Lambda : V_q[\mathscr{H}(H_1(\Sigma; G))] \to \mathrm{End}_{\mathbb{C}}(L^2(G\otimes\Lambda))$$

of the Schrödinger representation making the diagram

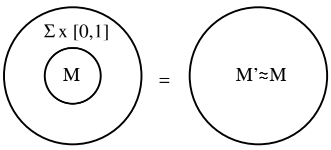
$$\begin{tikzcd} \mathscr{H}(H_1(\Sigma; G)) \arrow[r, "\pi_\Lambda"] \arrow[d] & \mathrm{U}(L^2(G\otimes\Lambda)) \arrow[d] \\ V_q[\mathscr{H}(H_1(\Sigma; G))] \arrow[r, "\tilde{\pi}_\Lambda"] & \mathrm{End}_{\mathbb{C}}(L^2(G\otimes\Lambda)) \end{tikzcd}$$

commute.

PROOF. Extend π_Λ by $\mathbb{C}$-linearity to a linear map $\mathbb{C}[\mathscr{H}(H_1(\Sigma; G))] \to \mathrm{End}_{\mathbb{C}}(L^2(G\otimes\Lambda))$, that we continue to denote π_Λ. We have $\pi_\Lambda((0,t)\cdot h) = \pi_\Lambda(0,t)\ \pi_\Lambda(h) = \chi(t)\pi_\Lambda(h)$, so $\pi_\Lambda(I_q) = 0$. The result follows. $\square$

4.5. The Schrödinger representation from Abelian skein theory. Let M be a compact oriented 3-manifold with boundary Σ. In this paragraph, we explain how the skein module $\widehat{\mathscr{A}}(\Sigma\times[0,1])$ acts on the skein module $\widehat{\mathscr{A}}(M)$. We then identity this action to "the" Schrödinger representation, as defined in Chap. 3, §3.

There is a natural gluing $M \cup (\Sigma\times[0,1]) \approx M$ defined by identifying $\partial M = \Sigma$ with $-\Sigma\times 0$. The result is a "thickened" manifold M which is homeomorphic to M.



LEMMA 4.7. *The Heisenberg skein algebra* $\mathscr{A}(\Sigma\times[0,1])$ *acts on* $\mathscr{A}(M)$ *by the map*

$$\mathscr{A}(M)\times\mathscr{A}(\Sigma\times[0,1]) \to \mathscr{A}(M\cup\ \Sigma\times[0,1]) = \mathscr{A}(M),$$

defined by

$$(\langle L\rangle,\ \langle L'\rangle) \mapsto \langle L\cup L'\rangle.$$

This action induces an action α_Λ of the reduced Heisenberg skein algebra $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ on $\widehat{\mathscr{A}}(M)$.

PROOF. This is a consequence of the definitions. □

Let H_g denote the standard handlebody of genus g.

PROPOSITION 4.5. *The action of the reduced Heisenberg skein algebra $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ on $\widehat{\mathscr{A}}(H_g)$ is induced by the Schrödinger representation π_Λ where Λ is the standard longitudinal Lagrangian in $H_1(\Sigma)$. More precisely, the following diagram is commutative:*

$$\begin{tikzcd}
\widehat{\mathscr{A}}(\Sigma \times [0,1]) \arrow[r, "\alpha_\Lambda"] \arrow[d, "\Psi"', "\simeq"] & \mathrm{End}_{\mathbb{C}}(\widehat{\mathscr{A}}(H)) \arrow[d, "\simeq"', "\delta"] \\
V_q[\mathscr{H}(H_1(\Sigma;G))] \arrow[r, "\tilde{\pi}_\Lambda"] & \mathrm{End}_{\mathbb{C}}(L^2(G \otimes \Lambda)) \\
\mathscr{H}(H_1(\Sigma;G)) \arrow[r, "\pi_\Lambda"] \arrow[u] & \mathrm{U}(L^2(G \otimes \Lambda)) \arrow[u, hook]
\end{tikzcd}$$

where the first vertical descending arrow is the isomorphism of Theorem 4.1*, the second vertical descending arrow is the isomorphism induced by δ (Cor.* 4.3*).*

PROOF. We have already seen that $\widehat{\mathscr{A}}(H_g)$ is isomorphic via δ to the algebra $L^2(G\otimes\Lambda) = \mathbb{C}[G\otimes\Lambda]$(Cor. 4.3). We have also seen that $\widehat{\mathscr{A}}(\Sigma\times[0,1])$ is isomorphic via Ψ to the reduced group algebra $V_q[\mathscr{H}(H_1(\Sigma;G))]$ of the finite Heisenberg algebra $\mathscr{H}(H_1(\Sigma;G))$ (Theorem 4.1). By Lemma 4.6, It suffices to identify the representation on the image of the finite Heisenberg algebra $\mathscr{H}(H_1(\Sigma;G))$.

Let $\langle m_i; g\rangle_\Sigma$ denote the skein in $\Sigma \times [0,1]$ represented by the i-th standard meridian of $\Sigma \times 1 \subset \Sigma \times [0,1]$ and colored by $x \in G$. Let $\langle l_j; h\rangle_{H_g}$ denote the skein in $H = H_g$ represented by the j-th standard longitude of H and colored by $h \in G$. Using skein calculus, we verify that

$$\langle m_i; g\rangle_\Sigma \cdot \langle l_j; h\rangle_H = \begin{cases} \langle l_j; h\rangle_H & \text{if } i \neq j; \\ e^{2\pi i b_q(g,h)} \langle l_j; h\rangle_H & \text{if } i = j. \end{cases}$$

(In the case when Σ has genus 1, a proof follows from Fig. 4.1. The general case is similar.) Note that $\beta_\Upsilon(i_* m_i, i_* l_j) = \delta_{ij}$. Therefore, if L is an arbitrary disjoint union of parallel meridians in $\Sigma \times 1$ and L' an arbitrary element in $\mathscr{S}(H)$,

$$\langle L\rangle_\Sigma \cdot \langle L'\rangle_H = e^{2\pi i (b_q \otimes \beta_\Upsilon)([L],[L'])} \langle L'\rangle_H. \tag{4.13}$$

Let $\langle l_i; g\rangle_\Sigma$ denote the skein in $\Sigma \times [0,1]$ represented by the i-th longitude of $\Sigma \times 1 \subset \Sigma \times [0,1]$ and colored by $x \in G$. Let $\langle l_j; h\rangle_{H_g}$ denote the skein in H_g represented by the j-th standard longitude of H_g and colored by $h \in G$. We have

$$\langle l_i; g\rangle_\Sigma \cdot \langle l_j; h\rangle_{H_g} = \langle l_i; g\rangle_{H_g} \cdot \langle l_j; h\rangle_{H_g}.$$

Therefore, if L is an arbitrary disjoint union of parallel longitudes in $\Sigma \times 1$ and L' an arbitrary element in $\mathscr{S}(H)$,

$$\langle L\rangle_\Sigma \cdot \langle L'\rangle_H = \langle L\rangle_H \cdot \langle L'\rangle_H. \tag{4.14}$$

Recall that $\delta_{\langle L\rangle_H \cdot \langle L'\rangle_H} = \delta_{[L]} \star \delta_{[L']} = \delta_{[L]+[L']}$. Therefore, longitudes acts as translations as required. □

4.6. The Weil representation from Abelian skein theory.

5. A direct proof of "Weil=TQFT" theorem

In order to identify the Weil representation, we use Proposition 7.1. It is therefore sufficient to identify the Weil representation on generators of $\mathrm{Sp}(H_1(\Sigma))$. We use the list (7.1) provided by Remark 7.1. In the sequel, we endow $H_1(\Sigma)$ with a geometric symplectic basis (See Fig. 1.1). There are three types of generators of $\mathrm{Sp}(H_1(\Sigma))$, so there are three cases to consider.

First case. Consider a diffeomorphism $h : \Sigma_- \to \Sigma_-$ such that $h(m_j^-) = l_j^-$ and $h(l_j^-) = -m_j^-$, $1 \leqslant j \leqslant g$. With respect to the symplectic basis $([m_1^-], \dots, [m_g^-], [l_1^-], \dots, [l_g^-])$, we have

$$\mathrm{Mat}_{[m_i^-],[l_i^-]}(h_*) = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}.$$

Hence h represents a generator of $\mathrm{Sp}(H_1(\Sigma))$ of the first type. By definition, Λ is the Lagrangian generated by m_j^+ and $h(m_j^-) = -l_j^-$, $1 \leqslant j \leqslant g$. Thus $i_*\Lambda$ is generated by $i_*([m_j^+]) = [m_j]$ and $i_*h_*([m_j^-]) = -i_*([l_j^-]) = -[l_j]$, $1 \leqslant j \leqslant g$. It follows that $G_\Lambda = H_1(M(h))/i_*\Lambda = 0$. In particular $T_\Lambda M(h) = 0$, $\lambda_\Lambda = 0$ and $\theta = 0$. Since $G \otimes G_\lambda = 0$, for any $x \in G^g$, the set $\mathcal{H}(x)$ consists of all elements $y \in G^g$. Let

$$\begin{aligned}\kappa_{xy} &= i_*^+\Big(\sum_j y_j \otimes l_j^+\Big) - i_*^-\Big(\sum_j x_j \otimes h(l_j^-)\Big)\\ &= i_*^+\Big(\sum_j y_j \otimes l_j^+\Big) + i_*^-\Big(\sum_j x_j \otimes m^-\Big)\end{aligned}$$

be the corresponding framed 1-boundary with coefficients in G. Since

$$\mathrm{lk}_\Lambda(i_*^+ l_j^+, i_*^+ l_j^{+'}) = \mathrm{lk}_\Lambda(i_*^- m_j^-, i_*^- m_j^{-'}) = 0,$$

we find

$$\begin{aligned}(q \otimes \mathrm{lk}_\Lambda)(\kappa_{xy}) =&(q \otimes \mathrm{lk}_\Lambda)(\sum_j y_j \otimes i_*^+ l_j^+ + x_j \otimes i_*^- m_j^-)\\ &= \sum_{j<k} b_q(y_j, x_k)\mathrm{lk}_\Lambda(i_*^+ l_j^+, i_*^- m_k^-).\end{aligned}$$

We have

$$\mathrm{lk}_\Lambda(i_*^+ l_j^+, i_*^- m_k^-) = \mathrm{lk}_\Lambda(i_*^- m_k^-, i_*^+ l_j^+) = \mathrm{lk}_\Lambda(i_*^+ m_k^+, i_*^+ l_j^+) = m_k^+ \bullet_{\Sigma_+} l_j^+ = \delta_{jk}.$$

Now apply Corollary 5.2. We obtain

$$\rho(h_*)\ e_x = |G|^{-g/2} \sum_{y \in G \otimes L_0} \chi((b_q \otimes \beta)(y,x))\ e_y.$$

This is the formula (7.2) as desired.

Second case. We have to consider a diffeomorphism $h : \Sigma_- \to \Sigma_-$ such that with respect to the symplectic basis $([m_1^-], \ldots, [m_g^-], [l_1^-], \ldots, [l_g^-])$,

$$\mathrm{Mat}_{[m_i^-],[l_i^-]}(h_*) = \begin{bmatrix} 1_g & 0 \\ B & 1_g \end{bmatrix}$$

where B is an symmetric integral square matrix of size g. Note that

$$\begin{bmatrix} 1_g & 0 \\ B & 1_g \end{bmatrix} \cdot \begin{bmatrix} 1_g & 0 \\ B' & 1_g \end{bmatrix} = \begin{bmatrix} 1_g & 0 \\ B + B' & 1_g \end{bmatrix}$$

and

$$\chi(q \otimes (B + B')) = \chi(q \otimes B + q \otimes B') = \chi(q \otimes B) \cdot \chi(q \otimes B').$$

It follows that it suffices to verify the formula for an elementary symmetric integral matrix B. Let $1 \leqslant i < j \leqslant g$. Let E_{ij} denotes the elementary matrix defined by $(E_{ij})_{kl} = \delta_{ik} \cdot \delta_{jl}$. Consider the case when $B = E_{ij} + E_{ji}$. Then Λ is generated by $[m_k^+]$ and

$$h_*([m_k^-]) = m_k^- + \sum_p (\delta_{ip}\delta_{jk} + \delta_{jp}\delta_{ik}) l_p = \begin{cases} m_i + l_j & \text{if } k = i; \\ m_j + l_i & \text{if } k = j; \\ m_j & \text{if } k \notin \{i, j\} \end{cases}$$

for $1 \leqslant k \leqslant g$. Thus $G_\Lambda M = H_1(M(h))/i_*\Lambda$ is the free abelian group of rank $g - 2$ generated by $[l_1], \ldots, \widehat{[l_i]}, \widehat{[l_j]}, \ldots, [l_g]$. (Here $\widehat{\ }$ denotes deletion.) In particular, $T_\Lambda M$ is trivial, $\lambda_\Lambda = 0$ and $\theta = 0$. By definition, $y \in \mathcal{H}(x)$ if and only if $i_*^+([y]) = i_*^-([x])$ in $G \otimes G_\Lambda M$. Since $G_\Lambda M$ is free, $\mathrm{Ker}(G \otimes H_1(M) \to G \otimes G_\Lambda M) = G \otimes \Lambda$. It follows that the map $i_*^\pm|_{G \otimes A^\pm} : G \otimes A^\pm \to G \otimes G_\Lambda M$ are injective. (Here we use the fact that Λ and $A^- \oplus A^+$ are transverse Lagrangians in $H_1(\partial M(h))$.) It follows that the equation $i_*^+([y]) = i_*^-([x])$ has a unique solution in y. With the identification $A^+ = L_0 = A^-$, this solution identifies $y = x$.

Third case.

5.1. A second proof.

6. A few computations and examples

7. A modular category

For the definition of a modular category and the construction of quantum invariants of 3-manifolds from modular categories, we refer to [**61**, Chap. 2, Chap. 3]. Let $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ be a biadditive pairing on a finite group G and $a : G \to \mathbb{Q}/\mathbb{Z}$ be a homomorphism such that $2a = 0$. In [**61**, p.77] (the second "toy example"), V. Turaev defines a modular tensor category $\mathcal{C}(G, a, b)$ as follows.

- Objects are elements of G.
- The set of morphisms is $\mathbb{C}$ between two identical elements of G and the singleton $\{0\}$ otherwise.
- Composition of morphisms is the usual product in $\mathbb{C}$.
- Tensor product of morphisms is induced by the usual products in G and $\mathbb{C}$ respectively.
- The *dual* object to an object $g \in G$ is g^{-1}.
- The *braiding* $c_{g,h} : g \otimes h = gh \to hg = h \otimes g$ is defined as $\exp(2\pi i b(g,h))$ where b is a fixed bilinear pairing.
- The *twist* $t_g : g \to g$ is defined as $\exp(2\pi i(b(g,g) + a(g)))$ where a.
- The *associator* $(gh)k \to g(hk)$ is the identity (trivial).

This category is known to be a *semi-simple ribbon* category with the set of *simple* objects being G itself. Indeed, the (xy)-th entry of the S-matrix is

$$\mathrm{Tr}(c_{x,y} \circ c_{y,x}) = \exp(2\pi i(b(x,y) + b(x,y) + a(x) + a(y)).$$

By definition a semi-simple category (over $\mathbb{C}$) is modular if the S-matrix is invertible over $\mathbb{C}$.

Proposition 7.1. *Let (G, a, b) be a triple as above with $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ nondegenerate. The following three statements are equivalent:*

(1) *The form*

$$\widetilde{b} : G \times G \to \mathbb{Q}/\mathbb{Z},\ (x,y) \mapsto b(x,y) + b(y,x)$$

is non-degenerate.

(2) *$a = 0$ and G has no cyclic (left or right) orthogonal summand of even order.*

(3) *The semi-simple category $\mathcal{C}(G, a, b)$ is modular.*

Proof. Preliminary observation: using adjoint maps one sees that $\widetilde{b}$ is nondegenerate if and only if the linking

$$(x,y) \mapsto \widetilde{b}(x,y) + a(x) + a(y)$$

is nondegenerate, which implies $a = 0$. (1) $\Longrightarrow$ (2): suppose (1) satisfied. Suppose that G has a cyclic orthogonal summand of even order $2k$ generated by $x \in G$. Then by [**8**, Lemma 28], x and $\widetilde{b}(x,x) = 2b(x,x)$ have the same order $2k$ in $\mathbb{Q}/\mathbb{Z}$. But this would imply that $b(x,x)$ has order $4k$, a contradiction since $2k\ b(x,x) = b(2k\ x, x) = b(0,x) = 0$. (2) $\Longrightarrow$ (1): suppose that the linking $\widetilde{b}$ is degenerate, i.e., $\mathrm{Ann}(\widetilde{b}) = \{x \in G \mid \widetilde{b}(x,-) = 0\} \neq \{0\}$. It induces a nondegenerate linking $\widetilde{b}'$ on the quotient $G/\mathrm{Ann}(\widetilde{b})$. Any section of the projection $G \to G/\mathrm{Ann}(\widetilde{b})$ induces an isomorphism of linkings

$$(G, \widetilde{b}) \simeq (G/\mathrm{Ann}(\widetilde{b}), \widetilde{b}') \oplus (\mathrm{Ann}(\widetilde{b}), 0).$$

By the previous result, since $\widetilde{b}'$ is nondegenerate, $G/\mathrm{Ann}(\widetilde{b}')$ has no cyclic orthogonal summand of even order. On the other hand, there is $x_0 \in \mathrm{Ann}(\widetilde{b}) - \{0\}$ such that $\widetilde{b}(x_0, -) = 0$. In particular,

$$\widetilde{b}(x_0, x_0) = b(x_{,0}, x_0) + b(x_0, x_0) = 0,$$

hence $2\ b(x_0, x_0) = 0$. This implies that the order of x_0 is even. It follows that $\mathrm{Ann}(\widetilde{b})$ has even order. Note that the direct sum decomposition of $\mathrm{Ann}(\widetilde{b})$ is orthogonal. Hence $\mathrm{Ann}(\widetilde{b})$ contains an orthogonal cyclic summand of even order. This is the desired result. (3) $\Longrightarrow$ (1): modularity is equivalent to the invertibility of the S-matrix. The latter is equivalent to the nondegeneracy of $(x, y) \mapsto \widetilde{b}(x, y) + a(x) + a(y)$ which implies $a = 0$ and nondegeneracy of $\widetilde{b}$ by the preliminary observation. The converse follows from [**5**, Prop. 1.1]. $\square$

Any triple (G, a, b) as above gives rise to a homogeneous quadratic form q on G defined, as the sum of the polarization of b and the homomorphism a, by: $q(x) = b(x, x) + a(x)$, $x \in G$. In this case, the bilinear linking associated to q is the symmetrized form built from b. Conversely:

LEMMA 7.1. *Any homogeneous quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$ is the sum of the polarization of some bilinear pairing b and a homomorphism $a : G \to \mathbb{Q}/\mathbb{Z}$ such that $2a = 0$ if and only if G has no cyclic orthogonal summand of even order.*

PROOF. If G has odd order then the image of q is an odd (cyclic) subgroup of $\mathbb{Q}/\mathbb{Z}$. Thus we can define $b(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$ for $x, y \in G$. Then $q(x) = b(x, x)$ for all $x \in G$. If G has even order, consider any quadratic form q on G that is nondegenerate. Then by [**8**, Lemma 29], there exists $x \in G$ of order k such that $q(x)$ has order $2k$ in $\mathbb{Q}/\mathbb{Z}$, while $b(x, x) + a(x)$ has order dividing k for any bilinear pairing $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ and homomorphism $a : G \to \mathbb{Q}/\mathbb{Z}$. Therefore, q is not the sum of the polarization of b and the homomorphism a. $\square$

We consider now the case when $\mathcal{C}(G, a, b)$ is not modular. A weaker condition than the invertibility of the S-matrix is known in order to construct a topological invariant from a semi-simple ribbon category (see [**5**, Prop. 1.6], [**6**, Appendix A]):

$$\sum_{x \in G} \exp(2\pi i(b(x, x) + a(x))) \neq 0. \tag{7.1}$$

This is a Gauss sum. Using Lemma 3.8, we see that Condition (7.1) is satisfied if and only if

$$b(x, x) + a(x) = 0 \quad \text{for all } x \in G \text{ such that } 2x \in \mathrm{Ann}(b) = \mathrm{Ker}(\widehat{b}). \tag{7.2}$$

Without loss of generality, assume that b is nondegenerate. Then (7.2) becomes

$$b(x, x) + a(x) = 0 \quad \text{for all } x \in G \text{ such that } 2x = 0. \tag{7.3}$$

Since $x \mapsto b(x, x)$ is a homomorphism on $\{x \in G \mid 2x = 0\}$, the condition (7.2), viewed as an equation in a, has always a solution in $a \in G^*$.

The next step consists in extending a quantum invariant to a topological quantum field theory. The modularity is used in a crucial way in this extension. However, in the case of $\mathcal{C}(G, a, b)$, the invertibility of the S-matrix is used to ensure that the resulting topological quantum field theory is non-degenerate (i.e., the cobordism invariant on a cylinder may vanish). It is not hard to see that in the case when $\mathcal{C}(G, a, b)$ is semi-simple and satisfies (7.1), there is a topological quantum field theory associated to $\mathcal{G}(G, a, b)$.

We mention an alternative modular category that produces (possibly up to a normalization real factor) the invariant τ above. The category in question – as opposed to the one above – is not strict and includes a non trivial associator $a : (xy)z \to x(yz)$ defined as $a_{x,y,z} = \exp(2\pi i h(x, y, z))$, $x, y, z \in G$, where h is a map : $G \times G \times G \to \mathbb{Q}/\mathbb{Z}$. The braiding is still defined as above $c_{x,y} : xy \to yx$ by $c_{x,y} = \exp(2\pi i b(x, y))$ but b is no longer necessarily biadditive. The maps b and h are required to satisfy the hexagon identity. It turns out that the pair (b, h) is an Abelian 3-cocycle in the sense of Eilenberg and McLane [**15**]. Computations of Abelian Eilenberg-McLane cohomology were performed by S. Eilenberg and S. McLane themselves [**15**] and in particular the identification of $H^3(A^1(G); \mathbb{Q}/\mathbb{Z})$ where $A^1(G)$ is a certain cell complex associated to G, and the set of homogeneous quadratic functions $G \to \mathbb{Q}/\mathbb{Z}$. Further computations were performed by A. Joyal and R. Street [**31**], F. Quinn [**51**] and others. The details of the construction of a modular category from this data are worked out by F. Quinn [**51**] and S.D. Stirling in [**59**]. The fundamental construction of Eilenberg and McLane has been generalized by C. Ospel to a nonabelian ("quasi-abelian") setting in [**48**], which in turn, has been further used by V. Turaev to construct enriched modular categories to incorporate a group action [**63**].

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