A direct proof that intuitionistic predicate calculus is complete with respect to presheaves of classical models

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Remarks

This direct completeness proof is essentially due to
Ivano Ciardelli (in TACL 2011 cf. reference at the end).

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A Logic?

formulas

proofs $\leftrightsquigarrow$ interpretations

A.1. Logic

Logic: language (trees with binding relations)

whose expressions can be true (or not):
wellformed expressions of a logical language have a meaning.

A.2. Intutionnistic logic vs. classical logic (the usual logic of mathematics)

Absence of Tertium no Datur, $A \vee \neg A$ does not always hold.

Disjunctive statemeents are stronger.

Existential statements are stronger.

Proof have a constructive meaning,
algorithms can be extracted from proofs.

A.3. Rules of intuitionist logic: structures

Structural rules

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ E_g$$

$$\frac{\Delta \vdash C}{A, \Delta \vdash C}\ A_g$$

$$\frac{\Gamma, A, A, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}\ C_g$$

A.4. Rules of intuitionistic logic: connectives

Axioms are $A \vdash A$ (if A then A...) for every A.

Negation $\neg A$ is just a short hand for $A \Rightarrow \perp$.

$$\frac{\Theta \vdash (A \wedge B)}{\Theta \vdash A} \wedge_e \qquad \frac{\Theta \vdash (A \wedge B)}{\Theta \vdash B} \wedge_e$$

$$\frac{\Theta \vdash A \quad \Delta \vdash B}{\Theta, \Delta \vdash (A \wedge B)} \wedge_d$$

$$\frac{\Theta \vdash (A \vee B) \quad A, \Gamma \vdash C \quad B, \Delta \vdash C}{\Theta, \Gamma, \Delta \vdash C} \vee_e$$

$$\frac{\Theta \vdash A}{\Theta \vdash (A \vee B)} \vee_d \qquad \frac{\Theta \vdash B}{\Theta \vdash (A \vee B)} \vee_d$$

$$\frac{\Theta \vdash A \quad \Gamma \vdash A \Rightarrow B}{\Gamma, \Theta \vdash B} \Rightarrow_e$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash (A \Rightarrow B)} \Rightarrow_d$$

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash \mathrm{C}} \perp_e$$

A.5. Differences

$A \vee \neg A$ does not hold for any A.

$\neg\neg B$ does not entail B.

However $\neg\neg(C \vee \neg C)$ hods for any C.

$\neg\forall x.\neg P(x)$ does not entail $\exists x P(x)$.

An example, in the language of rings: $\forall x.((x = 0) \vee \neg(x = 0))$ is not provable and their are concrete counter models.

$[\neg\forall x.((x = 0) \vee \neg(x = 0))] \to [\neg\forall x.\neg\neg((x = 0) \vee \neg(x = 0))]$
is also non provable.

B Topological models

B.1. Usual FOL models

One is given a language,
e.g. constants $(0, 1)$, functions $(+, *)$, and predicates $(\leqslant)$.

One is given a set $|M|$.

Constants are interpreted by elements of $|M|$,
n-ary functions symbols by n-ary applications from $|M|^n$ to $|M|$,
and n-ary predicates by parts of $|M|^n$.

Logical connectives and quantifiers are interpreted intuitively
(Tarskian truth: "$\wedge$" means "and", "$\forall$" means "for all" etc.).

B.2. Soundness

any provable formula is true for every interpretation

or:

when T entails F then any model that satisfies T satisfies F

B.3. Completeness

Completeness (a word that often encompass soundness):

a formula that is true in every interpretation is derivable

or

a formula F that is true in every model of T
is a logical consequence of T

e.g.
a formula F of ring theory is true in any ring
if and only if
F is provable from the axioms of ring theory

Soundness, completeness (and compacity)
are typical fo first order logic (as opposed to higher order logic).

B.4. Presheaves

A pre sheaf can de defined as a contravariant functor F

- from open subsets of a topological set (this partial order can be viewed as a category)
- to a category (e.g. sets, groups, rings):

Contravariant functor: when $U \subset V$ there is a restriction map ρ_V, U from $F(V)$ to $F(U)$ and $\rho_{U_3,U_2} \circ \rho_{U1,U_2} = \rho_{U_1,U_3}$ whenever is makes sense, i.e. $U_3 \subset U_2 \subset U_1$.

Example of pre-sheaf on the topological space R:
$U \mapsto F(U)$ the ring of bounded functions from U to R.

B.5. Sheaves

The presheaf (resp separated presheaf) is said to be a sheaf if every family of compatible elements has unique glueing:

- given a cover U_i of an open set U,
- with for every i an element $c_i \in F(U_i)$ such that for every pair i, j $\rho_{U_i,U_j}(c_i) = \rho_{U_j,U_i}(c_j)$
- there is a unique (resp. at most one) c in $F(u)$ such that $c_i = \rho_{U,U_i}(c)$.

Example of pre-sheaf on the topological space R:
$U \mapsto C(U,R)$ the ring of continuous functions from U to R.

B.6. Presheaf semantics

Grothendieck generalized the notion of topological space,using coverings.

A site is a category with every object is provided with various covering.

A covering of an object φ consists in a set of arrows $f_i, i \in \mathscr{I}$ with codomain φ_i — when the category is a preorder it is enough to know the domain of every f_i: there is at most one arrow from φ_i to φ.

1. $\varphi \lhd \{\varphi\}$;
2. if $\psi \leqslant \varphi$ and $\varphi \lhd \{\varphi_i \mid i \in \mathscr{I}\}$ then $\psi \lhd \{\psi \wedge \varphi_i \mid i \in \mathscr{I}\}$;
3. if $\varphi \lhd \{\varphi_i \mid i \in \mathscr{I}\}$ and if for each $i \in \mathscr{I}$, $\varphi_i \lhd \{\psi_{i,k} \mid k \in \mathscr{K}_i\}$, then $\varphi \lhd \{\psi_{i,k} \mid i \in \mathscr{I}, k \in \mathscr{K}_i\}$.

Who, when? 70's Joyal, Lawvere, Lambek,...

B.7. Presheaf semantics: models

A **presheaf model** M for $\mathscr{L}$ is a presheaf of first-order $\mathscr{L}-$structures over a Grothendieck site $(\mathscr{C}, \lhd)$:

- for any object u a first-order model M_u
- for any arrow $f : v \to u$ a homomorphism $\upharpoonleft_f : M_u \to M_v$

satisfying the following extra conditions.

Separateness For any elements a, b of M_u,
if there is a cover $u \lhd \{f_i : u_i \to u \,|\, i \in \mathscr{I}\}$
such that for all $i \in \mathscr{I}$ we have $a\upharpoonleft_{f_i} = b\upharpoonleft_{f_i}$,
then $a = b$.

Local character of atoms For any $n-$ary relation symbol R,
for any tuple $(a_1, \ldots, a_n)$ from M_u
if there is a cover $u \lhd \{f_i : u_i \to u \,|\, i \in \mathscr{I}\}$
such that for all $i \in \mathscr{I}$ we have $(a_1\upharpoonleft_{f_i}, \ldots, a_n\upharpoonleft_{f_i}) \in R_{u_i}$,
then $(a_1, \ldots, a_n) \in R_u$.

B.8. Presheaf semantics: Kripke-Joyal forcing — 1/3 atoms and conjunction

Formulas of $\mathscr{L}$ can be inductively interpreted on an object u of a given presheaf model M (v: assignment into M_u):

- $u \Vdash_v R(t_1, \ldots, t_n)$ iff $([t_1]_v, \ldots, [t_n]_v) \in R_u$.
- $u \Vdash_v t_1 = t_2$ iff $\quad [t_1]_v = [t_2]_v$.

- $u \Vdash_v \perp$ iff $u \lhd \emptyset$ ($M_\emptyset \Vdash \perp$)
- $u \Vdash_v \varphi \wedge \psi$ iff $u \Vdash_v \varphi$ and $u \Vdash_v \psi$.

B.9. Presheaf semantics: Kripke-Joyal forcing — 2/3 disjunction and existential

Formulas of $\mathscr{L}$ can be inductively interpreted on an object u of a given presheaf model M (ν: assignment into M_u):

- $u \Vdash_\nu \varphi \vee \psi$ iff there is a covering family $\{f_i : u_i \to u \mid i \in \mathscr{I}\}$ such that for any $i \in \mathscr{I}$ we have $u_i \Vdash_\nu \varphi$ or $u_i \Vdash_\nu \psi$.
- $u \Vdash_\nu \exists x \varphi \quad \Longleftrightarrow \quad$ there exist a covering family $\{f_i : u_i \to u \mid i \in \mathscr{I}\}$ and elements $a_i \in |M_{u_i}|$ for $i \in \mathscr{I}$ such that $u_i \Vdash_{\nu[x \mapsto a_i]} \varphi$ for any index i.

B.10. Presheaf semantics: Kripke-Joyal forcing — 3/3 implication and universal

Formulas of $\mathscr{L}$ can be inductively interpreted on an object u of a given presheaf model M (ν: assignment into M_u):

- $u \Vdash \varphi \to \psi$ iff for all $f : v \to u$, if $v \Vdash \varphi$ then $v \Vdash \psi$.
- $u \Vdash \neg\varphi$ iff for all $f : v \to u$, with $v \neq \emptyset$, $v \nVdash \varphi$.
- $u \Vdash_\nu \forall x \varphi$ iff for all $f : v \to u$ and all $a \in M_v$, $v \Vdash_{\nu[x \mapsto a]} \varphi$.

Notice that the usual Kripke semantics is obtained as a particular case when the underlying Grothendieck site is a poset equipped with the trivial covering $u \lhd \mathscr{F} \iff u \in \mathscr{F}$.

B.11. Properties of Kripke-Joyal forcing

Fonctoriality of $\Vdash$:
if $f_i : U_i \to U_j$ and $U_j \Vdash F(t_1, \dots, t_n)$ then $U_i \Vdash F(t_1^i, \dots, t_n^i)$ where t_k^j is simply the restriction of t_k to U_i.

Locality of validity:
we asked for the validity of atoms to be local, but Krike-Joyal forcing propagates this property to all formulae:
If there exist a covering of U by $f_i : U_i \to U$ and if for all i one has $U_i \Vdash F(t_1^i, \dots, t_n^i)$ then $U \Vdash F(t_1, \dots, t_n)$

Given a closed term (no variables) $\varphi \Vdash_{\nu, x \mapsto t^\varphi} \psi(x)$ iff $\overline{\varphi} \Vdash_\nu \psi(t)$.

B.12. Soundness

Whenever $\vdash F$ in IQC then any presheaf semantics satisfies F.

Whenever $\Gamma \vdash F$ in IQC then any presheaf semantics that satisfies Γ satisfies F as well.

The theory of rings, whose language has two binary functions $(+, .)$ two constants $0, 1$ and equality, can be interpreted in the presheaf on the topological space R which maps U to the ring $C_{U,R}$ of continuous functions from the open set U to R.

In this model, both $[\neg\forall x.((x = 0) \vee \neg(x = 0))]$ and $[\forall x \neg\neg((x = 0) \vee \neg(x = 0))]$ are both valid.

B.13. Soundness proof

Induction on the proof height, looking at every possible last rule, e.g. in natural deduction.

C The completeness part of completeness

C.1. Completeness for presheaf semantics

If every presheaf models satisfies φ
then φ is provable in **intuitionistic** logic.

Usually established by:

- equivalence with $\Omega-$models;
- construction of a canonical Kripke model.

C.2. Canonical model construction: the underlying site

Canonical site:

- **Category:** we take the Lindenbaum-Tarski algebra $\overline{\mathscr{L}}$
  - Objects: classes of provably equivalent formulas $\overline{\varphi}$.
  - Arrows: $\overline{\varphi} \leq \overline{\psi} \quad \Longleftrightarrow \quad \varphi \vdash \psi$
- **Grothendieck topology:** $\overline{\varphi} \lhd \{\psi_i\}_{i \in I}$ whenever

$$\forall \chi \left[\quad \varphi \vdash \chi \quad \textit{iff} \quad (\forall i \in I \quad \psi_i \vdash \chi) \right]$$

Think of the last line as $\varphi = \bigvee_i \psi_i$
(incorrect, because FOL formulae are finite!)

C.3. Properties of this site

The proposed site is actually a site
i.e. it enjoys the three properties.

1. $\varphi \lhd \{\varphi\}$;
2. if $\psi \vdash \varphi$ and $\varphi \lhd \{\varphi_i \mid i \in \mathscr{I}\}$ then $\psi \lhd \{\psi \wedge \varphi_i \mid i \in \mathscr{I}\}$;
3. if $\varphi \lhd \{\varphi_i \mid i \in \mathscr{I}\}$ and if for each $i \in \mathscr{I}$, $\varphi_i \lhd \{\psi_{i,k} \mid k \in \mathscr{K}_i\}$, then $\varphi \lhd \{\psi_{i,k} \mid i \in \mathscr{I}, k \in \mathscr{K}_i\}$.

C.4. Canonical model construction: the presheaf

- Put $t \equiv_\varphi t'$ in case $\varphi \vdash t = t'$.
- Denote by t^φ the equivalence class of t modulo $\equiv_\varphi$.

Canonical presheaf:

- Model $M_{\overline{\varphi}}$:
  1. Universe $|M_{\overline{\varphi}}|$:
     set of equivalence classes t^φ of closed terms;
  2. Function symbols: $f_{\overline{\varphi}}(\vec{t}^\varphi) = f(\vec{t})^\varphi$;
  3. Relation symbols: $\vec{t}^\varphi \in R_{\overline{\varphi}} \quad \Longleftrightarrow \quad \varphi \vdash R(\vec{t})$.
- Restriction. If $t^\psi \in M_{\overline{\psi}}$ and $\overline{\varphi} \leq \overline{\psi}$, put $t^\psi \upharpoonright_{\overline{\varphi}} = t^\varphi$.

C.5. The canonical presheaf is well defined

The canonical presheaf is separated. If two elements have the same restictions on each part of a cover, then they are equal.

The interpretation of atomic formulas is local. If an atomic formula holds on each part of a cover of U then it holds on U.

Observe that it is not a sheaf
(the glueing of compatible elements may not exist).

C.6. Method for the proof of completeness

$$\forall\psi\left[\ \forall\varphi\ [\ \textit{if} \quad \overline{\varphi} \Vdash \psi \quad \textit{then} \quad \varphi \vdash \psi]\ \right]$$

By induction on the formula ψ.

It is also possible to prove directly:

$$\forall\psi\left[\ \forall\varphi[\overline{\varphi} \Vdash \psi\ \textit{iff}\ \varphi \vdash \psi]\ \right]$$

What is fun is that soundness mainly uses introduction rules while completeness mainly uses elimination rules.

C.7. Variants

The method and the construction can be parametrised by a context Γ for obtaining what is called *strong completeness*:

The quotient on formula is not really needed.

Equality is not mandatory but pleasant.

Terms and constants can be eliminated in FOL with equality.

If there are no constants, one should add a denumerable set of constants. Indeed some proof steps need a *fresh constant*.

C.8. Future work

Can we construct a canonical sheaf and not just separated presheaf e.g. with the sheaf completion method that basically simply formally adds the missing global sections?
Does the proof works as well?

Initially the idea was to define couple models of first order linear logic, following some hints by Giovani Sambin, but nowadays not so many people are interested in such questions.

Thank you for your attention

Reference:

A Canonical Model for Presheaf Semantics Ivano Ciardelli *Topology, Algebra and Categories in Logic (TACL) 2011*, Jul 2011, Marseille. https://hal.inria.fr/inria-00618862/fr/