The genus of regular languages and other ideas from low-dimensional topology

Florian Deloup

Institut de Mathématiques de Toulouse, France

June 21, 2016

Joint work with Guillaume Bonfante.

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1) The genus of regular languages, 2012. Math. Str. Computer Sc., 2016.

2) The decidability of language genus computation, 2016. Available on ArXiv.

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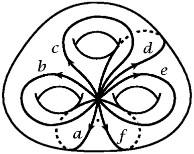

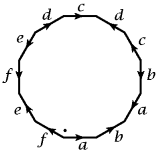

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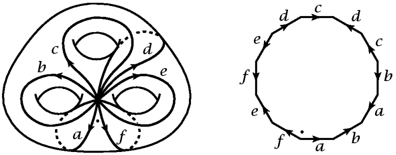

- Knots: encoding Reidemeister moves (1927) yields language(s). Particular cases: quandles, Wirtinger presentation of the fundamental group of the complement of a knot.

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Languages $\Longrightarrow$ Topology (as a tool to study languages): *topology as a language invariant*

This talk: language invariants from *low-dimensional topology*.

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Shape and space organization become central $\Longrightarrow$
Low-dimensional topology $\Longrightarrow$ Invariants of Languages from low-dimensional topology

Regular languages

Set-up:

- the class $\mathrm{Reg}_{\mathscr{A}}$ of regular languages on a finite alphabet $\mathscr{A}$.
- the class $\mathrm{DFA}_{\mathscr{A}}$ of deterministic finite automata on $\mathscr{A}$.

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- Union of two languages:
$(L, L') \mapsto L \cup L' = \{w \in A^* \mid w \in L, \mathrm{or}\ w \in L'\}$.
- Composition of two languages:
$(L, L') \mapsto LL' = \{ww' \mid w \in L, w' \in L'\}$.
- Star operation: $L \mapsto L^* = \bigcup_{n \geq 0} L^n$

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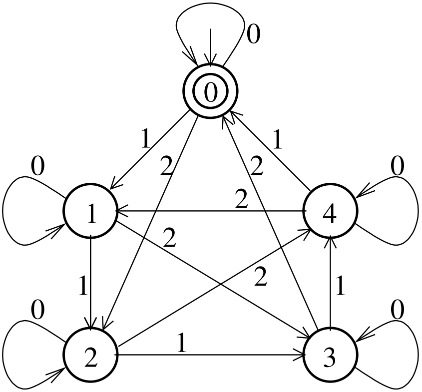

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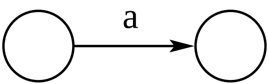
Decoration:

- label each directed edge (transition) by a letter of the alphabet $\mathscr{A}$.



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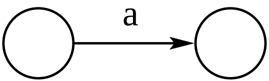
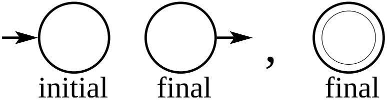


- distinguish special states: one initial state, one subset of final states.

Pictorial convention for initial and final states:



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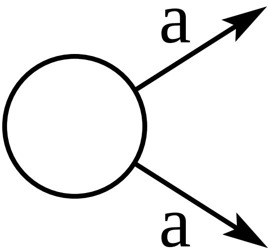
The automaton is *deterministic* if there is at most one transition labelled by a given letter.



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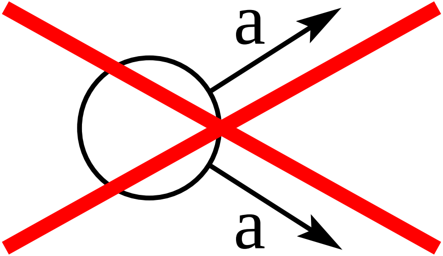

Kleene

Let $\mathtt{A} \in \mathrm{DFA}$. The language $L(\mathtt{A})$ computed by $\mathtt{A}$ is the set of all words $w \in \mathscr{A}^*$ read from (the sequence of labels of) a path starting at the initial state and ending at some final state of $\mathtt{A}$.

Theorem

The assignment $\mathtt{A} \mapsto L(\mathtt{A})$ *defines a surjective map* $\mathrm{DFA}_{\mathscr{A}} \to \mathrm{Reg}_{\mathscr{A}}$.

In the words of the topologists

Challenge: define “quantum invariants” of $L = L(\mathtt{A})$ (beyond the size of L), locally computable from a picture of any automaton $\mathtt{A}$ computing L. The computation from two equivalent automata should give the same invariant.

Why is it a challenge ? Nonlocal nature of the computation: two automata can be nonlocally equivalent.

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Let L be a regular language. There is a unique automaton $\mathtt{A} \in \mathrm{DFA}$ such that $L(\mathtt{A}) = L$ with number $|A|$ of states equal to $|L|$.

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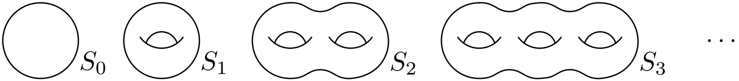
Such an automaton is called the *minimal* automaton of L.

Classification of closed oriented surfaces

Theorem (first stated ˜1850, proved ˜1920): The topological type of a closed oriented surface Σ is determined by one natural number $g(\Sigma) \in \mathbb{N}$.



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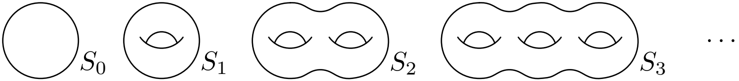

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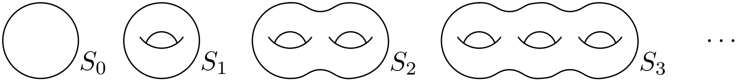

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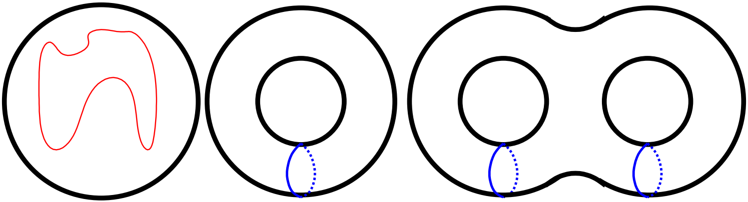
The genus $g(\Sigma)$ of Σ is the maximal number of mutually disjoint simple closed curves $C_1, \dots, C_g$ such that the complement $\Sigma - (C_1 \cup \dots \cup C_g)$ remains connected.

Examples



genus = 0 genus = 1 genus = 2 . . . . . .

Embedding an automaton into a closed oriented surface

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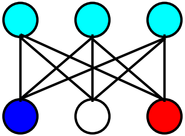
Definition

An embedding of a graph $G = (E, V)$ into a closed oriented surface Σ is a map $\varphi : (E, V) \to \Sigma$ sending injectively vertices to points, sending edges to simple arcs in Σ such that $\varphi(\partial e) = \partial\varphi(e)$ for any edge $e \in E$, $\varphi(e) \cap \varphi(e') = \varphi(\partial e) \cap \varphi(\partial e')$ for any pair $e, e' \in E$.

Example. The "Utility Graph" (complete bipartite $K_{3,3}$)



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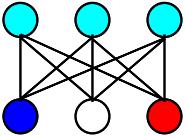

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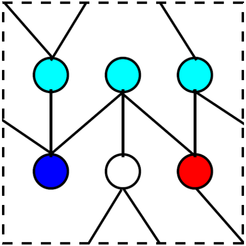

Genus of a regular language

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Let L be a regular language. The *genus* $g(L)$ is defined as

$$g(L) = \min\{g(\mathtt{A}) \mid \mathtt{A} \in \mathrm{DFA},\ L(\mathtt{A}) = L\}.$$

If $g(L) = 0$, then L is said to be *planar*.

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If $g(L) = 0$, then L is said to be *planar*.

Remark: the definition makes sense because any graph embeds into some closed oriented surface.

Recall: the simplest invariant of a regular language L is its *size*

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Question: relation between the genus and the size of a language ?

Basic observation: the automaton $\mathtt{A}$ for which a minimal embedding (with minimal genus) is realized *may not be* the minimal automaton.

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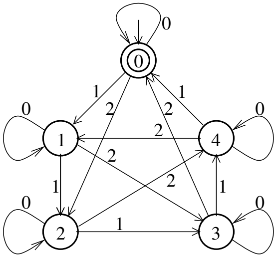
Alphabet: $\mathscr{A} = \{0, 1, 2\}$

Morphism: $\varphi : \mathscr{A}^* \to \mathbb{Z}/5\mathbb{Z}$ defined by $\varphi(aw) = \varphi(a) + \varphi(w)$ for any $a \in \mathscr{A}$, $w \in \mathscr{A}^*$.

Language: $L = \{w \in \mathscr{A}^* \mid \varphi(w) = 0 \mod 5\}$



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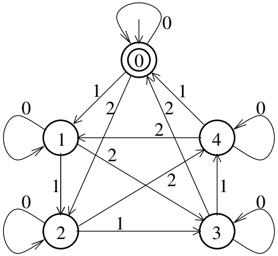

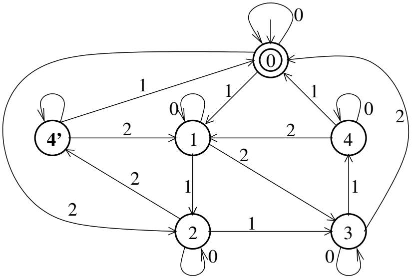

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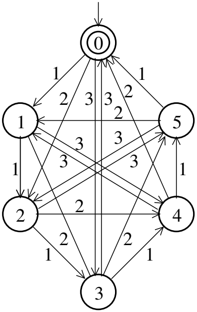

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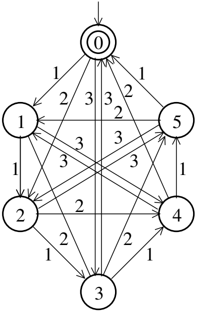

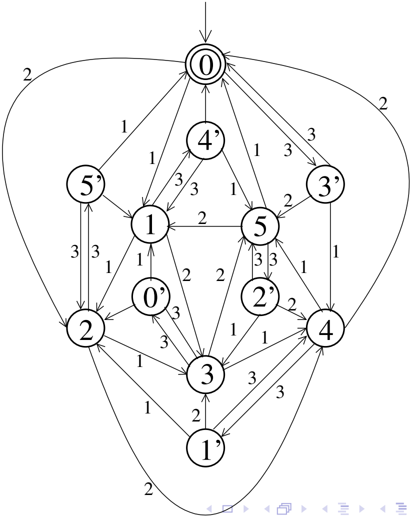

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The topological size $|L|_{\mathrm{top}}$ is regarded as "the cost" you are willing to pay for the simplest topological embedding of the representing automaton of L.

Question

Is there a universal bound $|L|_{\mathrm{top}} \leq f(|L|)$ for some explicit function f ?

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If such a function exists, it has to be at least exponential.

Theorem (2015)

There is a family of planar regular languages $(L_n)_{n\geq 1}$ such that for some $K > 2$, $|L_n|_{\mathrm{top}} = O(K^{|L_n|})$.

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The genus of a regular language is computable.

Partial positive answer:

Theorem (2012, 2015)

If the language has "no short cycles", the conjecture is true.

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Definition

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Theorem (2012)

Let L be a language on m letters. Assume that $m \geq 4$ and that L has no cycles of length ≤ 2. Then

$$1 + \frac{m-3}{6}|L| \leq g(L) \leq 1 + \frac{m-1}{2}|L|.$$

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Theorem (2012, 2015)

Let A be an alphabet of at most 2 letters. There exists a family of languages $(L_n)_{n\in\mathbb{N}}$ on alphabet A such that $g(L_n) = n$.

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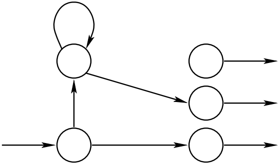
Remark. The minimal example of nonplanar language on two letters has 30 states. Conjecture. 30 is optimal.

The following definition is the "directed version" of Fellows' graph emulator (in connection with the planar finite cover conjecture in the 1980s).

Definition

Let $G = (E, V)$ be a directed graph. A *directed emulator* of G is a graph $\tilde{G} = (\tilde{E}, \tilde{V})$ such that there is a surjective simplicial map $\varphi : \tilde{G} \to G$ sending *surjectively outgoing edges* of each vertex $\tilde{v} \in \tilde{V}$ onto outgoing edges of the *image vertex* $\varphi(\tilde{v})$.



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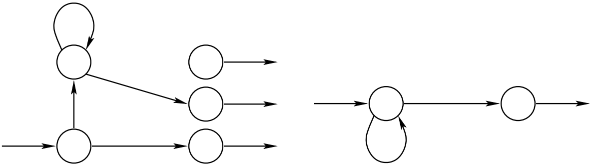

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More on this: come to Denis Kuperberg's talk.

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Graph width, *cohomology theories based on graphs*...