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Additivity for derivator K-theory

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Abstract

We prove the additivity theorem for the K-theory of triangulated derivators. This solves one of the conjectures made by Maltsiniotis in [G. Maltsiniotis, La K-théorie d'un dérivateur triangulé, in: Alexei Davydov, Michael Batanin, Michael Johnson, Stephen Lack, Amnon Neeman (Eds.), Categories in Algebra, Geometry and Physics, Conference and Workshop in honor of Ross Street's 60th Birthday, in: Contemp. Math., vol. 431, Amer. Math. Soc., 2007, pp. 341–368]. We also review some basic definitions and results in the theory of derivators in the sense of Grothendieck.

Keywords: K-theory; Derivators

Contents

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0. Introduction

Triangulated categories were introduced in the 1960s to study certain phenomena in homological algebra. Among the notable early achievements of the theory was Grothendieck's duality theorem; even the statement of the theorem makes no sense without derived categories. See [11] for more detail. In the forty years since then derived and triangulated categories have become pervasive in mathematics, appearing even in such unlikely places as mirror symmetry in mathematical physics.

Notwithstanding their many achievements, the consensus has always been that triangulated categories are deeply flawed. Ever since their introduction, in the 1960s, people have been acutely aware of their shortcomings. There is an extensive literature on the subject. Given that, right from the start, the consensus was that triangulated categories are inadequate, it is quite surprising how much has been achieved using them. Anyway, this article is about one of the proposed improvements on triangulated categories.

Over the years many people have tried to improve on the formalism of triangulated categories. The idea has always been to construct a gadget with a little more structure and more flexibility. There is an extensive list of candidates: stable model categories, A_∞-categories, DG-categories, stable Segal categories, quasicategories and triangulated derivators. Each of the constructions has its advantages and its advocates. Let us confine ourselves to saying that, among the options listed above, the striking feature of triangulated derivators is that they have the least added structure. All the other constructions functorially give rise to triangulated derivators.

To motivate our discussion of triangulated derivators let us briefly remind the reader of one of the defects which triangulated categories have. Given a triangulated category $\mathcal{T}$ we basically do not understand how to produce any others, except for trivial constructions (subcategories and quotients). In particular, if $\mathcal{T}$ is triangulated and X is an arbitrary small category, then the category $\underline{\mathsf{Hom}}(X, \mathcal{T})$ is not usually triangulated in any reasonable way. The idea of a triangulated derivator is to fix this problem by defining it away. It might be best to illustrate with an example what a triangulated derivator is.

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{D}ia$ be the 2-category of all finite partially ordered sets. Given any finite, partially ordered set X, that is given an object $X \in \mathcal{D}ia$, we can form $\underline{\mathsf{Hom}}(X^{\mathrm{op}}, \mathcal{A})$, the category of functors $X^{\mathrm{op}} \longrightarrow \mathcal{A}$. The category $\underline{\mathsf{Hom}}(X^{\mathrm{op}}, \mathcal{A})$ is abelian, and we are therefore free to form its derived category. Let

$$\mathbb{D}(X) = D\big(\underline{\mathsf{Hom}}(X^{\mathrm{op}}, \mathcal{A})\big).$$

In this way we obtain a 2-functor $\mathbb{D} : \mathcal{D}ia^{\mathrm{op}} \longrightarrow \mathcal{CAT}$. Whatever its properties, the 2-functor $\mathbb{D}$ is the prototype example of a *triangulated derivator*. Perhaps the most enlightening explanation might be that a triangulated derivator is a 2-functor $\mathbb{D}:\mathcal{D}ia^{\mathrm{op}} \longrightarrow \mathcal{CAT}$, satisfying a list of axioms that try to formalize the essential properties of the example given above. The list of axioms is presented, in some detail, in Section 1.

Among the axioms that a triangulated derivator must satisfy is the following key property. Suppose we are given a morphism $f : X \longrightarrow Y$ in $\mathcal{D}ia$; the fact that $\mathbb{D}$ is a functor says there is an induced map $f^* = \mathbb{D}(f):\mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$. A crucial feature of derivators is that this functor f^* must have a right and a left adjoint. We denote the right adjoint f_* and the left adjoint $f_!$. If we think of our motivating example, that is $\mathbb{D}(X)$ is the derived category of $\underline{\mathsf{Hom}}(X^{\mathrm{op}}, \mathcal{A})$, the existence of the adjoints says that, given a chain complex C of objects in $\underline{\mathsf{Hom}}(X^{\mathrm{op}}, \mathcal{A})$ and a morphism of partially ordered sets $f : X \longrightarrow Y$, then there are two chain complexes f_*C, $f_!C$ of objects in the abelian category $\underline{\mathsf{Hom}}(Y^{\mathrm{op}}, \mathcal{A})$ that come for free. If $Y = e$ is the 1-point, terminal category then the extension of C to f_*C is a homotopy limit, while the extension of C to $f_!C$ is a homotopy colimit. In the world of triangulated derivators, homotopy limits and colimits exist and are part of the structure. In the next couple of paragraphs we will explain how, starting from a triangulated derivator $\mathbb{D}$, one can construct new derivators and maps between them. As we already noted, in the world of triangulated categories no such construction is known.

Let $\Box$ be the partially ordered set given by the commutative square

$$\begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,1). \end{array}$$

The subcategory $\ulcorner$ is

$$\begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \\ (1,0), & & \end{array}$$

and $\ulcorner$ has a subcategory I which is

$$(0,0) \longleftarrow (0,1).$$

Let $f : I \longrightarrow \ulcorner$ and $g : \ulcorner \longrightarrow \Box$ be the inclusions. Given an object $a \in \mathbb{D}(I)$ we can form first the object $f_*a \in \mathbb{D}(\ulcorner)$, and then $g_! f_*a \in \mathbb{D}(\Box)$. In the special case where $\mathbb{D}$ is our motivating derivator, that is where $\mathbb{D}(X) = D(\underline{\mathsf{Hom}}(X^{\mathrm{op}}, \mathcal{A}))$, it is simple enough to work out explicitly what $g_! f_*a$ is. The object a is a functor from I^{op} to the category of chain complexes in $\mathcal{A}$. That is, we have a map of chain complexes in $\mathcal{A}$

$$C_{0,0} \xrightarrow{i} C_{0,1}.$$

Replacing a by an isomorphic object in $\mathbb{D}(I)$ we may assume the map i is a (split) monomorphism in each degree. The object $f_*a \in \mathbb{D}(\ulcorner)$ turns out to be the diagram of chain complexes

$$\begin{array}{ccc} C_{0,0} & \xrightarrow{\ i\ } & C_{0,1} \\ \downarrow & & \\ 0 & & \end{array}$$

and the object $g_! f_* a \in \mathbb{D}(\square)$ is nothing more nor less than the diagram

$$\begin{array}{ccc} C_{0,0} & \xrightarrow{\ i\ } & C_{0,1} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \frac{C_{0,1}}{C_{0,0}}. \end{array}$$

In other words the object $g_! f_* a \in \mathbb{D}(\square)$ can be thought of as a short exact sequence of chain complexes, or as a triangle.

So far we have seen how to construct short exact sequences; in this paragraph we form a derivator $\mathbb{E}\mathrm{xact}(\mathbb{D})$ of short exact sequences in $\mathbb{D}$, and some maps out of it. We begin with the maps. Given a derivator $\mathbb{D}$ it is possible to define a new derivator $\mathbb{D}_\square$ by the formula $\mathbb{D}_\square(X) = \mathbb{D}(X \times \square)$. The inclusions of the objects $(0,0) \in \square$ and $(1,1) \in \square$ induce functors

$$\mathbb{D}(X \times \square) \xrightarrow{p_{(0,0)}} \mathbb{D}\big(X \times (0,0)\big), \qquad \mathbb{D}(X \times \square) \xrightarrow{p_{(1,1)}} \mathbb{D}\big(X \times (1,1)\big).$$

That is, we have two maps $p_{(0,0)} : \mathbb{D}_\square \longrightarrow \mathbb{D}$ and $p_{(1,1)} : \mathbb{D}_\square \longrightarrow \mathbb{D}$. We can also form yet another derivator $\mathbb{E}\mathrm{xact}(\mathbb{D})$. For every X we make $\mathbb{E}\mathrm{xact}(\mathbb{D})(X)$ a full subcategory of $\mathbb{D}_\square(X)$. The objects of $\mathbb{E}\mathrm{xact}(\mathbb{D})(X)$ are given by the following rule:

$$\mathbb{E}\mathrm{xact}(\mathbb{D})(X) = \left\{ b \in \mathbb{D}_\square(X) \;\middle|\; \begin{array}{l} \text{there exists } a \in \mathbb{D}(X \times I)\text{, and an} \\ \text{isomorphism } b \simeq (1_X \times g)_!(1_X \times f)_* a \end{array} \right\}.$$

We have two composites

$$\begin{array}{ccccc} & & & \overset{p_{(0,0)}}{\nearrow} & \mathbb{D} \\ \mathbb{E}\mathrm{xact}(\mathbb{D}) & \hookrightarrow & \mathbb{D}_\square & & \\ & & & \underset{p_{(1,1)}}{\searrow} & \mathbb{D}. \end{array}$$

This concludes the formal facts about derivators which we need in order to state our main theorem.

To each derivator $\mathbb{D}$, Keller and Maltsiniotis assigned a K-theory space $K(\mathbb{D})$. The above diagram of derivators induces a diagram of spaces

$$\begin{array}{ccccc} & & & \xrightarrow{K(p_{(0,0)})} & K(\mathbb{D}) \\ K(\mathbb{E}\mathrm{xact}(\mathbb{D})) & \longrightarrow & K(\mathbb{D}_\square) & & \\ & & & \xrightarrow[K(p_{(1,1)})]{} & K(\mathbb{D}). \end{array}$$

The main theorem of this article is that the map $K(\mathbb{E}\mathrm{xact}(\mathbb{D})) \longrightarrow K(\mathbb{D}) \times K(\mathbb{D})$, induced by the above, is a homotopy equivalence. This was conjectured by Maltsiniotis [20, Conjecture 3, p. 8]; he called it the "additivity conjecture." It should be mentioned that Garkusha [6] proved a partial result; he showed that the additivity conjecture is true for the class of triangulated derivators that come from complicial biWaldhausen categories in the sense of Thomason and Trobaugh [32]. The proof we give here works for general triangulated derivators; see Theorem 3.17.

We should say a little about the history of the subject. The notion of derivator was introduced by Grothendieck in [8,9]. Independently of Grothendieck but later B. Keller [15] and J. Franke [5] studied similar constructions. A. Heller [12–14] was aware of Grothendieck's earlier work, but his theory is still quite independent in spirit. Inspired by the work of A. Grothendieck and J. Franke, G. Maltsiniotis defined the notion of triangulated derivator; his axioms Der 1–Der 7 are given in Section 1 of this paper. He furthermore proved that these axioms are sufficient to give rise to a canonical structure of triangulated category; see Theorem 1.17 and Corollary 7.10 for more precise statements, and [19] for the proofs. During the years 2001–2002 Cisinski, Keller and Maltsiniotis undertook an intensive study of the subject in the *Algèbre et topologie homotopiques* seminar at the University of Paris 7. One development to come out of this seminar was the idea of associating a K-theory to every triangulated derivator $\mathbb{D}$. The idea, as we have already said, was due to Keller and Maltsiniotis. In his manuscript [20], on derivator K-theory, Maltsiniotis states three conjectures. The main theorem of this article will prove his additivity conjecture.

Lastly we should explain the structure of the article. After the introduction comes Section 1, which contains the axioms of a triangulated derivator. Then follows the K-theoretic component of the article. Using the techniques developed by Neeman, in his articles on triangulated K-theory [21–28,30], we give a very simple proof of additivity; it is so formal that there is not much to check to see that it works in the derivator context. It also works to give a new proof of additivity in other contexts, for example for Waldhausen's K-theory; but there already are many other proofs of the additivity theorem for Waldhausen's K-theory.

There are facts about triangulated derivators which come up in the proof. These are not difficult to check, but the checking does involve developing techniques for dealing with triangulated derivators. The theory becomes very easy if the derivator happens to be the motivating example we discussed earlier, but constructing it from the axioms requires some work. In Sections 2–5 we give the homotopy theoretic component of the proof, leaving the lemmas about triangulated derivators till later. We are careful to highlight such occurrences, for the reader's convenience. Every time we use an assertion about triangulated derivators, leaving the proof till later, we warn the reader with a Caution, and give references to the later parts of the article where the assertion is proved. Starting with Section 6 we study the formalism of triangulated derivators, and provide the proofs promised in the Cautions of Sections 2–5.

Our treatment of triangulated derivators is minimal; we develop almost only those aspects of the theory we absolutely need. The only exception is Appendix A, which proves that $K(\mathbb{D})$ is always an infinite loop space. This is not a fact we need in the proof of additivity. Since Garkusha, in his paper [6], said that for a general triangulated derivator he could not make $K(\mathbb{D})$ into a spectrum,[1] we felt we ought to include the argument here. But the infinite loop space structure is irrelevant to the proof of additivity, and hence it is consigned to Appendix A.

A fuller account, of the theory of triangulated derivators, should appear in [18,19].

1. Triangulated derivators

In this section, we recall the definition of a (triangulated) derivator following Grothendieck [8,9]. The reader will find a more complete exposition in [18–20]. The theory of derivators is also very close to A. Heller's theory of homotopy theories (see [12–14]), and the theory of triangulated derivators is similar (and in some sense equivalent) to the notion of systems of triangulated categories defined by Franke [5].

1.1. We denote by $\mathcal{C}at$ the 2-category of small categories. The empty category will be written $\emptyset$, and the 1-point category (i.e. the category with one object and one identity morphism) will be written e. If X is a small category, X^{op} is the opposite category associated to X. If $u : X \longrightarrow Y$ is a functor, and if y is an object of Y, one defines the category X/y as follows. The objects of X/y are the pairs (x, f), where x is an object of X, and f is a map in Y from $u(x)$ to y. A map from (x, f) to (x', f') in X/y is a map $\xi : x \longrightarrow x'$ in X such that $f'u(\xi) = f$. The composition law in X/y is induced by the composition law in X. Dually, one defines $y\backslash X$ by the formula $y\backslash X = (X^{\mathrm{op}}/y)^{\mathrm{op}}$. We then have canonical functors

$$X/y \longrightarrow X \quad \text{and} \quad y\backslash X \longrightarrow X$$

defined by the projection $(x, f) \longmapsto x$. One can check easily that one gets the following pullback squares of categories

$$\begin{array}{ccc} X/y & \longrightarrow & X \\ {\scriptstyle u/y}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ Y/y & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} y\backslash X & \longrightarrow & X \\ {\scriptstyle y\backslash u}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ y\backslash Y & \longrightarrow & Y. \end{array}$$

If X is any category we let $p_X : X \longrightarrow e$ be the canonical functor. Given any object x of X we will write $x : e \longrightarrow X$ for the unique functor which sends the object of e to x. As we will often want to think of small categories as spaces the objects of X, or equivalently the functors $e \longrightarrow X$, will be called the *points* of X.

If X and Y are two categories we denote by $\underline{\mathsf{Hom}}(X, Y)$ the category of functors from X to Y.

1.2. We assume that the reader is familiar with the basic notions of 2-categories and of 2-functors (see for example [17, XII.3]). If $\mathcal{C}$ is a 2-category one writes $\mathcal{C}^{\mathrm{op}}$ for its dual 2-category: $\mathcal{C}^{\mathrm{op}}$ has

[1] Garkusha shows that $K(\mathbb{D})$ is a spectrum if $\mathbb{D}$ comes from a biWaldhausen complicial category.

the same objects as $\mathcal{C}$ and, for any two objects X and Y, the category $\underline{\mathsf{Hom}}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ of 1-arrows from X to Y in $\mathcal{C}^{\mathrm{op}}$ is $\underline{\mathsf{Hom}}_{\mathcal{C}}(Y, X)^{\mathrm{op}}$.

A *category of diagrams* is a full subcategory $\mathcal{D}ia$ of the category of small categories satisfying the following axioms.

D0 Any finite partially ordered set (seen as a category) is in $\mathcal{D}ia$.
D1 $\mathcal{D}ia$ is stable under finite sums and under pullbacks.
D2 For any X in $\mathcal{D}ia$ and any point x of X, the categories X/x and $x\backslash X$ are in $\mathcal{D}ia$.
D3 For any X in $\mathcal{D}ia$, X^{op} is in $\mathcal{D}ia$.
D4 For any Grothendieck fibration (in the sense of [7, Exposé VI]) $p : X \longrightarrow Y$, such that Y is in $\mathcal{D}ia$, and such that all the fibers of p are in $\mathcal{D}ia$, the category X is in $\mathcal{D}ia$.

We will consider $\mathcal{D}ia$ as a 2-category with the structure induced by the canonical structure of 2-category on $\mathcal{C}at$. Some examples of categories of diagrams are $\mathcal{C}at$, the 2-category $\mathcal{C}at_f$ of finite categories and the 2-category $\mathcal{P}oset_f$ of finite partially ordered sets.

In all that follows we assume that a fixed category of diagrams $\mathcal{D}ia$ is given.

Definition 1.3. A *prederivator* (of domain $\mathcal{D}ia$) is a strict 2-functor from $\mathcal{D}ia^{\mathrm{op}}$ to the 2-category of categories

$$\mathbb{D} : \mathcal{D}ia^{\mathrm{op}} \longrightarrow \mathcal{CAT}.$$

More explicitly: For any small category $X \in \mathcal{D}ia$ one has a category $\mathbb{D}(X)$. For any functor $u : X \longrightarrow Y$ in $\mathcal{D}ia$ one gets a functor

$$u^* = \mathbb{D}(u) : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X).$$

For any morphism of functors

$$X \underset{v}{\overset{u}{\underset{\Downarrow \alpha}{\rightrightarrows}}} Y,$$

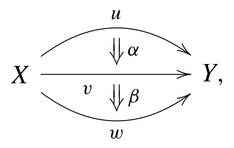
one has a morphism of functors

$$\mathbb{D}(X) \underset{v^*}{\overset{u^*}{\underset{\alpha^* \Uparrow}{\leftleftarrows}}} \mathbb{D}(Y).$$

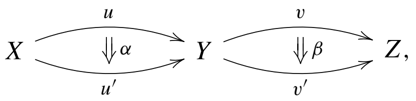
Of course, all these data have to verify some coherence conditions, namely:

(a) For any composable maps in $\mathcal{D}ia$, $X \xrightarrow{u} Y \xrightarrow{v} Z$,

$$(vu)^* = u^* v^*, \qquad 1_X^* = 1_{\mathbb{D}(X)}.$$

(b) For any composable 2-cells in $\mathcal{D}ia$, 

$$(\beta\alpha)^* = \alpha^*\beta^*, \qquad 1_u^* = 1_{u^*}.$$

(c) For any 2-diagram in $\mathcal{D}ia$, 

$$(\beta\alpha)^* = \alpha^*\beta^*.$$

1.4. Let X be a small category in $\mathcal{D}ia$ and let x be an object of X. Given an object F of $\mathbb{D}(X)$ we will write $F_x = x^*(F)$. The object F_x will be called the *fiber of F at the point x*.

1.5. For a prederivator $\mathbb{D}$, define its *opposite* to be the prederivator $\mathbb{D}^{\mathrm{op}}$ given by the formula

$$\mathbb{D}^{\mathrm{op}}(X) = \mathbb{D}\big(X^{\mathrm{op}}\big)^{\mathrm{op}} \quad \text{for all } X \in \mathcal{D}ia\,.$$

Example 1.6. Let $\mathcal{M}$ be any category. We obtain a prederivator

$$X \longmapsto \mathcal{M}(X)$$

where $\mathcal{M}(X) = \underline{\mathsf{Hom}}(X^{\mathrm{op}}, \mathcal{M})$ is the category of presheaves over X with values in $\mathcal{M}$.

Let $\mathcal{M}$ be a category endowed with a class of maps called weak equivalences (for example $\mathcal{M}$ can be the category of bounded complexes in a given abelian category, and the weak equivalences can be the quasi-isomorphisms). For any category X in $\mathcal{D}ia$, we define the weak equivalences in $\mathcal{M}(X)$ to be the morphisms of presheaves which are termwise weak equivalences in $\mathcal{M}$. We can then define $\mathbb{D}_{\mathcal{M}}(X)$ as the localization of $\mathcal{M}(X)$ by the weak equivalences. It is clear that, for any functor $u : X \longrightarrow Y$ in $\mathcal{D}ia$, the inverse image functor

$$\mathcal{M}(Y) \longrightarrow \mathcal{M}(X), \qquad F \longmapsto u^*(F) = F \circ u.$$

respects weak equivalences, so that it induces a well-defined functor

$$u^* : \mathbb{D}_{\mathcal{M}}(Y) \longrightarrow \mathbb{D}_{\mathcal{M}}(X).$$

The 2-functoriality of localization implies that we have a prederivator $\mathbb{D}_{\mathcal{M}}$. All the (pre)derivators we know can be obtained this way. Note that, by definition, the category $\mathbb{D}_{\mathcal{M}}(e)$ is the homotopy category of $\mathcal{M}$, that is the universal category obtained from $\mathcal{M}$ by inverting the weak equivalences.

Definition 1.7. Let $\mathbb{D}$ be a prederivator. A map $u : X \longrightarrow Y$ in $\mathcal{D}ia$ has a *cohomological direct image functor* (respectively a *homological direct image functor*) in $\mathbb{D}$ if the inverse image functor

$$u^* : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$$

has a right adjoint (respectively a left adjoint)

$$u_* : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y) \quad \big(\text{respectively } u_! : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)\big),$$

called the *cohomological direct image functor* (respectively *homological direct image functor*) associated to u.

1.8. Let X be a category in $\mathcal{D}ia$, and $p = p_X : X \longrightarrow e$. If p has a cohomological direct image functor in $\mathbb{D}$, one defines for any object F of $\mathbb{D}(X)$ the object of global sections of F as

$$\Gamma_*(X, F) = p_*(F).$$

The object $\Gamma_*(X, F)$ can be viewed as some kind of cohomology of the 'space' X with coefficients in F. This also corresponds to what algebraic topologists think of as the homotopy limit of F indexed by the category X. Dually, if p has a homological direct image in $\mathbb{D}$ then, for any object F of $\mathbb{D}(X)$, one sets

$$\Gamma_!(X, F) = p_!(F),$$

which can be regarded as the homology of X with coefficients in F or as a homotopy colimit, according to the taste of the reader.

1.9. Let $\mathbb{D}$ be a prederivator, and let

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle u'}\downarrow & \swarrow \alpha & \downarrow{\scriptstyle u} \\ Y' & \xrightarrow[\ w\ ]{} & Y \end{array}$$

be a 2-diagram in $\mathcal{D}ia$. By 2-functoriality one obtains the following 2-diagram

$$\begin{array}{ccc} \mathbb{D}(X') & \xleftarrow{\ v^*\ } & \mathbb{D}(X) \\ {\scriptstyle u'^*}\uparrow & \alpha^* \nearrow & \uparrow{\scriptstyle u^*} \\ \mathbb{D}(Y') & \xleftarrow[\ w^*\ ]{} & \mathbb{D}(Y). \end{array}$$

If we assume that the functors u and u' both have cohomological direct images in $\mathbb{D}$ one can define the *base change morphism* induced by α

$$\beta : w^* u_* \longrightarrow u'_* v^*$$

$$\begin{array}{ccc} \mathbb{D}(X') & \xleftarrow{\;v^*\;} & \mathbb{D}(X) \\ {\scriptstyle u'_*}\big\downarrow & \beta \;\Nwarrow & \big\downarrow{\scriptstyle u_*} \\ \mathbb{D}(Y') & \xleftarrow[\;w^*\;]{} & \mathbb{D}(Y) \end{array}$$

as follows. The counit $u^*u_* \longrightarrow 1_{\mathbb{D}(X)}$ induces a morphism $v^*u^*u_* \longrightarrow v^*$, and by composition with α^*u_*, a morphism $u'^*w^*u_* \longrightarrow v^*$. This gives β by adjunction.

This construction will be used in the following situation. Let $u : X \longrightarrow Y$ be a map in $\mathcal{D}ia$, and let y be a point of Y. According to 1.1 we have a functor $j : X/y \longrightarrow X$, defined by the formula $j(x, f) = x$, where $f : u(x) \longrightarrow y$ is a morphism in Y. If $p : X/y \longrightarrow e$ is the canonical map one obtains the 2-diagram below, where α denotes the 2-cell defined by the formula $\alpha_{(x,f)} = f$

$$\begin{array}{ccc} X/y & \xrightarrow{\;j\;} & X \\ {\scriptstyle p}\big\downarrow & \Swarrow\; \alpha & \big\downarrow{\scriptstyle u} \\ e & \xrightarrow[\;y\;]{} & Y. \end{array}$$

With the notations of 1.8, the associated base change morphism gives rise to a canonical morphism

$$u_*(F)_y \longrightarrow \Gamma_*(X/y, F/y)$$

for any object F of $\mathbb{D}(X)$, where $F/y = j^*(F)$. Dually one has canonical morphisms

$$\Gamma_!(y\backslash X, y\backslash F) \longrightarrow u_!(F)_y$$

where $y\backslash F = k^*(F)$ and k denotes the canonical functor from $y\backslash X$ to X.

1.10. Let X and Y be two categories in $\mathcal{D}ia$. Using the 2-functoriality of $\mathbb{D}$, one defines a functor

$$d_{X,Y} : \mathbb{D}(X \times Y) \longrightarrow \underline{\mathsf{Hom}}\big(X^{\mathrm{op}}, \mathbb{D}(Y)\big)$$

as follows. Setting $X' = X \times Y$, we have a canonical functor

$$\underline{\mathsf{Hom}}(Y, X')^{\mathrm{op}} \longrightarrow \underline{\mathsf{Hom}}\big(\mathbb{D}(X'), \mathbb{D}(Y)\big)$$

which defines a functor

$$\underline{\mathsf{Hom}}(Y, X')^{\mathrm{op}} \times \mathbb{D}(X') \longrightarrow \mathbb{D}(Y)$$

and then a functor

$$\mathbb{D}(X') \longrightarrow \underline{\mathsf{Hom}}\big(\underline{\mathsf{Hom}}(Y, X')^{\mathrm{op}}, \mathbb{D}(Y)\big).$$

Using the canonical functor

$$X \longrightarrow \underline{\mathsf{Hom}}(Y, X \times Y), \qquad x \longmapsto \big(y \longmapsto (x, y)\big),$$

this gives the desired functor.

In particular, for any category X in $\mathcal{D}ia$, one gets a functor

$$d_X = d_{X,e} : \mathbb{D}(X) \longrightarrow \underline{\mathsf{Hom}}\big(X^{\mathrm{op}}, \mathbb{D}(e)\big).$$

If F is an object of $\mathbb{D}(X)$, then $d_X(F)$ is the presheaf on X with values in $\mathbb{D}(e)$ defined by

$$x \longmapsto F_x.$$

Given a presheaf G on X with value in $\mathbb{D}(e)$, we will say that an object F of $\mathbb{D}(X)$ is *locally of shape* G if $d_X(F)$ is isomorphic (as a presheaf) to G.

Definition 1.11. A *derivator* is a prederivator $\mathbb{D}$ with the following properties.

Der 1 (*Non-triviality axiom*). For any finite set I and any family $\{X_i,\ i \in I\}$ of categories in $\mathcal{D}ia$, the canonical functor

$$\mathbb{D}\Big(\coprod_{i \in I} X_i\Big) \longrightarrow \prod_{i \in I} \mathbb{D}(X_i)$$

is an equivalence of categories.

Der 2 (*Conservativity axiom*). For any category X in $\mathcal{D}ia$, the family of functors

$$x^* : \mathbb{D}(X) \longrightarrow \mathbb{D}(e), \qquad F \longmapsto x^*(F) = F_x,$$

corresponding to the points x of X is conservative. In other words: If $\varphi : F \longrightarrow G$ is a morphism in $\mathbb{D}(X)$, such that for any point x of X the map $\varphi_x : F_x \longrightarrow G_x$ is an isomorphism in $\mathbb{D}(e)$, then φ is an isomorphism in $\mathbb{D}(X)$.

Der 3 (*Direct image axiom*). Any functor in $\mathcal{D}ia$ has a cohomological direct image functor and a homological direct image functor in $\mathbb{D}$ (see 1.7).

Der 4 (*Base change axiom*). For any functor $u : X \longrightarrow Y$ in $\mathcal{D}ia$, any point y of Y and any object F in $\mathbb{D}(X)$, the canonical base change morphisms (see 1.9)

$$u_*(F)_y \longrightarrow \Gamma_*(X/y, F/y) \quad \text{and} \quad \Gamma_!(y\backslash X, y\backslash F) \longrightarrow u_!(F)_y$$

are isomorphisms in $\mathbb{D}(e)$.

Der 5 (*Essential surjectivity axiom*). Let I be the category corresponding to the graph

$$0 \longleftarrow 1.$$

For any category X in $\mathcal{D}ia$, the functor

$$d_{I,X} : \mathbb{D}(I \times X) \longrightarrow \underline{\mathsf{Hom}}\big(I^{\mathrm{op}}, \mathbb{D}(X)\big)$$

(see 1.10) is full and essentially surjective.

1.12. A functor $j : U \longrightarrow X$ is an *open immersion* if it is injective on objects, fully faithful, and if, for any morphism $x \longrightarrow j(u')$ in the category X, we have that $x = j(u)$ for some $u \in U$. Dually a functor $i : Z \longrightarrow X$ is a *closed immersion* if $i^{\mathrm{op}} : Z^{\mathrm{op}} \longrightarrow X^{\mathrm{op}}$ is an open immersion. One can show easily that open immersions and closed immersions are stable by composition and pullback.

Definition 1.13. A derivator $\mathbb{D}$ is *pointed* if it satisfies the following property.

Der 6 (*Exceptional axiom*). For any closed immersion $i : Z \longrightarrow X$ in $\mathcal{D}ia$, the cohomological direct image functor

$$i_* : \mathbb{D}(Z) \longrightarrow \mathbb{D}(X)$$

has a right adjoint

$$i^! : \mathbb{D}(X) \longrightarrow \mathbb{D}(Z)$$

called the *exceptional inverse image functor* associated to i. Dually, for any open immersion $j : U \longrightarrow X$ in $\mathcal{D}ia$, the homological direct image functor

$$j_! : \mathbb{D}(U) \longrightarrow \mathbb{D}(X)$$

has a left adjoint

$$j^? : \mathbb{D}(X) \longrightarrow \mathbb{D}(U)$$

called the *coexceptional inverse image functor* associated to j.

1.14. Let $\square$ be the category given by the commutative square

$$\begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,1). \end{array}$$

We are interested in two of its subcategories. The subcategory $\lrcorner$ is

$$\begin{array}{ccc} & & (0,1) \\ & & \uparrow \\ (1,0) & \longleftarrow & (1,1), \end{array}$$

and $\ulcorner$ is the subcategory

$$\begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \\ (1,0). & & \end{array}$$

We thus have two inclusion functors

$$\sigma : \lrcorner \longrightarrow \square \quad \text{and} \quad \tau : \ulcorner \longrightarrow \square$$

(σ is an open immersion and τ a closed immersion). A *global commutative square* in $\mathbb{D}$ is an object of $\mathbb{D}(\square)$. A global commutative square C in $\mathbb{D}$ is thus locally of shape

$$\begin{array}{ccc} C_{0,0} & \longrightarrow & C_{0,1} \\ \downarrow & & \downarrow \\ C_{1,0} & \longrightarrow & C_{1,1} \end{array}$$

in $\mathbb{D}(e)$.

A global commutative square C in $\mathbb{D}$ is *cartesian* (or a *homotopy pullback square*) if, for any global commutative square B in $\mathbb{D}$, the canonical map

$$\mathsf{Hom}_{\mathbb{D}(\square)}(B, C) \longrightarrow \mathsf{Hom}_{\mathbb{D}(\lrcorner)}\big(\sigma^*(B), \sigma^*(C)\big)$$

is bijective. Dually, a global commutative square B in $\mathbb{D}$ is *cocartesian* (or a *homotopy pushout square*) if, for any global commutative square C in $\mathbb{D}$, the canonical map

$$\mathsf{Hom}_{\mathbb{D}(\square)}(B, C) \longrightarrow \mathsf{Hom}_{\mathbb{D}(\ulcorner)}\big(\tau^*(B), \tau^*(C)\big)$$

is bijective.

As $\square$ is isomorphic to its opposite $\square^{\mathrm{op}}$, one can see that a global commutative square in $\mathbb{D}$ is cartesian (respectively cocartesian) if and only if it is cocartesian (respectively cartesian) as a global commutative square in $\mathbb{D}^{\mathrm{op}}$.

Definition 1.15. A derivator $\mathbb{D}$ is *triangulated* if it is pointed and satisfies the following axiom:

Der 7 (*Stability axiom*). A global commutative square in $\mathbb{D}$ is cartesian if and only if it is cocartesian.

Example 1.16. By [3], any Quillen model category gives rise to a derivator with $\mathcal{D}ia = \mathcal{C}at$ (defined as in 1.6).

Similarly any complicial biWaldhausen category in the sense of Thomason and Trobaugh [32] gives rise to a triangulated derivator with $\mathcal{D}ia = \mathcal{P}oset_f$; see [2]. One can use this to define, for any exact category $\mathcal{E}$, a triangulated derivator $\mathbb{D}^b\,\mathcal{E}$ such that $\mathbb{D}^b\,\mathcal{E}(e)$ is the derived category of $\mathcal{E}$. See [16] for a direct proof.

Theorem 1.17 *(Maltsiniotis). For any triangulated derivator $\mathbb{D}$, the category $\mathbb{D}(e)$ has a canonical structure of a triangulated category.*

The proof of Theorem 1.17 will appear in [19]. We also note that a very similar result has been proved by Franke [5].

Remark 1.18. Maltsiniotis' proof is explicit; it tells us concretely how to construct the distinguished triangles in $\mathbb{D}(e)$. The reader can find a description in 7.9. It may be shown from Theorem 1.17 that, for any category X in $\mathcal{D}ia$ (not only $X = e$), the category $\mathbb{D}(X)$ is triangulated; also the functors $j^*: \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$ are all triangulated. See Corollary 7.10.

2. Regions

Let $\mathbb{D}$ be any triangulated derivator. Maltsiniotis associated to it a K-theory space $K(\mathbb{D})$, and it is time to remind ourselves how this is done. Since we want to set up the simplicial preliminaries in a framework general enough so that we can prove something, we will divide this into three sections. In this section and the two that follow we will explain some of the simplicial background we will need for the proof of the Maltsiniotis additivity conjecture [20, Conjecture 3, p. 8]. We try to keep the notation as consistent as possible with Neeman's work on triangulated K-theory [21–28]. However, familiarity with Neeman's work is not assumed; the account we give here is hopefully sufficiently self-contained to help the non-experts. The reader might also wish to look at [30], which gives another elementary account of triangulated K-theory. The survey in [30] is more ambitious than the short version we will attempt here; the longer and more thorough description tries to outline the results, the methods and the open problems in the field.

In this section we discuss the subcategories of $\mathcal{D}ia$ that arise in the definition of K-theory. The category $\mathbb{Z}$ will be the ordered set $\mathbb{Z}$ of all integers; the objects are the integers $n \in \mathbb{Z}$, and the morphisms are given by the usual formula

$$\mathsf{Hom}(m,n) = \begin{cases} \{1\} & \text{if } m \leqslant n, \\ \emptyset & \text{otherwise.} \end{cases}$$

The subcategories we care about will all be subcategories of $\mathbb{Z}^3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Let us therefore make this a definition:

Definition 2.1. We define the following three concepts:

(i) A *region* is a full subcategory of $\mathbb{Z}^3$.
(ii) Let $k \in \mathbb{Z}$ be any integer. The *slice* ${}_k\mathcal{R} \subset \mathcal{R}$ of the region $\mathcal{R} \subset \mathbb{Z}^3$ is the full subcategory

$${}_k\mathcal{R} = \big\{(x, y, k) \in \mathcal{R}\big\}.$$

(iii) A *morphism* from a region $\mathcal{R} \subset \mathbb{Z}^3$ to a region $\mathcal{R}' \subset \mathbb{Z}^3$ is a functor $F : \mathcal{R} \longrightarrow \mathcal{R}'$ satisfying the following restriction. First of all for every $k \in \mathbb{Z}$ we must have $F({}_k\mathcal{R}) \subset {}_k\mathcal{R}'$; that is, F respects the slices. But furthermore, for every $k \in \mathbb{Z}$ there exists a commutative square

$$\begin{array}{ccc} {}_k\mathcal{R} & \xrightarrow{F} & {}_k\mathcal{R}' \\ \downarrow & & \downarrow \\ \mathbb{Z}^2 & \xrightarrow[f_1 \times f_2]{} & \mathbb{Z}^2 \end{array}$$

where $f_1 : \mathbb{Z} \longrightarrow \mathbb{Z}$ and $f_2 : \mathbb{Z} \longrightarrow \mathbb{Z}$ are two functors (that is increasing maps). Note that $f_1 = {}_k f_1$ and $f_2 = {}_k f_2$ may depend on k.

Remark 2.2. In this way the regions $\mathcal{R} \subset \mathbb{Z}^3$ form a category (even a 2-category), which we will call $\mathcal{R}egions$.

Example 2.3. For example we can take the region $\mathcal{R}_{mn} = [0, n] \times [0, m] \times [0]$ in $\mathbb{Z}^3$. The picture is

$$\begin{array}{ccccc} (0,m) & \longrightarrow & \cdots & \longrightarrow & (n,m) \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ (0,0) & \longrightarrow & \cdots & \longrightarrow & (n,0). \end{array}$$

Note that since the region $\mathcal{R}_{mn}$ is planar, that is contained in $\mathbb{Z}^2 \subset \mathbb{Z}^3$, we omit the third coordinate in the picture.

As is customary, we let Δ be the category whose objects are the finite totally-ordered sets, and whose morphisms are the order-preserving maps.

Definition 2.4. An *r-fold cosimplicial region* is a functor

$$\Delta^r \longrightarrow \mathcal{R}egions.$$

Example 2.5. Let us denote by $[0, n]$ the object of Δ

$$0 < 1 < 2 < \cdots < n.$$

The natural embedding is a functor $[0, n] \longrightarrow \mathbb{Z}$. Any object of Δ is canonically isomorphic to a unique $[0, n] \subset \mathbb{Z}$. The map sending $[0, n] \in \Delta$ to

$$[0,n] \times [0] \times [0] \subset \mathbb{Z}^3$$

is a functor $F : \Delta \longrightarrow \mathcal{R}egions$.

Example 2.6. Slightly more interesting is the fact that we can concatenate. There is a functor $D : \Delta \times \Delta \longrightarrow \Delta$ which takes a pair of ordered sets S, T to the disjoint union $S \cup T$, with the order that any element of S is less than any element of T. We can iterate this to form functors $D^i : \Delta^{i+1} \longrightarrow \Delta$, by the rule

$$D^i = D\bigl(D^{i-1} \times 1\bigr).$$

The resulting functor D^i takes an object of Δ^{i+1}, that is $(i+1)$ ordered sets $S_0, S_1, \dots, S_i$, and forms the disjoint union, with the order that the elements in S_j are less than the elements of S_k if and only if $j < k$.

Example 2.5 gave us a functor $F : \Delta \longrightarrow \mathcal{R}egions$, and the paragraph above produced a functor $D^i : \Delta^{i+1} \longrightarrow \Delta$. Composing the two we have a functor $F \circ D^i : \Delta^{i+1} \longrightarrow \mathcal{R}egions$. We have an $(i+1)$-fold cosimplicial region.

Remark 2.7. Despite the many cosimplicial structures, this region is still very 1-dimensional. The points in this region all have coordinates $(i, 0, 0)$ in $\mathbb{Z}^3$. After all the map $F \circ D^i : \Delta^{i+1} \longrightarrow \mathbb{Z}^3$ still factors through $F : \Delta \longrightarrow \mathcal{R}egions$ whose image is 1-dimensional.

Example 2.8. Next we do the 2-dimensional version. We have a functor $F : \Delta \longrightarrow \mathcal{R}egions$ where $\mathcal{R}egions$ is the category of regions in $\mathbb{Z}^3$. Of course the $\mathbb{Z}^3$ is a little ridiculous since the image lies in $\mathbb{Z} \subset \mathbb{Z}^3$. The functor we plan to name $F \times F$ will be a functor

$$F \times F : \Delta \times \Delta \longrightarrow \mathcal{R}egions.$$

It will take the pair $[0,n], [0,m] \in \Delta$ to $F([0,n]) \times F([0,m]) \subset \mathbb{Z}^6$. Very concretely, $F \times F$ takes the pair of objects $[0,n], [0,m] \in \Delta$ to

$$[0,n] \times [0,m] \times [0]^4$$

which we are free to consider as lying in $\mathbb{Z}^3 \subset \mathbb{Z}^6$. The image is nothing other than the region $\mathcal{R}_{mn}$ of Example 2.3. There is a functor $F \times F : \Delta \times \Delta \longrightarrow \mathcal{R}egions$ with

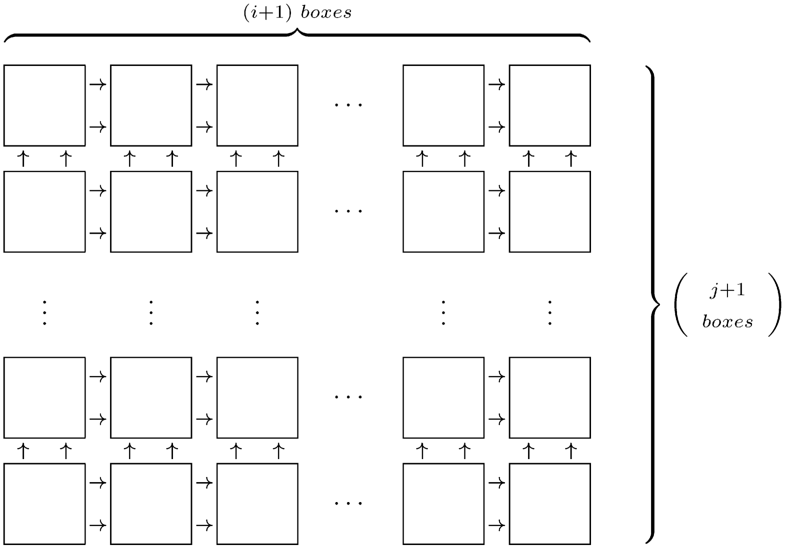
$$\{F \times F\}\bigl([0,n],[0,m]\bigr) = \mathcal{R}_{mn}.$$

We have a 2-dimensional example of a 2-fold cosimplicial region.

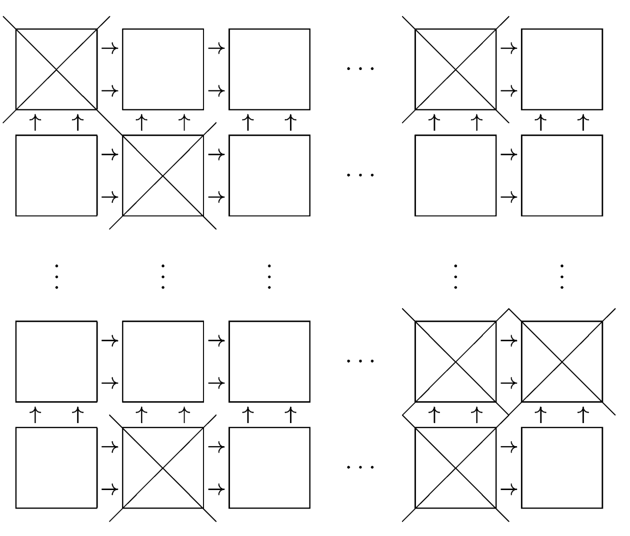
Remark 2.9. We can also combine this with the concatenation functor of Example 2.6, forming the $(i+j+2)$-fold cosimplicial regions $\{F \circ D^i\} \times \{F \circ D^j\}$. That is, we consider the composite

$$\Delta^{i+1} \times \Delta^{j+1} \xrightarrow{D^i \times D^j} \Delta \times \Delta \xrightarrow{F \times F} \mathcal{R}egions.$$

Notation 2.10. Our notation for the $(i + j + 2)$-fold cosimplicial regions of Remark 2.9 will be



The cosimplicial regions we will consider are all very specific cosimplicial subregions of the regions $\{F \circ D^i\} \times \{F \circ D^j\}$ of Remark 2.9. One way to obtain a cosimplicial subregion is simply to delete some of the boxes; the region



should hopefully be self-explanatory. It is also a functor $\Delta^{i+1} \times \Delta^{j+1} \longrightarrow \mathcal{R}egions$. Any object of $\Delta^{i+1} \times \Delta^{j+1}$ is mapped to the subregion indicated; the parts crossed out are left out of the region.

The next construction is slightly more subtle.

Construction 2.11. Example 2.8 gave us a 2-fold cosimplicial region, that is a functor $F \times F : \Delta \times \Delta \longrightarrow \mathcal{R}egions$. The functor takes the pair of objects $[0,n],[0,m] \in \Delta$ to the region $[0,n] \times [0,m] \times [0] \subset \mathbb{Z}^3$. Of course we can consider the composite

$$\Delta \xrightarrow{\text{diagonal}} \Delta \times \Delta \xrightarrow{F \times F} \mathcal{R}egions$$

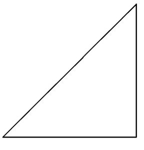
getting a 1-fold cosimplicial region. The object $[0,n] \in \Delta$ is mapped to

$$[0,n] \times [0,n] \times [0] \subset \mathbb{Z}^3.$$

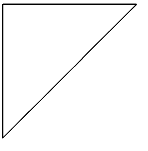
The virtue of diagonalizing is that now the sets of points

$$A_n = \big\{(x,y,0) \in [0,n] \times [0,n] \times [0] \bigm| y \leqslant x\big\},$$
$$B_n = \big\{(x,y,0) \in [0,n] \times [0,n] \times [0] \bigm| y \geqslant x\big\}$$

define cosimplicial subregions. Given any morphism $\varphi : [0,m] \longrightarrow [0,n]$ in Δ, the map $F(\varphi) \times F(\varphi)$, which is just $\varphi \times \varphi : [0,m] \times [0,m] \longrightarrow [0,n] \times [0,n]$, clearly carries $A_m \subset \mathcal{R}_{mm}$ into $A_n \subset \mathcal{R}_{nn}$ and $B_m \subset \mathcal{R}_{mm}$ into $B_n \subset \mathcal{R}_{nn}$. Thus we have two functors $A, B : \Delta \longrightarrow \mathcal{R}egions$, with A taking $[0,n]$ to A_n and B taking $[0,n]$ to B_n. They give cosimplicial subregions. Understandably enough we denote these subregions diagramatically by



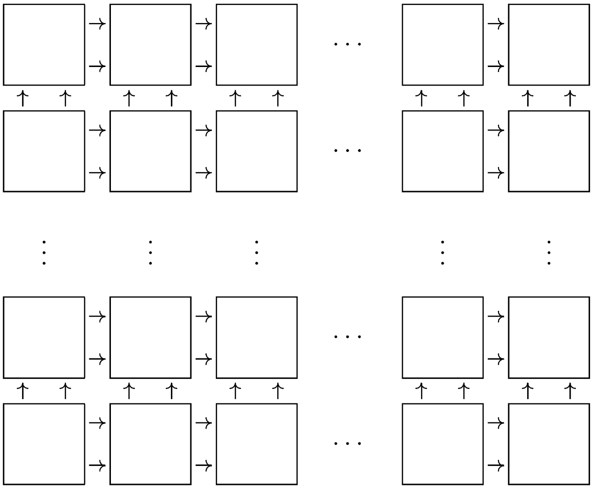
and



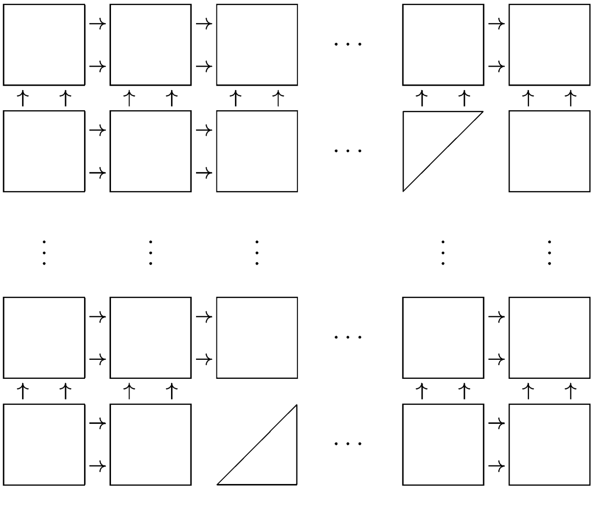
Remark 2.12. We next want to combine this with the concatenation. Recall that in Remark 2.9 we considered the amusing cosimplicial regions that can be obtained as $\{F \circ D^i\} \times \{F \circ D^j\}$. This gives a functor $\Delta^{i+1} \times \Delta^{j+1} \longrightarrow \mathcal{R}egions$. There is nothing to stop us from embedding a Δ by some diagonal. Consider the composite

$$\begin{array}{ccc}
\{\Delta^a \times \Delta^b\} \times \Delta \times \{\Delta^{i-a} \times \Delta^{j-b}\} & & \\
\Big\downarrow {\scriptstyle 1 \times \{\text{diagonal}\} \times 1} & & \\
\{\Delta^a \times \Delta^b\} \times \{\Delta \times \Delta\} \times \{\Delta^{i-a} \times \Delta^{j-b}\} & = & \Delta^{i+1} \times \Delta^{j+1} \\
 & & \Big\downarrow {\scriptstyle D^i \times D^j} \\
 & & \Delta \times \Delta \\
 & & \Big\downarrow {\scriptstyle F \times F} \\
 & & \mathcal{R}egions.
\end{array}$$

As we learned in Notation 2.10 we will be denoting this cosimplicial region by



Until now we have been thinking of this as a region possessing $(i+j+2)$ cosimplicial structures, but we have now forced the action on the $(a+1)$ column and $(b+1)$ row to be diagonal. This means that, in the $(a+1, b+1)$ box the two cosimplicial subregions of Construction 2.11 make sense. We are now free to draw subregions



The meaning of each triangle is that

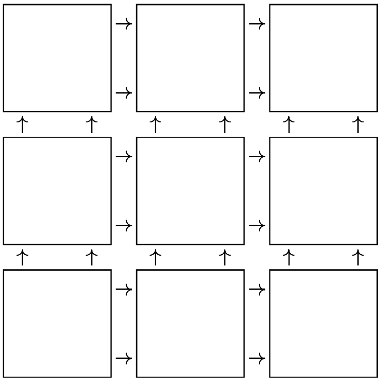
(i) In the row and column of the triangle the cosimplicial action is diagonal.
(ii) In the intersection of the row and column we take the subregion of Construction 2.11.

Remark 2.13. So far all our regions have been contained in a plane in $\mathbb{Z}^3$. Given any planar cosimplicial region, that is a functor $G : \Delta^r \longrightarrow \mathbb{Z}^2$, we can make it 3-dimensional by multiplying by a fixed subset $I \subset \mathbb{Z}$. That is, we define $G' : \Delta^r \longrightarrow \mathbb{Z}^3$ by the formula

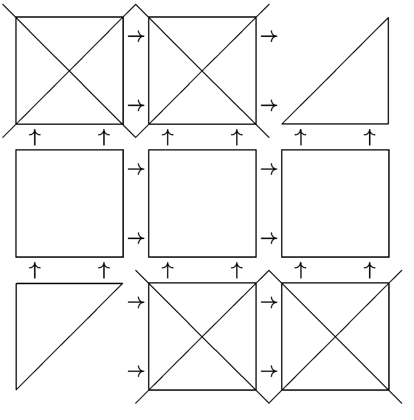
$$G'(x) = G(x) \times I \subset \mathbb{Z}^3.$$

In our constructions I will be $I = [0, 1] \subset \mathbb{Z}$.

Remark 2.14. It is now time to leave the world of generalities and become specific. Our proof of the Maltsiniotis conjecture will study the following regions. In Remark 2.9 and Notation 2.10 we learned how to form the 6-fold cosimplicial region

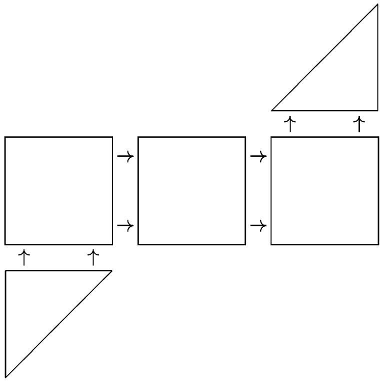


In Notation 2.10 and Construction 2.11 we learned how to form subregions, either by leaving out one of the boxes or by taking a triangular piece. The region we really wish to consider is

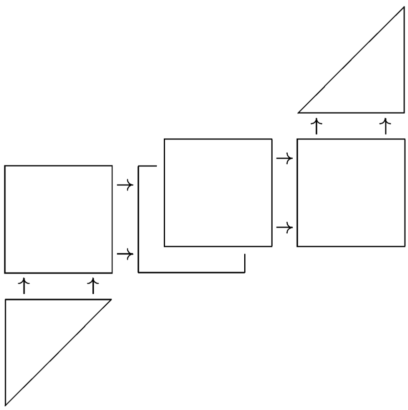


Note that this subregion has only four cosimplicial structures. We start with a 6-fold cosimplicial region, but to form the triangular subregions we had to diagonalize. In our specific case the action on the first row and column is diagonal, as is the action on the third row and column. This reduces the 6-fold to a 4-fold cosimplicial region.

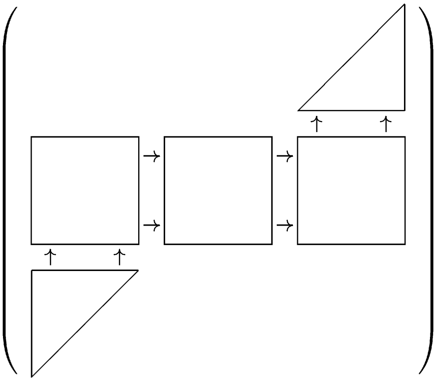
Since all of our regions in the plane $\mathbb{Z}^2$ will be subregions of the above, we will slightly simplify the notation and write the region



As indicated in Remark 2.13 we will consider the cosimplicial region in $\mathbb{Z}^3$ obtained by multiplying by $I = [0, 1]$. And we will actually study a subregion of this. The real region we wish to look at will be

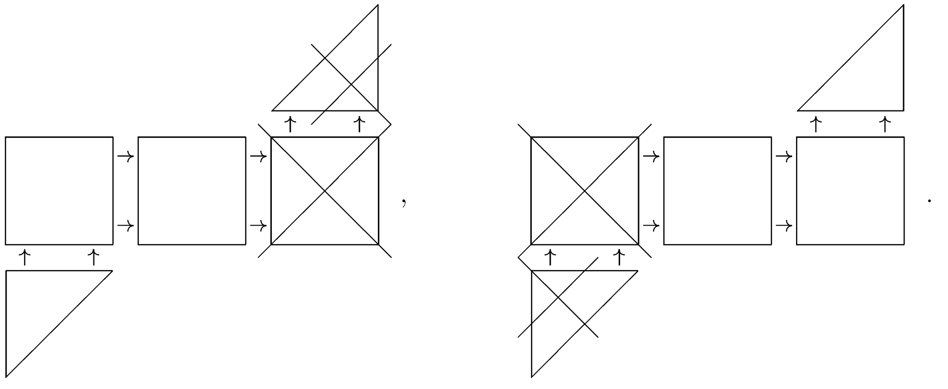


What this picture means is that in the cosimplicial region in $\mathbb{Z}^3$

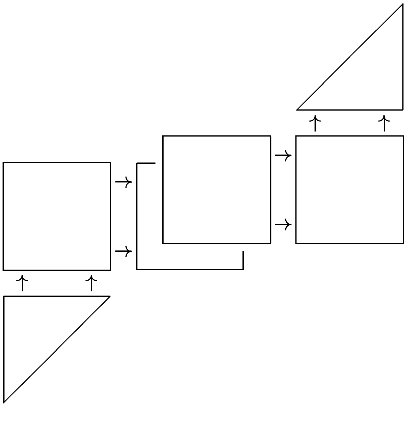


$\times [0, 1]$

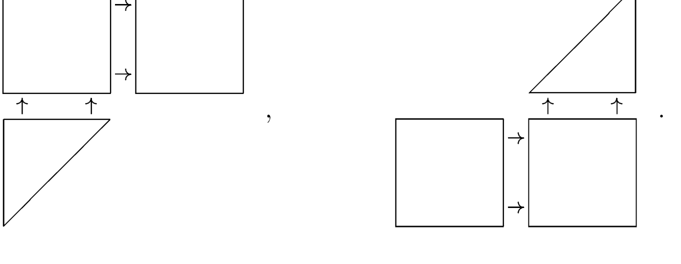
we consider the subregion whose intersection with $\mathbb{Z}^2 \times [0]$ and $\mathbb{Z}^2 \times [1]$ are, respectively,



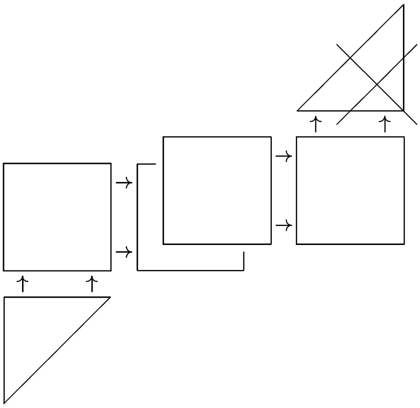
Notation 2.15. Our 4-fold cosimplicial region will, as already said, be denoted



Its intersections with $\mathbb{Z}^2 \times [0]$ and $\mathbb{Z}^2 \times [1]$ will be written



For all other subregions, and we will have to consider several, we will explicitly write which subregions are deleted; for example



has a meaning which should hopefully be obvious.

3. K-theory

In Section 2 we defined cosimplicial regions. Given a cosimplicial region F and a derivator $\mathbb{D}$ there is a way to form a simplicial set, whose geometric realization is a topological space with a homotopy which might be interesting. We will describe the recipe in this section. The K-theory of the derivator $\mathbb{D}$ is defined this way, and the idea of Section 2 was to set up the machinery in sufficient generality to be able to prove something about this K-theory. We begin with the key definition.

Definition 3.1. Let $H : \Delta^r \longrightarrow \mathcal{R}egions$ be an r-fold cosimplicial region in $\mathbb{Z}^3$. Let $\mathbb{D} : \mathcal{D}ia^{\mathrm{op}} \longrightarrow \mathcal{CAT}$ be a prederivator. Suppose that $\mathcal{D}ia$ contains the image of the functor H; this happens, for example, if $H(x) \subset \mathbb{Z}^3$ is finite for any object $x \in \Delta^r$. We form the r-fold simplicial groupoid $g(\mathbb{D}, H)$ by declaring that, for every $x \in \Delta^r$,

$$g(\mathbb{D}, H)(x) = \bigl\{\text{groupoid of isomorphisms in } \mathbb{D}\bigl(H(x)^{\mathrm{op}}\bigr)\bigr\}.$$

That is we take the category $\mathbb{D}(H(x)^{\mathrm{op}})$, and consider $g(\mathbb{D}, H)(x)$ to be the groupoid of all isomorphisms in this category.

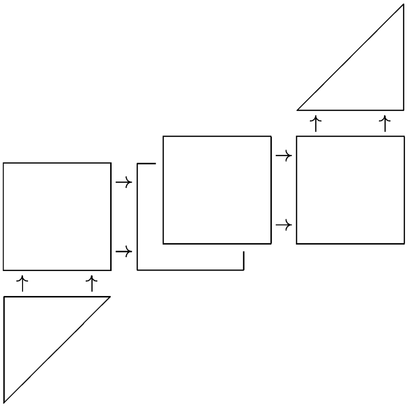
Remark 3.2. Let us first note that the variance is right. Since $\mathbb{D} : \mathcal{D}ia^{\mathrm{op}} \longrightarrow \mathcal{CAT}$ is contravariant, the composite

$$\bigl\{\Delta^r\bigr\}^{\mathrm{op}} \xrightarrow{H^{\mathrm{op}}} \{\mathcal{R}egions\}^{\mathrm{op}} \xrightarrow{\mathbb{D}} \mathcal{CAT}$$

does have the right variance; it is a functor $\mathbb{D} \circ H^{\mathrm{op}} : \{\Delta^{\mathrm{op}}\}^r \longrightarrow \mathcal{CAT}$. We have an r-fold simplicial category, and the subcategory of isomorphisms is an r-fold simplicial groupoid.

Remark 3.3. The cosimplicial regions to which we wish to apply the constructions are the ones of Section 2, most especially the ones of Remark 2.14. Let $H : \Delta^r \longrightarrow \mathcal{R}egions$ be such an r-fold cosimplicial region. We next define the simplicial groupoid we really care about:

Construction 3.4. Suppose $\mathbb{D}$ is a triangulated derivator, not just any prederivator. Let $H : \Delta^r \longrightarrow \mathcal{R}egions$ be one of the 4-fold cosimplicial categories of Remark 2.14; that is H is a subregion of the cosimplicial region we have been denoting



In Definition 3.1 we defined a 4-fold simplicial groupoid $g(\mathbb{D}, H)(x)$. Inside this simplicial groupoid we wish to consider a simplicial subgroupoid. It is a full subcategory of $g(\mathbb{D}, H)(x)$, of all objects $A \in g(\mathbb{D}, H)(x)$ satisfying the following restrictions:

(i) The restriction of $A \in \mathbb{D}(H(x)^{\mathrm{op}})$ to the diagonal lines in the boundary of the region vanishes.
(ii) Suppose we consider the slices of the region $H(x)$, as in Definition 2.1(ii). We remind the reader: if $k \in \mathbb{Z}$ is any integer, then the slice ${}_kH(x)$ is the full subcategory

$$ {}_kH(x) = \bigl\{(x, y, k) \in H(x)\bigr\}. $$

In our case only ${}_0H(x)$ and ${}_1H(x)$ are non-empty. Take any square contained in ${}_kH(x)$; it gives a map $\square \longrightarrow H(x)$. The restriction of $A \in \mathbb{D}(H(x)^{\mathrm{op}})$ to $\mathbb{D}(\square^{\mathrm{op}})$ is a global commutative square, and we insist that all such squares should be cartesian (or equivalently cocartesian by Der 7; see Definition 1.15).

Remark 3.5. Perhaps it would help to rewrite more explicitly the conditions (i) and (ii) above. For $x \in \Delta^r$ we have a region $H(x) \subset \mathbb{Z}^3$, and the region $H(x)$ consists of some points $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ with $\gamma \in \{0, 1\}$. We could consider separately the cases $\gamma = 0$ and $\gamma = 1$; that is what we meant when we said, in (ii) above, that we consider the "slices" ${}_0H(x)$ and ${}_1H(x)$. If we separately consider the two slices what we have is two regions in the plane, and it is easy enough to concretely visualize such regions. The two regions are pictured, in explicit coordinates, on the following page:

$$
\begin{array}{ccccccc}
 & & & & & & (e,e,1) \\
 & & & & & & \uparrow \\
 & & & & & \iddots & \vdots \\
 & & & & & & \uparrow \\
 & & & & (d,d,1) & \longrightarrow \cdots \longrightarrow & (e,d,1) \\
 & & & & \uparrow & & \uparrow \\
(b,y,1) & \longrightarrow \cdots \longrightarrow & (c,y,1) & \longrightarrow & (d,y,1) & \longrightarrow \cdots \longrightarrow & (e,y,1) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(b,x,1) & \longrightarrow \cdots \longrightarrow & (c,x,1) & \longrightarrow & (d,x,1) & \longrightarrow \cdots \longrightarrow & (e,x,1)
\end{array}
$$

$$
\begin{array}{ccccccc}
(0,y,0) & \longrightarrow \cdots \longrightarrow & (a,y,0) & \longrightarrow & (b,y,0) & \longrightarrow \cdots \longrightarrow & (c,y,0) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(0,x,0) & \longrightarrow \cdots \longrightarrow & (a,x,0) & \longrightarrow & (b,x,0) & \longrightarrow \cdots \longrightarrow & (c,x,0) \\
\uparrow & & \uparrow & & & & \\
(0,a,0) & \longrightarrow \cdots \longrightarrow & (a,a,0) & & & & \\
\uparrow & & & & & & \\
\vdots & \iddots & & & & & \\
\uparrow & & & & & & \\
(0,0,0) & & & & & &
\end{array}
$$

Given any inclusion of categories $\delta : J \longrightarrow H(x)$ there is an induced map $\mathbb{D}(\delta^{\mathrm{op}}) : \mathbb{D}(H(x)^{\mathrm{op}}) \longrightarrow \mathbb{D}(J^{\mathrm{op}})$. We can impose restrictions on the objects $A \in \mathbb{D}(H(x)^{\mathrm{op}})$ by specifying that, for certain inclusions $\delta : J \longrightarrow H(x)$, the image of $A \in \mathbb{D}(H(x)^{\mathrm{op}})$ by the functor $\mathbb{D}(\delta^{\mathrm{op}}) : \mathbb{D}(H(x)^{\mathrm{op}}) \longrightarrow \mathbb{D}(J^{\mathrm{op}})$ should satisfy some properties. This is precisely the nature of the conditions imposed in Construction 3.4(i) and (ii). In Construction 3.4 we define a full subcategory of $\mathbb{D}(H(x)^{\mathrm{op}})$ by imposing restrictions on the objects. An object $A \in \mathbb{D}(H(x)^{\mathrm{op}})$ belongs to the subcategory if it satisfies (i) and (ii), both of which are stated in terms of certain maps $\delta : J \longrightarrow H(x)$. More explicitly, we require

(i) If, in the picture on the previous page for the category $H(x)$, we let $J \subset H(x)$ be the full subcategory of all points

$$J = \big\{(i,i,0) \bigm| 0 \leqslant i \leqslant a\big\} \cup \big\{(j,j,1) \bigm| d \leqslant j \leqslant e\big\}$$

then the restriction of $A \in \mathbb{D}(H(x)^{\mathrm{op}})$ to $\mathbb{D}(J^{\mathrm{op}})$ is isomorphic to zero. The reader is reminded that $\mathbb{D}(J^{\mathrm{op}})$ has a zero object; see 7.9.

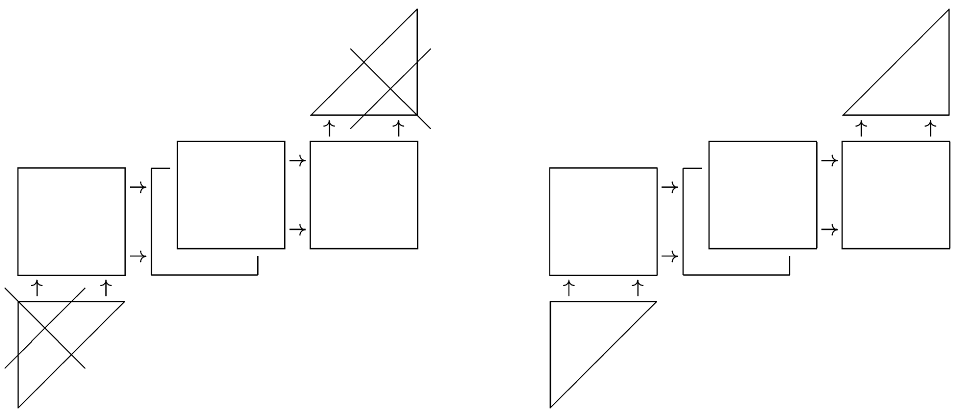
(ii) Suppose we choose five integers $i \leqslant i'$, $j \leqslant j'$ and $k \in \{0,1\}$. Suppose that the following square

$$\begin{array}{ccc} (i,j',k) & \longrightarrow & (i',j',k) \\ \uparrow & & \uparrow \\ (i,j,k) & \longrightarrow & (i',j,k) \end{array}$$

is entirely contained in $H(x)$. It gives a map $\delta : \square \longrightarrow H(x)$, and therefore any object $A \in \mathbb{D}(H(x)^{\mathrm{op}})$ restricts to a global commutative square $\mathbb{D}(\delta^{\mathrm{op}})(A) \in \mathbb{D}(\square^{\mathrm{op}}) = \mathbb{D}(\square)$. For A to lie in our subcategory we require that all such global commutative squares be cartesian (= cocartesian).

Notation 3.6. In Construction 3.4 we defined a simplicial subgroupoid of the simplicial groupoid $g(\mathbb{D}, H)$; in Remark 3.5 we elaborated and explained the construction in more detail. We have not given this groupoid a name. The reason is that we will freely confuse it with H. The symbol for the cosimplicial region H will also be used, interchangeably, to stand for the simplicial subgroupoid of $g(\mathbb{D}, H)$ given in Construction 3.4. Since the derivator $\mathbb{D}$ will be fixed throughout this should not cause confusion.

This means that, from now on, the symbols

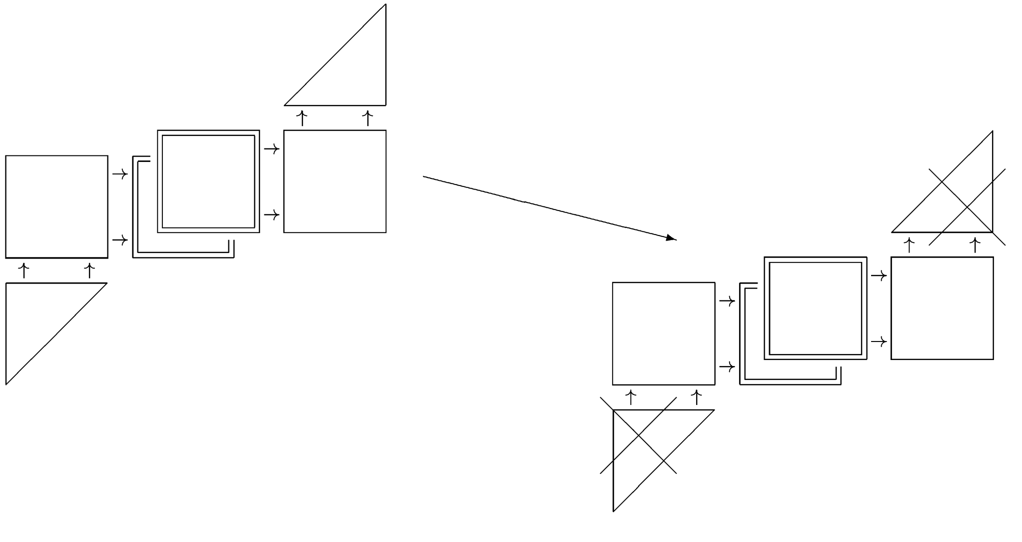


can stand either for the 4-fold cosimplicial regions or for the 4-fold simplicial groupoids. The inclusion of cosimplicial regions induces, by the contravariance of $\mathbb{D}$, a map of simplicial groupoids in the opposite direction. From the direction of the arrows it should be obvious whether we mean inclusions of cosimplicial regions or maps of simplicial groupoids.

Remark 3.7. Our regions have many cosimplicial structures. Since we agreed that they are all subregions of a 4-fold cosimplicial region, they all have 4 independent cosimplicial structures. The simplicial groupoids are therefore 4-fold simplicial groupoids; as simplicial sets they have five commuting simplicial structures (there is one that comes from realizing the groupoid). Our aim is to study some maps between them and show that these maps induce homotopy equivalences.

It is helpful to exploit the many simplicial structures. To prove that a map between 5-fold simplicial sets is a homotopy equivalence it suffices to show that, after realizing some subset of the 5 simplicial structures, the map of simplicial spaces is a homotopy equivalence. We will use this often.

This means we need a notation for the simplicial structures which we plan to realise. Our notation will be the following: we *do not* touch any of the regions surrounded by a double box, and realise all other simplicial structures. Thus the map



is the map of 5-fold simplicial sets induced by the obvious inclusion. But we realize only 3 of the simplicial structures; we realize the groupoid structure, as well as the two structures which leave alone the highlighted region. In the explicit drawing of the region, given with coordinates in Remark 3.5, the integers a and $e-d$ are allowed to change but the integers $c-b$ and $y-x$ are held fixed.

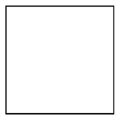
Remark 3.8. The careful reader, examining the diagram in Remark 3.5 giving the explicit description of $H(x) \subset \mathbb{Z}^3$ in terms of coordinates, will note that the integers b, d and x are arbitrary and correspond to our choice of embedding of the region in the plane. The integers a, $c-b$, $e-d$ and $y-x$, on the other hand, are not arbitrary. The diagram is a description of the region $H(x)$ with $x \in \Delta^4$, and

$$x = [0,a] \times [0,c-b] \times [0,e-d] \times [0,y-x].$$

When we fail to realize two of the simplicial structures we do not allow two of the integers to change. In the map pictured in Remark 3.7 the integers we leave untouched are $c - b$ and $y - x$. The result is a map of bisimplicial spaces.

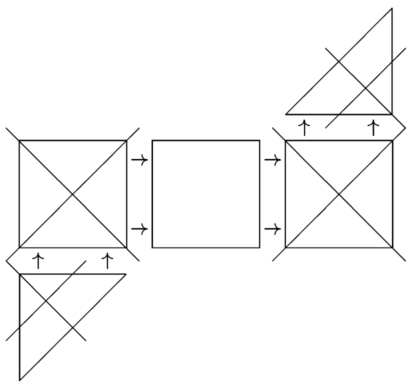
Remark 3.9. The definition of the simplicial groupoid can be extended to regions more general than the subregions of our favorite 4-fold cosimplicial region of Remark 2.14. We leave the obvious extension to the reader.

Definition 3.10. The K-theory of the triangulated derivator $\mathbb{D}$ is defined to be the loop space of the geometric realization of the 2-fold simplicial groupoid

$Q(\mathbb{D}) =$  .

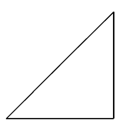
Remark 3.11. It follows from Corollary A.8 that the K-theory of a triangulated derivator is an infinite loop space.

Remark 3.12. We promised the reader that all our regions will be subregions of the 4-fold cosimplicial region of Remark 2.14. We have told no lies: the simplicial set of Definition 3.10 can also be described as



In this description it is a 4-fold simplicial groupoid, but two of the simplicial structures are degenerate; even at the level of regions two of the cosimplicial structures have no effect.

Remark 3.13. It is also possible to define the K-theory of the triangulated derivator $\mathbb{D}$ using a Waldhausen-like simplicial set. We define

$S(\mathbb{D}) =$  .

It can be shown that $S(\mathbb{D})$ and $Q(\mathbb{D})$ are homotopy equivalent. We will not need the result, hence will not include a proof. But it is not difficult to give a proof using the techniques we develop.

Caution 3.14. We are now about to use theorems concerning derivators, whose proofs will come in the later part of the paper. The theorems tell us about constructions of new triangulated derivators out of old ones. Let us be given a triangulated derivator $\mathbb{D}$. We will use now and prove later the following:

(i) For any object $W \in \mathcal{D}ia$, the formula

$$\mathbb{D}_W(X) = \mathbb{D}(X \times W)$$

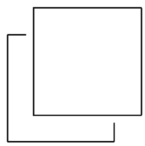
produces a triangulated derivator $\mathbb{D}_W$.

(ii) An object $F \in \mathbb{D}_W(\square) = \mathbb{D}(\square \times W)$ is cartesian if and only if, for every point $w \in W$, the object $F_w = (1_X \times w)^*(F)$ is cartesian in $\mathbb{D}(\square)$.

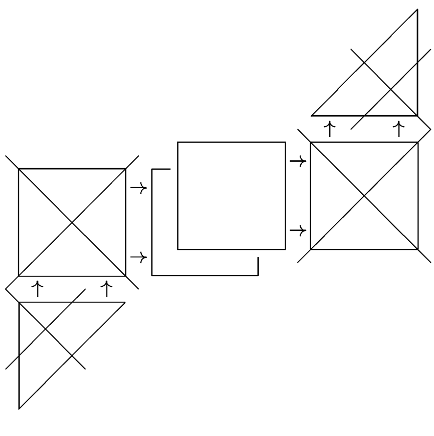
(iii) There is a construction of a triangulated derivator $\mathbb{E}\mathrm{xact}(\mathbb{D})$. It is a subderivator of the derivator $\mathbb{D}_\square$.

For (i) and (ii) the reader is referred to Section 7; for (iii) the reader is referred to Section 11.

Remark 3.15. Putting $W = I = [0, 1]$ in Caution 3.14(i) we note that, from a triangulated derivator $\mathbb{D}$, Proposition 7.8 allows us to form another triangulated derivator $\mathbb{D}_I$, given by the formula $\mathbb{D}_I(X) = \mathbb{D}(X \times [0, 1])$. The K-theory space of $\mathbb{D}_I$ is the loop space of the geometric realization of the simplicial groupoid



Keeping our promise only to consider subregions of the 4-fold cosimplicial region of Remark 2.14, the K-theory of $\mathbb{D}_I$ can also be described as the loop space of the geometric realization of the simplicial groupoid



Remark 3.16. In identifying the K-theory of $\mathbb{D}_I$ with the spaces of Remark 3.15 we are making use of Caution 3.14(ii), applied to the case $W = I = [0, 1]$. An object belongs to the simplicial groupoid defining the K-theory of $\mathbb{D}_I$ provided certain objects in $\mathbb{D}_I(\Box) = \mathbb{D}(\Box \times [0, 1])$ are cartesian. Caution 3.14(ii) tells us that this is equivalent to checking that each of the restrictions to $\mathbb{D}(\Box \times [0])$ and $\mathbb{D}(\Box \times [1])$ is cartesian.

The main theorem of this article is the following, which proves the Maltsiniotis additivity conjecture:

Theorem 3.17. *Let $\mathbb{D}$ be a triangulated derivator. Let $\mathbb{E}\mathrm{xact}(\mathbb{D})$ be the derivator of short exact sequences in $\mathbb{D}$ (see* 11.8*). Let*

$$(s \times s)^* : \mathbb{E}\mathrm{xact}(\mathbb{D}) \longrightarrow \mathbb{D}, \qquad (t \times t)^* : \mathbb{E}\mathrm{xact}(\mathbb{D}) \longrightarrow \mathbb{D}$$

be the morphisms of 11.11. *Then the natural map*

$$\big(Q\big((s \times s)^*\big), Q\big((s \times s)^*\big)\big) : Q\big(\mathbb{E}\mathrm{xact}(\mathbb{D})\big) \longrightarrow Q(\mathbb{D}) \times Q(\mathbb{D})$$

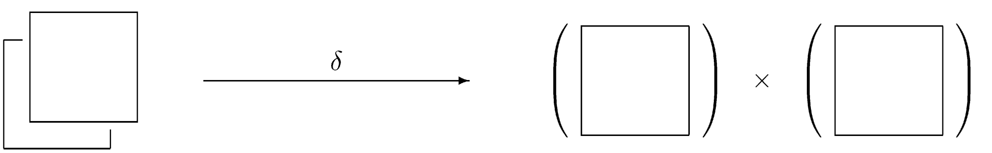
is a homotopy equivalence.

Caution 3.18. Strictly speaking the theorem is meaningless until after reading Section 11, since the derivator $\mathbb{E}\mathrm{xact}(\mathbb{D})$ and the morphisms of derivators

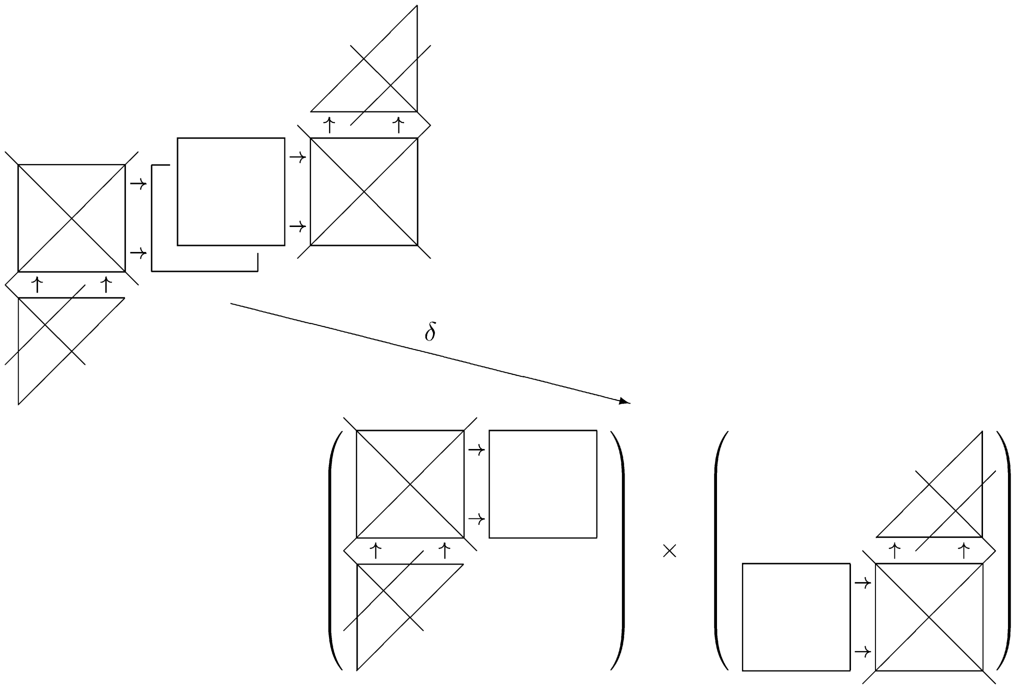
$$(s \times s)^* : \mathbb{E}\mathrm{xact}(\mathbb{D}) \longrightarrow \mathbb{D}, \qquad (t \times t)^* : \mathbb{E}\mathrm{xact}(\mathbb{D}) \longrightarrow \mathbb{D}$$

will not be defined until then. But the results of Section 11 permit us to reformulate the main theorem as in Remark 3.19 below, and the rephrased version makes sense without having to read Section 11.

Remark 3.19. By Proposition 11.12, Theorem 3.17 is equivalent to the assertion that the natural map



is a homotopy equivalence. In our way of viewing all the regions as subregions of the 4-fold cosimplicial region of Remark 2.14, this map becomes



The next few sections are devoted to proving δ a homotopy equivalence.

4. An easy example of a homotopy

We could immediately launch into the proof that the map δ of Remark 3.19 is a homotopy equivalence. The purpose of this section is to proceed a little more gently, reminding the reader of simplicial homotopies generally, and more particularly of the simplicial homotopies we will use.

Recall that an r-fold simplicial set is a functor $F:\{\Delta^{\mathrm{op}}\}^r \longrightarrow \mathcal{S}et$, and a simplicial map between two r-fold simplicial sets is a natural transformation $\varphi: F \Longrightarrow F'$, where $F, F':\{\Delta^{\mathrm{op}}\}^r \longrightarrow \mathcal{S}et$ are two functors. Simplicial homotopies are slightly more confusing in general. But there is a cheap way to produce them, which suffices for this article. In this section we will explain the very easy, simple homotopies we will use.

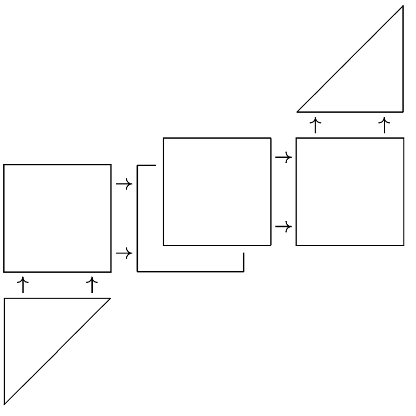
The first observation is that we are dealing with 2-categories. Let us make this precise.

Remark 4.1. Suppose we have a 2-category $\mathcal{C} \subset \mathcal{C}at$ and a functor $G:\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}et$. One way to obtain an r-fold simplicial set is to take a functor $H:\Delta^r \longrightarrow \mathcal{C}$ and form the composite

$$\{\Delta^{\mathrm{op}}\}^r \xrightarrow{H^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{G} \mathcal{S}et.$$

And perhaps the most important thing to keep in mind is that all the simplicial sets we have been considering are of this type. The category $\mathcal{R}egions \subset \mathcal{C}at$ of regions in $\mathbb{Z}^3$ is a 2-category, and in

Section 2 we learned how to form several interesting functors $H : \Delta^r \longrightarrow \mathcal{R}egions$. For most of what follows we can confine ourselves to the subcategory of $\mathcal{R}egions$ consisting of subregions of our favorite region



That is, the most important example to keep in mind is the following. We define the category $\mathcal{C} \subset \mathcal{R}egions$ to have for its objects the subregions of the above, and the morphisms are maps of regions preserving the diagonal lines in the boundary; see Remark 3.5(i) for the meaning of the diagonal lines in the boundary. The 2-morphisms in $\mathcal{C}$ are all the natural transformations between 1-morphisms in $\mathcal{C}$. Because all the objects in $\mathcal{C}$ are finite the 2-category $\mathcal{C}$ is contained in $\mathcal{D}ia$.

Given a triangulated derivator $\mathbb{D}$ and an object $I \in \mathcal{C}$, the recipe of Section 3 constructs for us a groupoid $G(I)$. The objects are the objects in $\mathbb{D}(I^{\mathrm{op}})$ satisfying the restrictions of Construction 3.4(i) and (ii), and the morphisms are the isomorphisms in $\mathbb{D}(I^{\mathrm{op}})$. This defines $G(I)$ for objects $I \in \mathcal{C}$; we leave it to the reader to check that the definition extends to morphisms in $\mathcal{C}$. It defines a functor $G : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{G}roupoid$. All the simplicial groupoids we will consider are of the form $G \circ H^{\mathrm{op}}$ for some $H : \Delta^r \longrightarrow \mathcal{C}$, and all the simplicial sets are the nerves of the simplicial groupoids $G \circ H^{\mathrm{op}}$.

In this paper we will need to look at r-fold simplicial sets where $r > 1$. As is customary in this subject we reduce the case of r-fold simplicial gadgets, with $r > 1$, to the special case where $r = 1$. The trick is to allow the categories to change. That is we look at simplicial objects F in a category $\mathcal{T}$. A simplicial object in $\mathcal{T}$ is a functor $F : \Delta^{\mathrm{op}} \longrightarrow \mathcal{T}$. Thus an r-fold simplicial set becomes a simplicial object in the category of $(r-1)$-fold simplicial sets.

Remark 4.2. As in Remark 4.1 let $\mathcal{C} \subset \mathcal{C}at$ be a 2-category. Let $G : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{T}$ be some fixed functor. In Remark 4.1 we saw that a functor $H : \Delta \longrightarrow \mathcal{C}$ gives in $\mathcal{T}$ a simplicial object $GH^{\mathrm{op}} : \Delta^{\mathrm{op}} \longrightarrow \mathcal{T}$. What we wish to note here is

(i) Given two functors $H, H' : \Delta \longrightarrow \mathcal{C}$ and a natural transformation $\varphi : H \longrightarrow H'$, then $G\varphi$ is a simplicial map of simplicial objects in $\mathcal{T}$: it is a natural transformation $GH'^{\mathrm{op}} \longrightarrow GH^{\mathrm{op}}$.
(ii) Next we want a method to produce homotopies. Suppose we are given two 1-morphisms $H, H' : \Delta \longrightarrow \mathcal{C}$, two 2-morphisms $\varphi, \varphi' : H \longrightarrow H'$ and a 3-morphism $\alpha : \varphi \Longrightarrow \varphi'$. We would like to conclude that α induces a homotopy $G\varphi \Longrightarrow G\varphi'$. This is true, as long as the following hypothesis holds:

(iii) (Technical condition to guarantee that (ii) holds):

(i) In $\mathcal{C}at$ we must be given a map $\lambda : H[0,1] \longrightarrow [0,1]$. Given a pair of morphisms in Δ $y \longrightarrow x$, $y \longrightarrow [0,1]$ we produce maps

$$Hy \longrightarrow Hx, \qquad Hy \longrightarrow H[0,1] \xrightarrow{\lambda} [0,1],$$

that is we have in $\mathcal{C}at$ a map $Hy \longrightarrow [0,1] \times Hx$. The composite with $\alpha_x : [0,1] \times Hx \longrightarrow H'x$ gives us in $\mathcal{C}at$ the map $Hy \longrightarrow H'x$; we insist that this be a morphism in $\mathcal{C} \subset \mathcal{C}at$.

(ii) Note that in the category Δ there are two maps $[0] \longrightarrow [0,1]$; let us denote them i_0, i_1. Let $x \in \delta$ be an arbitrary object. Since the object $[0] \in \Delta$ is terminal there is a unique map $p : x \longrightarrow [0]$. We can therefore form the two composites

$$Hx \xrightarrow{\binom{Hp}{1}} H[0] \times Hx \xrightarrow{Hi_0 \times 1} H[0,1] \times Hx \xrightarrow{\lambda \times 1} [0,1] \times Hx \xrightarrow{\alpha_x} H'x,$$
$$Hx \xrightarrow{\binom{Hp}{1}} H[0] \times Hx \xrightarrow{Hi_1 \times 1} H[0,1] \times Hx \xrightarrow{\lambda \times 1} [0,1] \times Hx \xrightarrow{\alpha_x} H'x.$$

The first composite must be the map $\varphi_x : Hx \longrightarrow H'x$, and the second must be $\varphi'_x : Hx \longrightarrow H'x$.

Remark 4.3. Let us very briefly remind the reader how to construct the homotopy of Remark 4.2(ii). To give a natural transformation $H \longrightarrow H'$ is to give, for every object $x \in \Delta$, a morphism $\varphi_x : Hx \longrightarrow H'x$. In the category $\mathcal{C} \subset \mathcal{C}at$ the morphisms are functors. For every $x \in \Delta$ we have a functor $\varphi_x : Hx \longrightarrow H'x$. To give a 3-morphism $\alpha : \varphi \Longrightarrow \varphi'$ is to give, for every object $x \in \Delta$, a 2-morphism in $\mathcal{C}$ of the form $\varphi_x \Longrightarrow \varphi'_x$. Another way of saying this is that for every object x we give, in $\mathcal{C}at$, a functor $\alpha_x : [0,1] \times Hx \longrightarrow H'x$. Of course these functors satisfy compatibilities as we vary $x \in \Delta^r$, which we are suppressing.

In general a simplicial homotopy takes a pair of morphisms in Δ

$$y \longrightarrow x, \qquad y \longrightarrow [0,1]$$

to a map $Hy \longrightarrow H'x$ in $\mathcal{C}$. Remark 4.2(iii)(i) gives the recipe for producing this morphism. We leave it to the reader to check the compatibilities that make this a simplicial homotopy.

This homotopy connects two simplicial maps. The fact that these maps are φ and φ' is a computation we are leaving to the reader; it comes down to computing the map $Hy \longrightarrow H'x$ for the special cases where the pair $y \longrightarrow x$, $y \longrightarrow [0,1]$ factors as

$$y \longrightarrow x, \qquad y \longrightarrow [0] \xrightarrow{i} [0,1]$$

with $i = i_0$ or i_1. The hint is that the computation reduces to the condition stipulated in Remark 4.2(iii).

Remark 4.4. This way of constructing homotopies is very limited. The authors know of very few homotopies which can be defined this way. The fact that these are the only homotopies we will use in the proof of additivity is an indication of the very formal nature of our proof.

In practice all our homotopies are very special cases of the above. We now discuss this.

Example 4.5. Recall the functor $D : \Delta \times \Delta \longrightarrow \Delta$ of Example 2.6. The functor takes a pair of objects $S, T \in \Delta$ to the disjoint union $S \cup T$, with the order that every element of S is less than every element of T. Now suppose we fix an object $S \in \Delta$, and consider the functor $D(S, -) : \Delta \longrightarrow \Delta$ which takes $T \in \Delta$ to $D(S, T)$. We have produced a functor $H = D(S, -) : \Delta \longrightarrow \Delta$, and $\Delta \subset \mathcal{C}at$ is a 2-category.

Now the identity gives a 1-morphism $1 : H \longrightarrow H$. There is another 1-morphism $\varphi : H \longrightarrow H$. To define it we must give, for every object $T \in \Delta$, a morphism $\varphi_T : HT \longrightarrow HT$. That is, for every T we must produce a morphism $S \cup T \longrightarrow S \cup T$ in Δ. The formula is:

$$\varphi_T(i) = \begin{cases} i & \text{if } i \in S, \\ \max(S) & \text{if } i \in T. \end{cases}$$

In this formula $\max(S)$ means the maximum of the totally ordered finite set S.

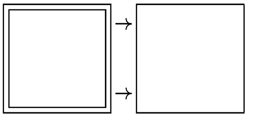
Because the order in $S \cup T$ is such that every element of T is bigger than $\max(S)$ we conclude that $\varphi_T(i) \leqslant i$ for every $i \in S \cup T$. This produces for us, for every T, a natural transformation $\alpha_T : \varphi_T \longrightarrow 1$. Taking the collection of all α_T, as T varies over the objects of Δ, this assembles to a 3-morphism $\alpha : \varphi \Longrightarrow 1$.

Finally we observe that the technical assumptions of Remark 4.2(iii) are satisfied. The object $H[0, 1]$ is, by definition of $H : \Delta \longrightarrow \Delta$, the object $S \cup [0, 1]$. We define a map $\lambda : H[0, 1] \longrightarrow [0, 1]$ by the formula

$$\lambda(i) = \begin{cases} 0 & \text{if } i \in S, \\ i & \text{if } i \in [0, 1]. \end{cases}$$

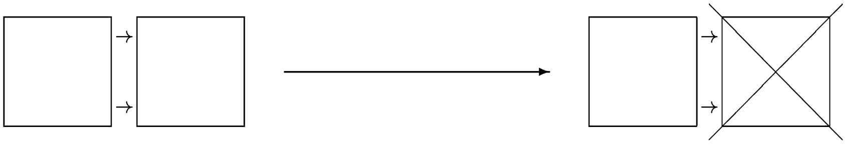
The conditions of Remark 4.2(iii) are easy.

Example 4.6. For a fixed object $S \in \Delta$, Example 4.5 produces a 1-morphism $H : \Delta \longrightarrow \Delta$, a 2-morphism $\varphi : H \longrightarrow H$, a 3-morphism $\alpha : \varphi \Longrightarrow 1$, and the conditions of Remark 4.2(iii) hold. To make this more immediately applicable to our favorite simplicial sets we multiply by an object $R \in \Delta$. Note that our embedding $F \times F : \Delta \times \Delta \longrightarrow \mathcal{R}egions$, given in Example 2.8, permits us to map objects in $\Delta \times \Delta$ into regions in $\mathbb{Z}^2 \subset \mathbb{Z}^3$. The regions we are considering are of the form $R \times D(S, -)$ with R and S fixed; in our notation the functor $T \mapsto F(R) \times F(D(S, T))$, which is a functor $\Delta \longrightarrow \mathcal{R}egions$, would be written



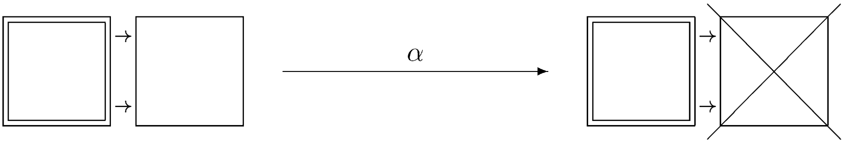
This is a functor (= 1-morphism) taking $T \in \Delta$ to $F(R) \times F(H(T))$. Multiplying the 2-morphisms and 3-morphisms of Example 4.5 by $R \in \Delta$ and applying $F \times F$, we have a 2-morphism $FR \times F\varphi : FR \times FH \longrightarrow FR \times FH$, a 3-morphism $FR \times F\alpha : FR \times F\varphi \Longrightarrow 1$, and the technical conditions of Remark 4.2(iii) hold. We have a simplicial homotopy, and it connects the identity with a map $FR \times F\varphi$, which is easy enough to compute. An immediate consequence is

Lemma 4.7. *The map of* 3*-fold simplicial groupoids*



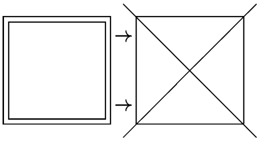
induces a homotopy equivalence.

Proof. As we have already observed in Remark 3.7, it suffices to prove the stronger assertion that the map

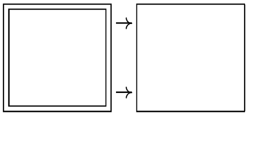


induces a homotopy equivalence. That is, we realize only the simplicial structure leaving unaffected the box on the left. In fact, we will not even realise the groupoid structure. We assert that the map is a homotopy equivalence after realizing just one of the four simplicial structures.

Of course on the 4-fold simplicial set



the only simplicial structure being realized is degenerate. The realization gives us a discrete 3-fold simplicial space. To show that α is a homotopy equivalence we need to show that the homotopy type of the simplicial set



is also discrete. In other words we need to find homotopies which contract the many components of this space.

The point is that Example 4.6 gives us homotopies, and they are all contractions. For each R and S we have a homotopy. It connects the identity to a map we called $FR \times F\varphi$. The map is explicit, and now we will show it to be a contraction.

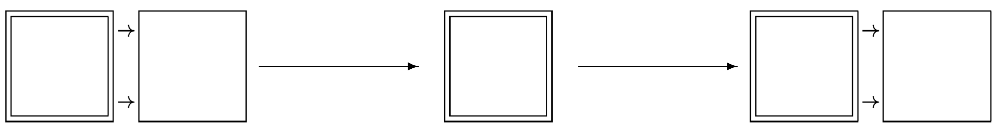
The map induced by $FR \times F\varphi$ is eminently computable. The natural transformation φ is given, for every object $T \in \Delta$, by a map $\varphi_T : S \cup T \longrightarrow S \cup T$. This map φ_T has the explicit formula

$$\varphi_T(i) = \begin{cases} i & \text{if } i \in S, \\ \max(S) & \text{if } i \in T. \end{cases}$$

For each T the map $\varphi_T : S \cup T \longrightarrow S \cup T$ factors through

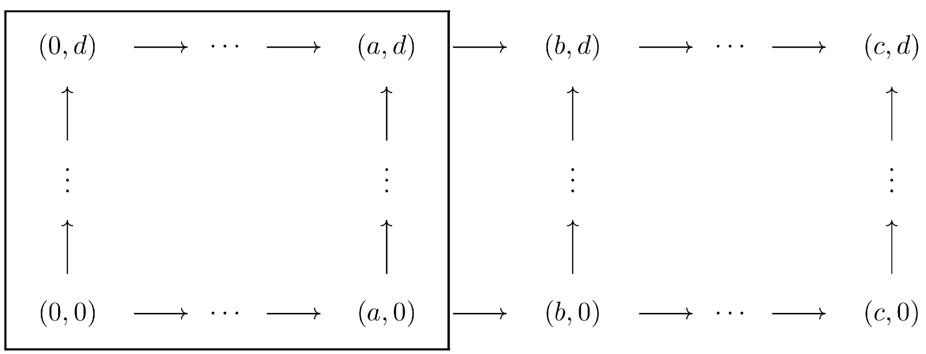
$$S \cup T \longrightarrow S \longrightarrow S \cup T$$

and the factoring is compatible with morphisms $T \longrightarrow T'$ in Δ. We have that φ factors, as a functor on Δ, through the above. If we multiply by R we have a factorization

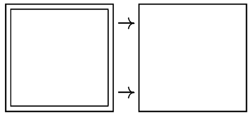


We conclude that φ, which is homotopic to the identity, factors through the trivial space. □

Remark 4.8. It might help the reader to write the homotopy more explicitly, in coordinates. For any $T \in \Delta$ let the region $H(T) = F(R) \times F(D(S, T))$, with $R, S \in \Delta$ fixed as above, be given in coordinates by



The solid box tells us that the integers a, d are fixed. They correspond to the fixed $R, S \in \Delta$. All points (i, j) with $0 \leqslant i \leqslant a$ and $0 \leqslant j \leqslant d$ lie in the region, independently of the choice of $T \in \Delta$.

The simplices in the simplicial set  (recall that we use the same symbol for the region as for the simplicial set) are strings of isomorphisms of objects in $\mathbb{D}(H(T)^{\mathrm{op}})$ for some $T \in \Delta$. An object in $\mathbb{D}(H(T)^{\mathrm{op}})$ induces a functor $M : H(T) \longrightarrow \mathbb{D}(e)$ and this functor is what, in 1.10, we referred to as the local shape of the object. It is sometimes illuminating to draw the local shape of the objects we are talking about.

In coordinates our region $H(T) = F(R) \times F(D(S,T))$ is pictured above. A functor $M : H(T) \longrightarrow \mathbb{D}(e)$ is a commutative diagram in $\mathbb{D}(e)$

$$\begin{array}{ccccccccccc}
M(0,d) & \longrightarrow & \cdots & \longrightarrow & M(a,d) & \longrightarrow & M(b,d) & \longrightarrow & \cdots & \longrightarrow & M(c,d) \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
M(0,0) & \longrightarrow & \cdots & \longrightarrow & M(a,0) & \longrightarrow & M(b,0) & \longrightarrow & \cdots & \longrightarrow & M(c,0)
\end{array}$$

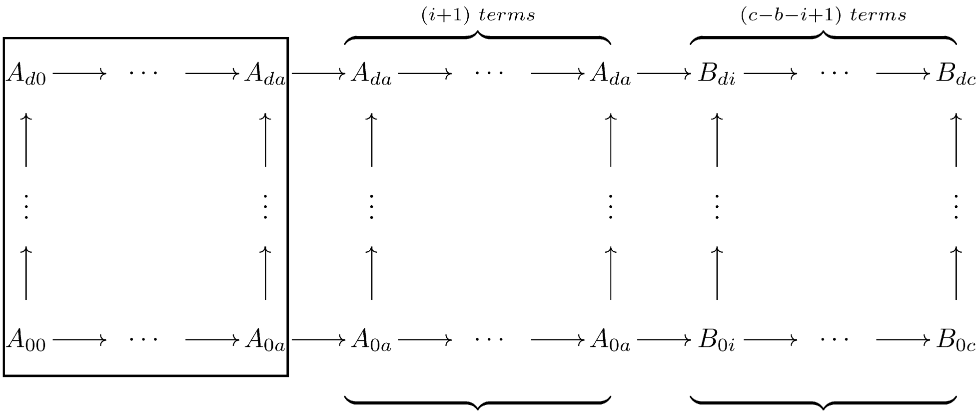
When we wish to write such diagrams we will adopt the following conventions.

(i) We will write M_{ji} for $M(i,j)$.

(ii) In a region made up of several boxes, as above, we will use different letters to denote the objects $M(i,j)$ for (i,j) belonging to different boxes. Thus in the above we will write $A_{ji} = M(i,j)$ if $0 \leqslant i \leqslant a$, and $B_{ji} = M(i,j)$ if $b \leqslant i \leqslant c$. Our local shape of the object in $\mathbb{D}(H(x)^{\mathrm{op}})$ becomes the commutative diagram in $\mathbb{D}(e)$

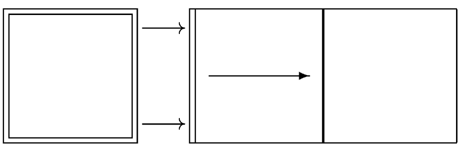
$$\begin{array}{ccccccccccc}
A_{d0} & \longrightarrow & \cdots & \longrightarrow & A_{da} & \longrightarrow & B_{db} & \longrightarrow & \cdots & \longrightarrow & B_{dc} \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
A_{00} & \longrightarrow & \cdots & \longrightarrow & A_{0a} & \longrightarrow & B_{0b} & \longrightarrow & \cdots & \longrightarrow & B_{0c}
\end{array}$$

With these conventions, a typical cell in the homotopy of the proof of Lemma 4.7 is the diagram in $\mathbb{D}(e)$



The reader can hopefully see that this is nothing but the usual contraction to the initial object.

Remark 4.9. Many of our homotopies will be minor variants of the homotopy of the proof of Lemma 4.7. By "minor variants" we mean that we feel free to dualize, and to concatenate to larger simplicial regions as in Remark 2.9. Because this homotopy is ubiquitous we need a notation for it. Our notation will be



Remark 4.10. The reader might be puzzled why we adopt the curious conventions of Remark 4.8. Why do we switch the subscripts in $A_{ji} = M(i, j)$, and why do we use different letters for different parts of the region?

The first reason is that we wish to be consistent with the notation of [21–28]. Of course this is a copout; the reader would still need a satisfactory explanation why, in the previous articles, we adopted this curious notation.

The reason for using different letters for the different parts of the diagram should be clear; we want to make it visually apparent which part of the region we are in. It also permits us to recycle the subscripts. Instead of using subscripts j lying in the ranges $0 \leqslant i \leqslant a$ and $b \leqslant i \leqslant c$ for the various $M(i, j)$, we can put $A_{ji} = M(i, j)$ if $0 \leqslant i \leqslant a$ and $B_{ji} = M(b + i, j)$ if $0 \leqslant i \leqslant c - b$. The effect is that the arbitrary choice of the integer b disappears. The possible subscripts i for B_{ji} range naturally over the integers in $[0, c - b]$, and $[0, c - b]$ is the object in Δ that the data came from.

And finally we come to the switching of the indices. The logic is that when we represent a point in $\mathbb{Z}^2$ it is customary to write it as (i, j) where i is the x-coordinate and j is the y-coordinate. When we write a matrix A_{ji} the convention is the reverse; usually j stands for the row and i for the column that the entry is in. That is j is usually the y-coordinate, and i the x-coordinate. We are trying to be simultaneously consistent with both conventions.

Remark 4.11. In Remark 4.10 we said that many of our homotopies are minor variants of the homotopy of the proof of Lemma 4.7. This is correct; we can make the statement stronger by saying there is exactly one other homotopy we plan to use. Once again this homotopy is given by a 3-morphism in the category of functors $\Delta \longrightarrow \mathcal{C}$, as in Remark 4.2. Let us explain.

Let us again fix an object $S \in \Delta$. Now we consider a functor $H : \Delta \longrightarrow \mathcal{C}at$ taking an object $T \in \Delta$ to the ordered set $HT = S \cup \{[0, 1] \times T\}$, where the order is that every element of S is smaller than every element of $[0, 1] \times T$, while in each of S and $[0, 1] \times T$ the order is obvious.

Next we define a 2-morphism $\varphi : H \longrightarrow H$. For every $T \in \Delta$ we need to produce a functor $\varphi_T : HT \longrightarrow HT$. The formula is

$$\text{If } s \in S, \quad \varphi_T(s) = s;$$

$$\text{If } t \in T, \quad \varphi_T\big((i, t)\big) = \begin{cases} \max(S) & \text{if } i = 0, \\ (1, t) & \text{if } i = 1. \end{cases}$$

The formula makes it clear that $\varphi_T(j) \leqslant j$ for all $j \in HT$, hence there is a (unique) map $\alpha_T : \varphi_T \longrightarrow 1$. It defines a 3-morphism.

Finally we need to establish the technical conditions of Remark 4.2(iii). For $T = [0, 1]$ we have that $HT = S \cup \{[0, 1] \times T\} = S \cup \{[0, 1] \times [0, 1]\}$. We define a map $\lambda : H[0, 1] \longrightarrow [0, 1]$ by the formula

$$H(s) = 0 \qquad \text{if } s \in S,$$

$$H(i, j) = j \quad \text{if } (i, j) \in [0, 1] \times [0, 1].$$

We leave it to the reader to check the condition of Remark 4.2(iii).

Remark 4.12. The construction in Remark 4.11 gives a simplicial homotopy. As in Example 4.6, we can multiply by a fixed object $R \in \Delta$ and return to familiar territory. We explain this with coordinates.

Let us embed the fixed objects $R, S \in \Delta$ respectively as $[0, d], [0, a] \subset \mathbb{Z}$. Let us embed the object $T \in \Delta$ as $[b, c] \subset \mathbb{Z}$, with $a < b$. The category $HT = S \cup \{[0, 1] \times T\}$ can be embedded in $\mathbb{Z} \times \mathbb{Z}$ as

$$
\begin{array}{ccccccc}
 & & (b,1) & \longrightarrow & \cdots & \longrightarrow & (c,1) \\
 & & \uparrow & & & & \uparrow \\
\boxed{(0,0) \longrightarrow \cdots \longrightarrow (a,0)} & \longrightarrow & (b,0) & \longrightarrow & \cdots & \longrightarrow & (c,0)
\end{array}
$$

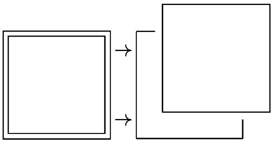
Multiplying by the fixed $R = [0, d] \subset \mathbb{Z}$ we embed $\mathcal{R} = R \times HT$ as the union of the two slices ${}_0\mathcal{R}$ and ${}_1\mathcal{R}$ (see Definition 2.1 for slices), pictured in coordinates below:

$$
\begin{array}{|ccccc|ccccccc}
\hline
(0,d,0) & \longrightarrow & \cdots & \longrightarrow & (a,d,0) & \longrightarrow & (b,d,0) & \longrightarrow & \cdots & \longrightarrow & (c,d,0) \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
(0,0,0) & \longrightarrow & \cdots & \longrightarrow & (a,0,0) & \longrightarrow & (b,0,0) & \longrightarrow & \cdots & \longrightarrow & (c,0,0) \\
\hline
\end{array}
$$

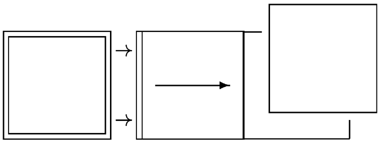
and

$$
\begin{array}{ccccc}
(b,d,1) & \longrightarrow & \cdots & \longrightarrow & (c,d,1) \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
(b,0,1) & \longrightarrow & \cdots & \longrightarrow & (c,0,1)
\end{array}
$$

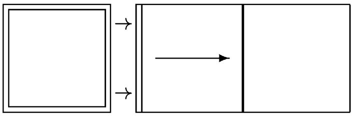
In other words the region is nothing other than



What we learn is that on this region there is a homotopy, the product with $R \in \Delta$ of the homotopy of Remark 4.11. We will denote this homotopy by the symbol



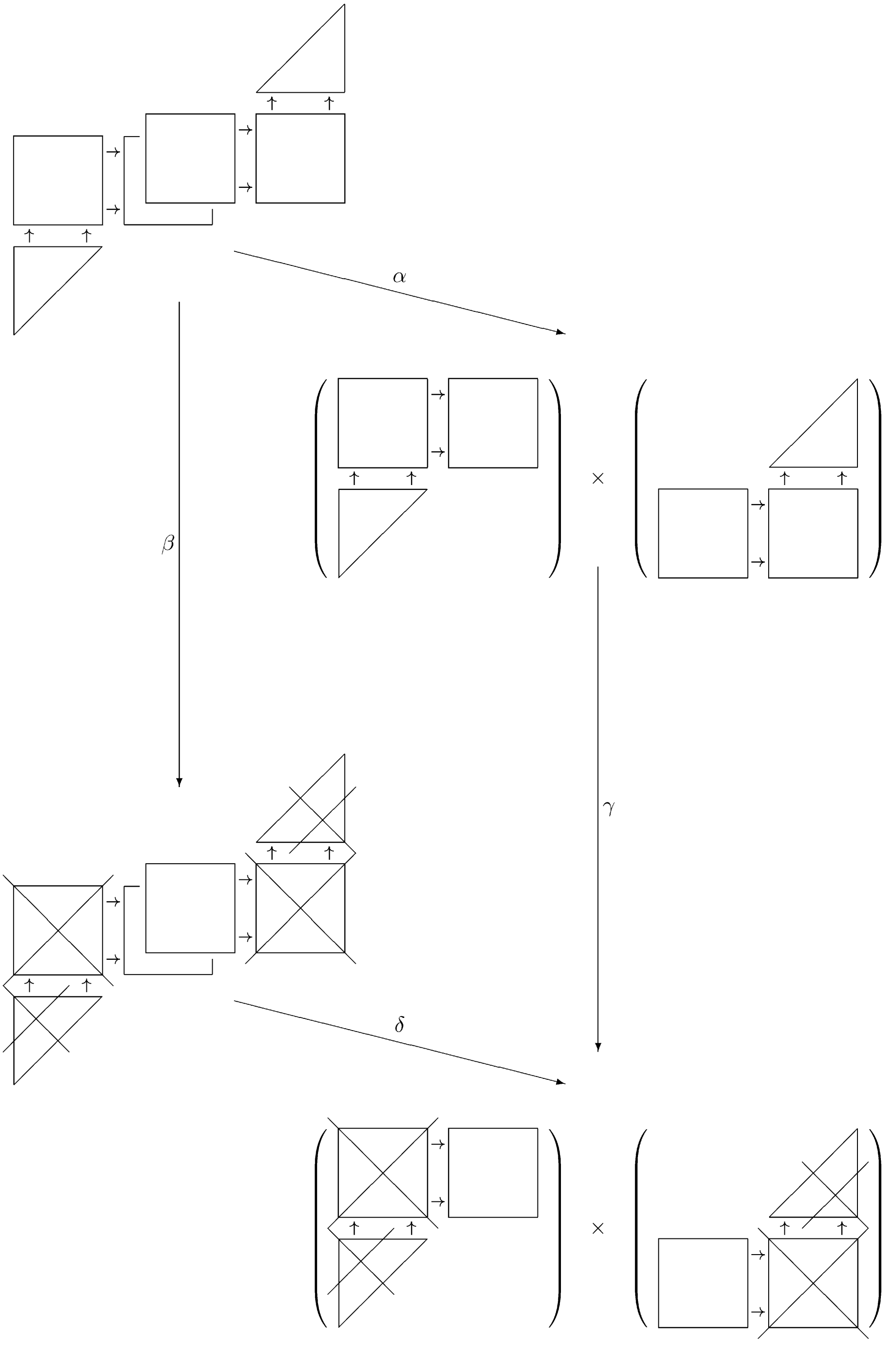
We will feel free to use this homotopy, its dual, and concatenate it into larger regions as in Remark 2.9. On the slice ${}_0\mathcal{R}$ of our region $\mathcal{R}$ the homotopy is the contraction to the initial object, that is the homotopy



of Example 4.6 and Remarks 4.8 and 4.9. On the slice ${}_1\mathcal{R}$ we have the trivial homotopy, connecting the identity map to the identity map.

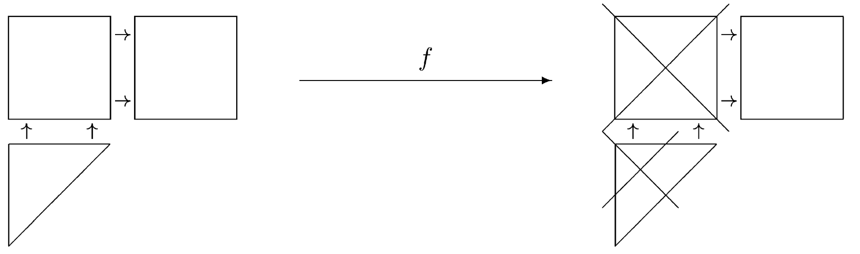
5. The proof

It is now time to get down to the proof. We need to prove that the map δ of Remark 3.19 induces a homotopy equivalence. To do so we embed δ in the following commutative diagram:

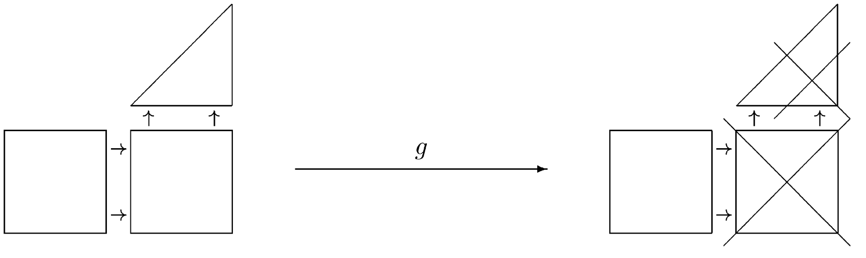


We need to prove that δ is a homotopy equivalence. The commutativity of the diagram means it suffices to show that α, β and γ are homotopy equivalences.

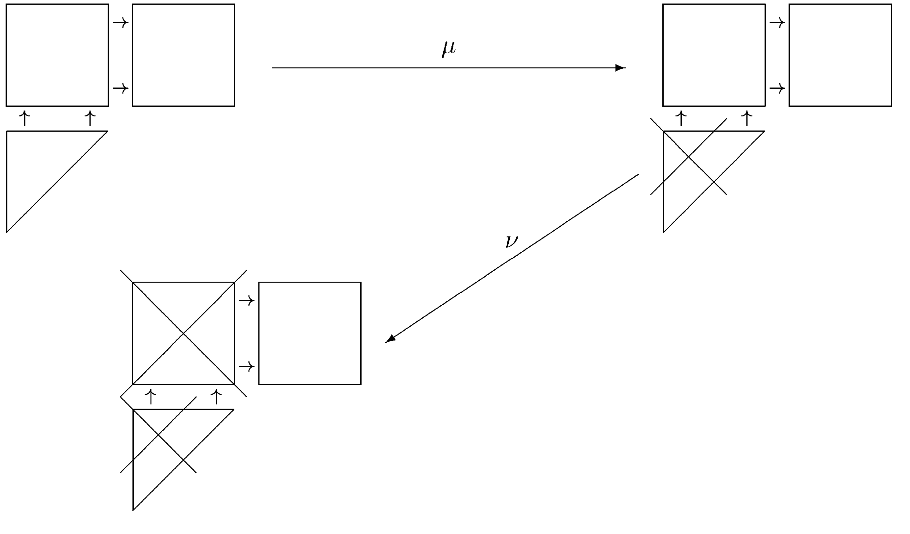
The maps β and γ are similar, so we will deal with them first. Let us take γ. The map γ is a product $\gamma = f \times g$ of two projections, as below:



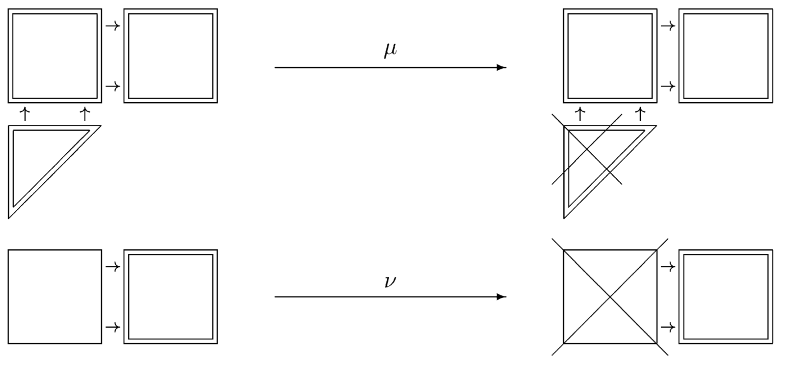
and



We want to show that the two projections f and g are homotopy equivalences. The two being dual, it suffices to show that f is a homotopy equivalence. We factor the map f as a composite



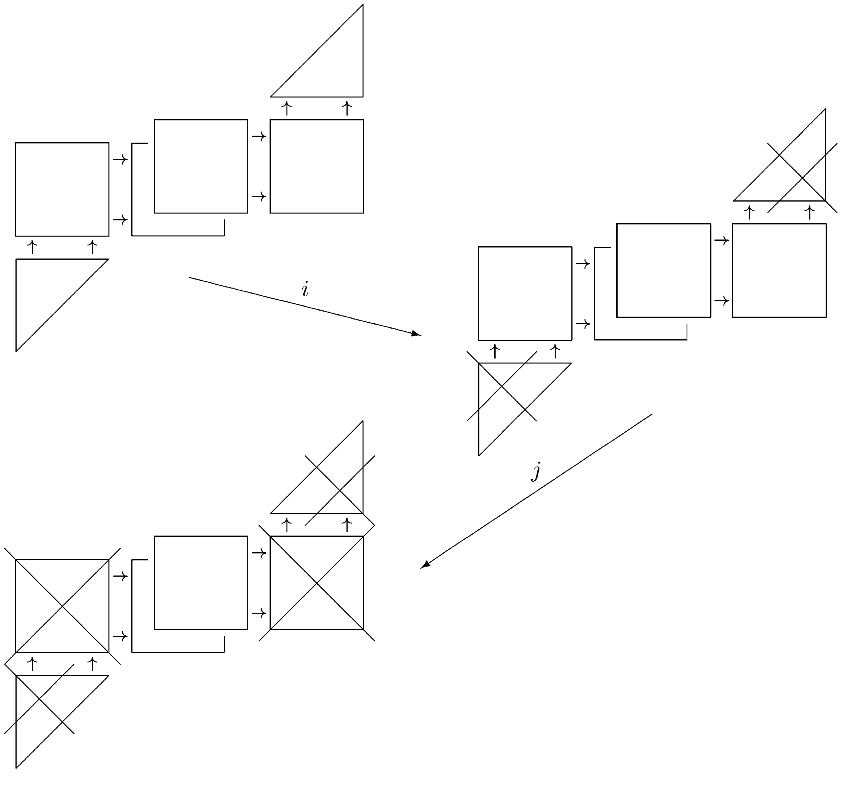
and it suffices to show that μ and ν are homotopy equivalences. For each of μ and ν we realise only one simplicial structure. In the notation of Remark 3.7 we prove that the two maps



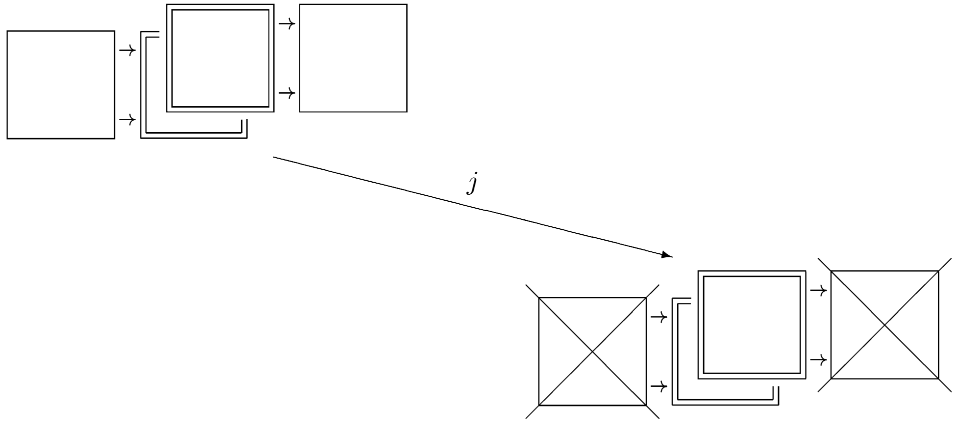
are both homotopy equivalences. The fact that ν is a homotopy equivalence is dual to Lemma 4.7. We need to show that μ is a homotopy equivalence; we note

Caution 5.1. This is one of the points that requires us to know a little more about derivators. After developing the theory of triangulated derivators, we will show that the map μ induces an equivalence of groupoids. See Proposition 8.13.

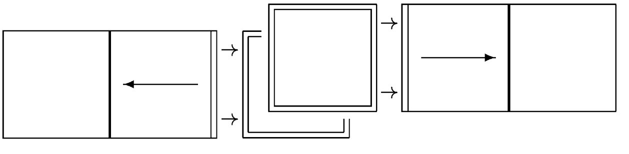
So far we have shown that, in the diagram on page 1422, the map γ is a homotopy equivalence. Next we move to the map β; the map β is the composite of



and it suffices to prove i and j are homotopy equivalences. Once again, we are free to realise any of the simplicial structures we wish. For the map j we realize the simplicial structures as indicated below

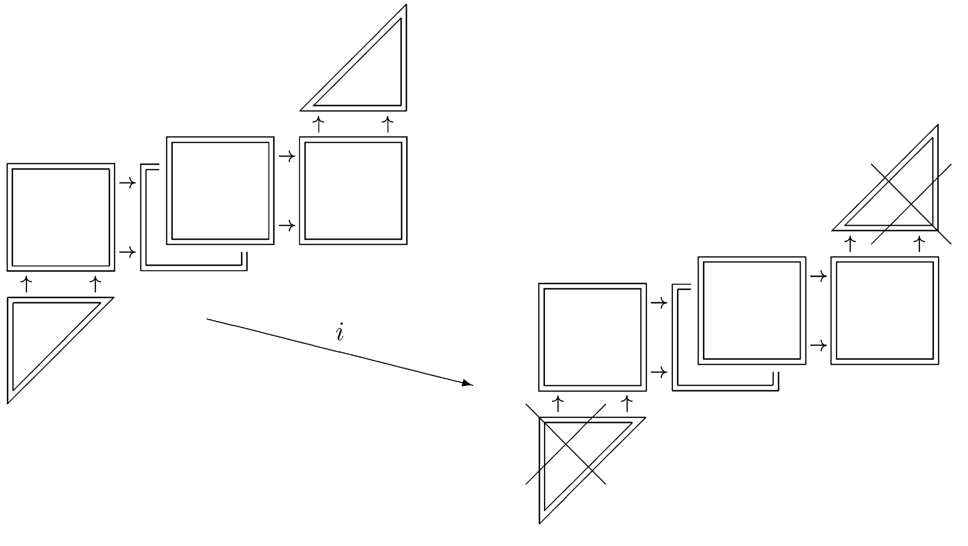


There is a homotopy which, in the notation of Remark 4.9 and the obvious dual, we would write



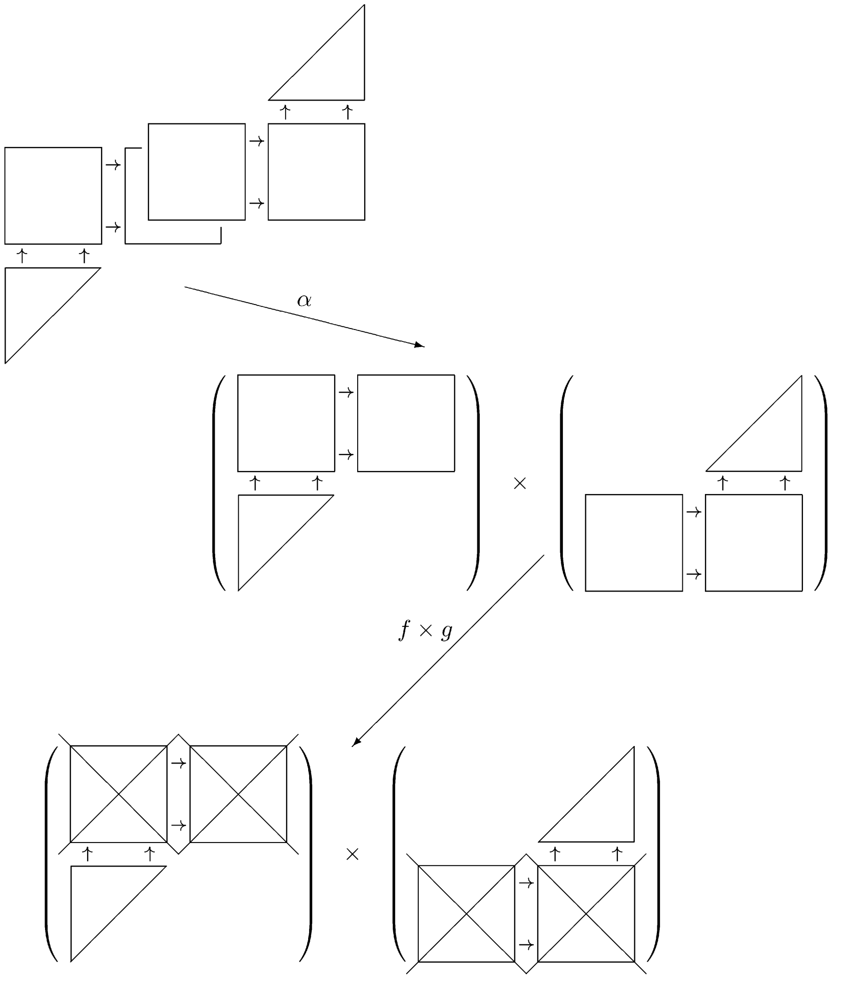
This homotopy shows, as in the proof of Lemma 4.7, that the map j is a homotopy equivalence.

The map i is more subtle. We realize only the groupoid structure; that is, we consider the simplicial map

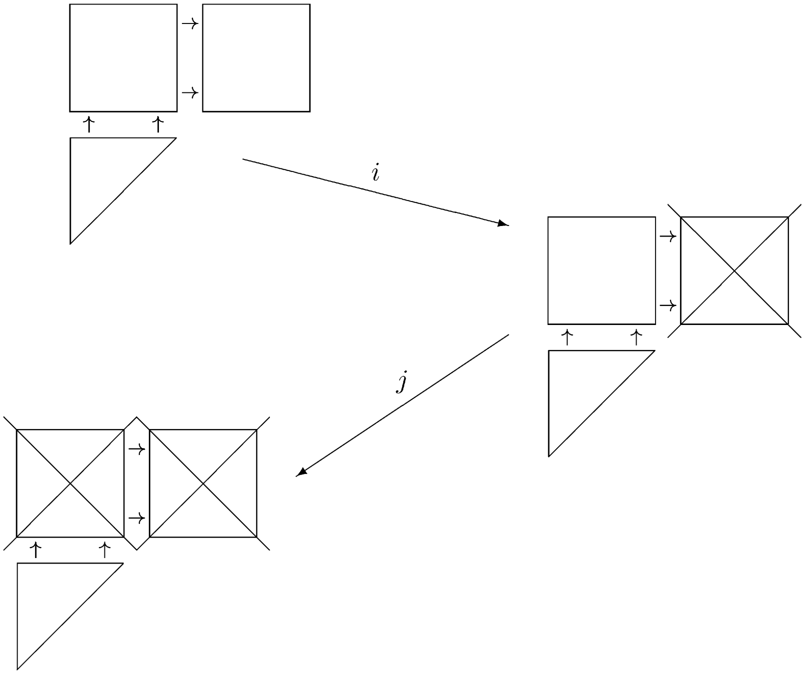


Caution 5.2. Once again, to prove that this map is a homotopy equivalence is more subtle and requires information we do not yet have, about triangulated derivators. As in Caution 5.1, Proposition 8.13 will show us that the map i is an equivalence of groupoids.

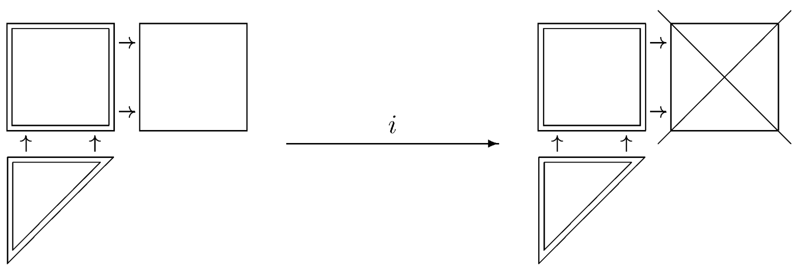
Combining the results so far we have established that, in the diagram on page 1422, the maps β and γ are homotopy equivalences. Only the map α remains. We first note that in the diagram



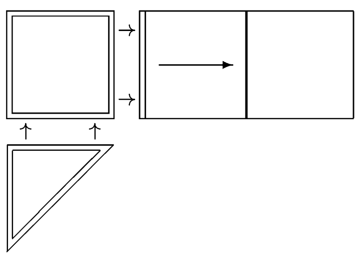
it suffices to prove that both the map $f \times g$ and the composite $(f \times g) \circ \alpha$ are homotopy equivalences; it then formally follows that so is α. To see that each of f and g are homotopy equivalences we first note that the two are dual to each other, hence it suffices to consider f. To show that f induces a homotopy equivalence first note that it can be factored as the composite



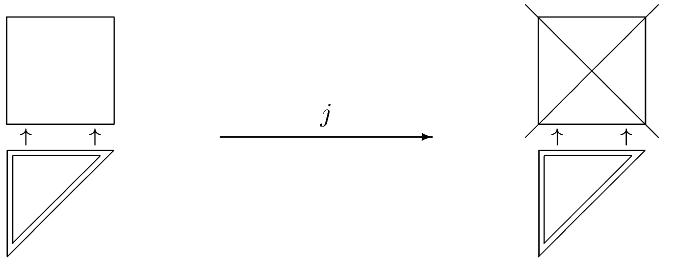
and it suffices to show that each of i and j induces a homotopy equivalence. For i we study the map



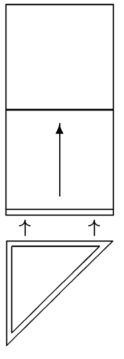
and, as in the proof of Lemma 4.7, the homotopy



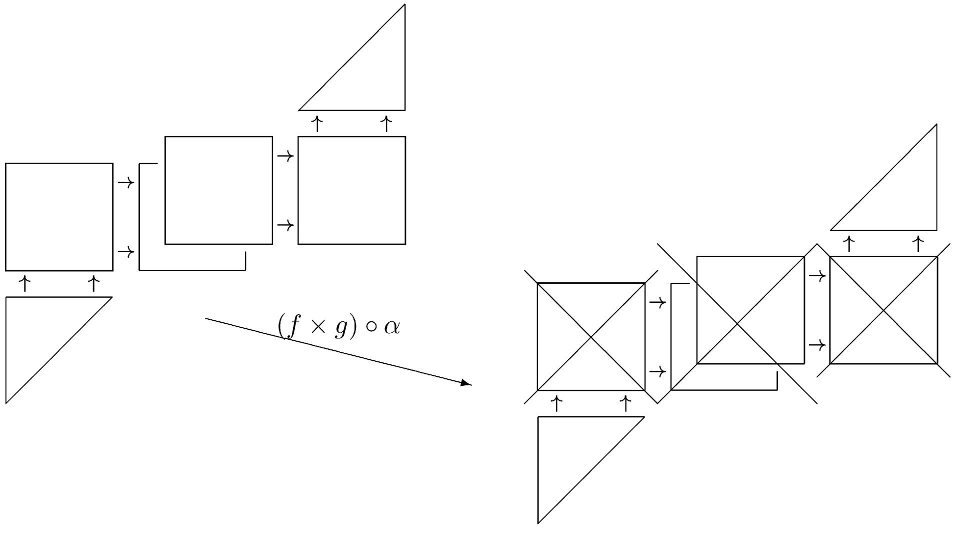
contracts the fibers. For the map j the simplicial structure we realize is as below



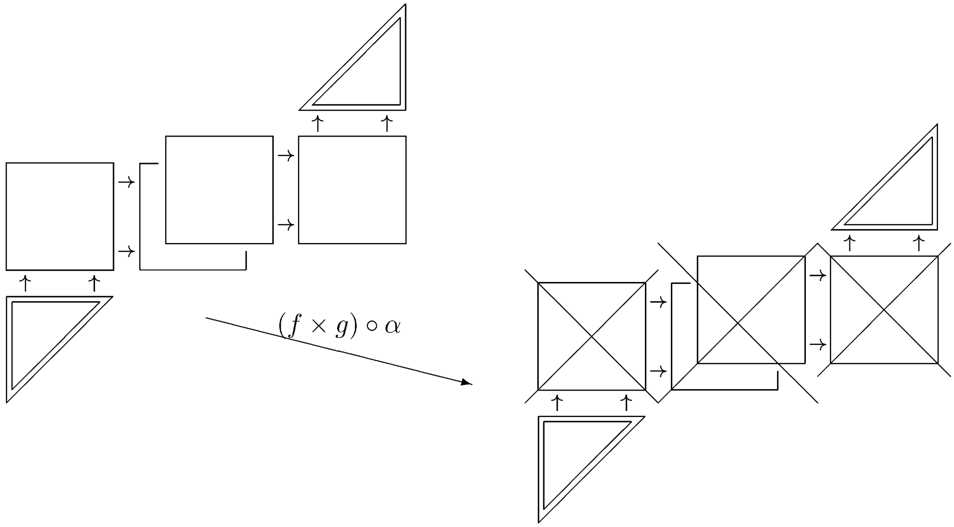
and the fiber is contracted by the homotopy



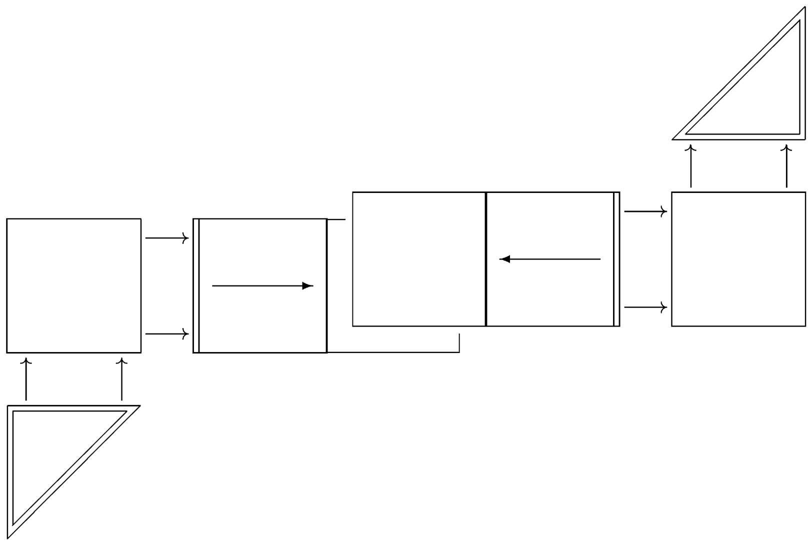
It remains therefore to show that the composite $(f \times g) \circ \alpha$ is a homotopy equivalence. The map is just



As always, we are free to realise any simplicial structures we wish. The map we will show to be a homotopy equivalence is

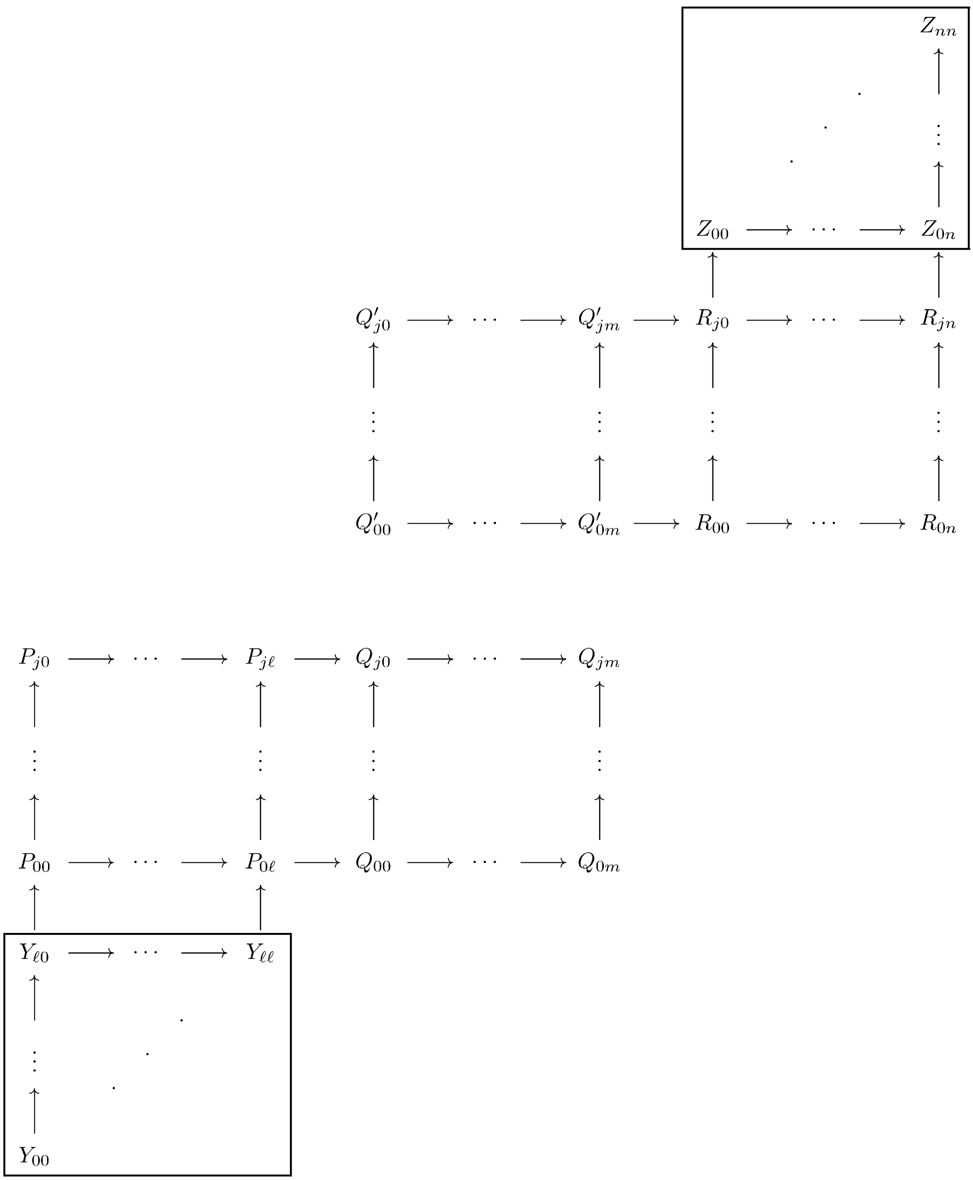


We will show that the simplicial set on the left contains a deformation retract on which the map $(f \times g) \circ \alpha$ is an equivalence of groupoids. The key is to produce the deformation retraction. To do it we need four homotopies. The first two of these are, in the notation of Remark 4.12,



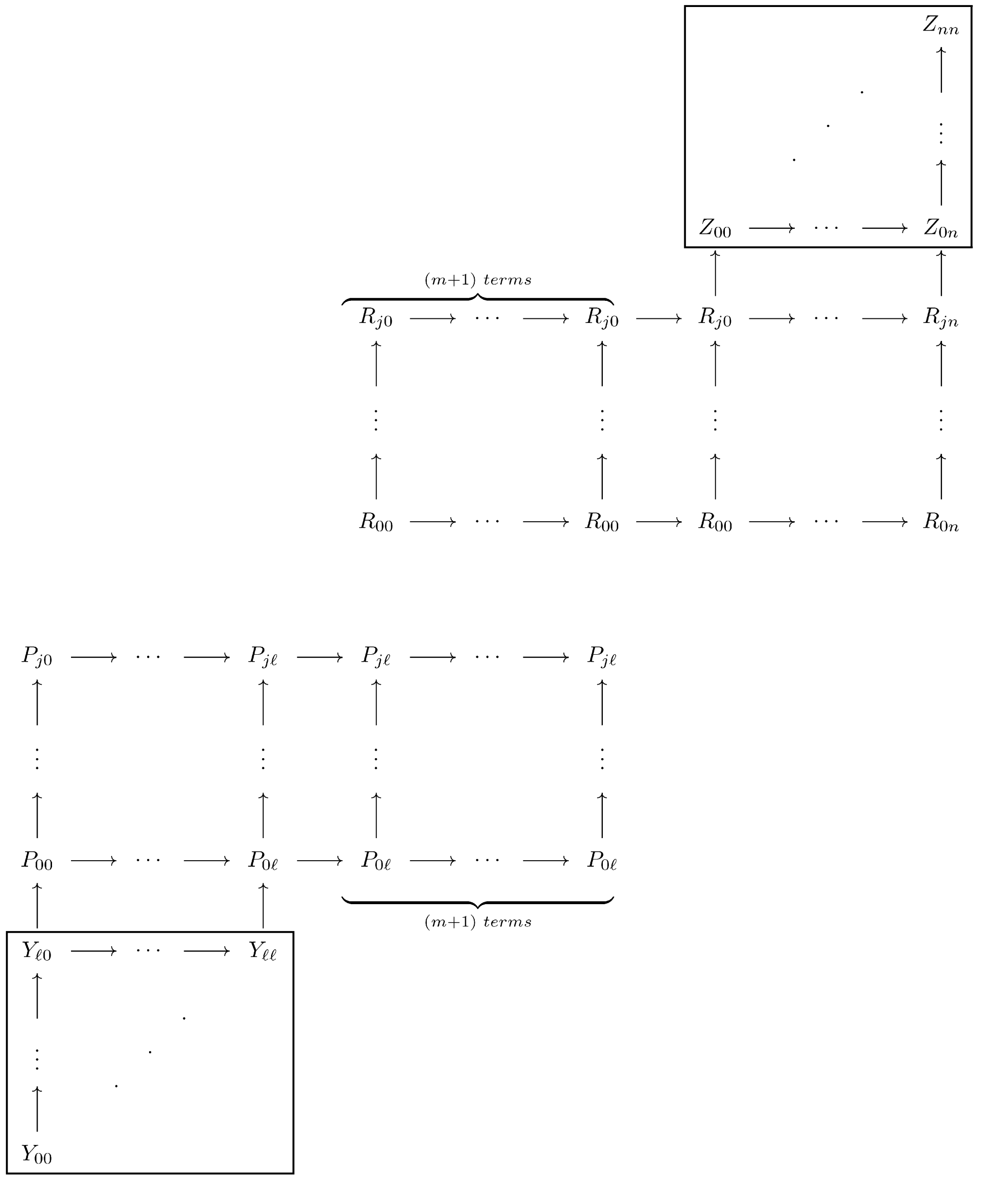
We saw quite explicitly how to compute these homotopies in Section 4. What is relevant to us now is that the homotopies connect the identity to a map φ. We will describe φ by giving its local shape; see 1.10. Given an object $x \in \Delta^2$ our cosimplicial region gives $H(x) \subset \mathbb{Z}^3$, and a simplex in the corresponding simplicial set is a string of isomorphisms of objects in $\mathbb{D}(H(x)^{\mathrm{op}})$. We split

$H(x)$ into its two slices ${}_0H(x)$ and ${}_1H(x)$, which have been drawn explicitly, with coordinates, in Remark 3.5. With the conventions of Remark 4.8 the local shape is a pair of diagrams in $\mathbb{D}(e)$

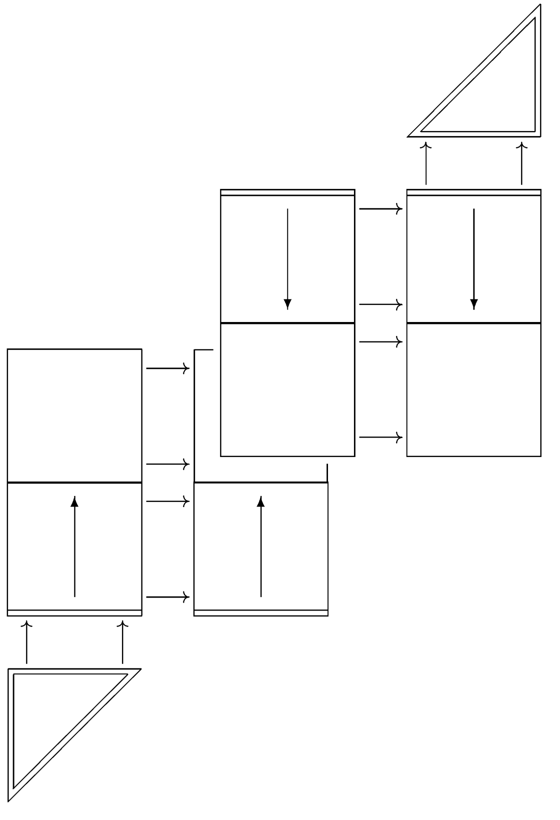


together with a map from the lower diagram to the upper diagram.

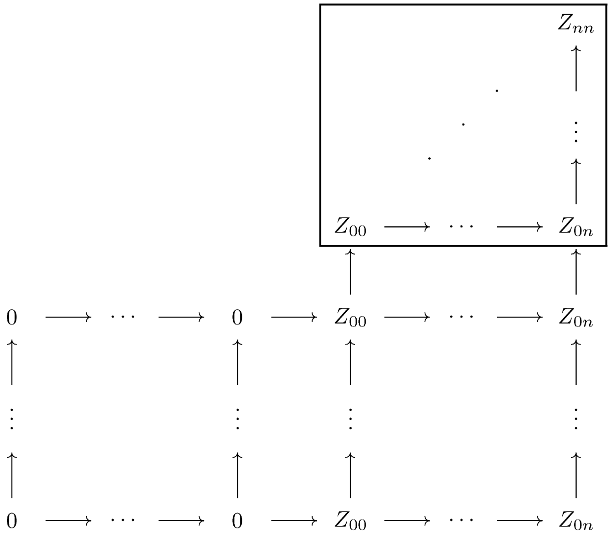
The simplicial map φ, which is homotopic to the identity, takes the simplex with the local shape above to the simplex

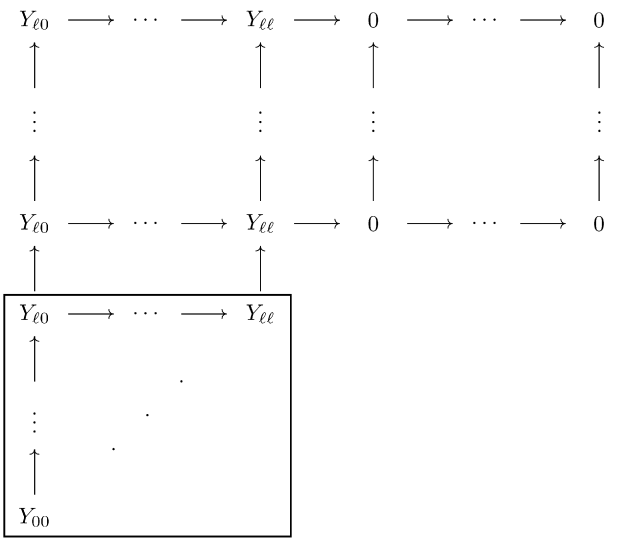


In other words the effect of the map φ is to replace the $Q_{ij} \longrightarrow Q'_{ij}$ by $P_{i\ell} \longrightarrow R_{i0}$. We can follow this by the homotopy of Remark 4.12



This homotopy connects the map φ above to a map ψ. Now remember that $Y_{\ell\ell}$ and Z_{00} are both isomorphic to the zero object in $\mathbb{D}(e)$; see Construction 3.4(i). The map ψ takes an object in $\mathbb{D}(H(x))$, with local shape pictured on page 1430, to the very special cell





We have a deformation retraction of the simplicial set onto a subset. And now we note

Caution 5.3. The objects, with local shape as above, decompose functorially as direct sums of objects vanishing on all but the top right and objects vanishing on all but the bottom left. This is true, but requires a more subtle understanding of triangulated derivators; the fact may be found in Proposition 9.7.

Modulo Proposition 9.7 we have that the map $(f \times g) \circ \alpha$ is homotopic to an equivalence of groupoids, finishing the proof of Maltsiniotis' conjecture.

6. Cofinality and base change maps

This section sets up the elementary lemmas which we will use, time and again, in manipulating (triangulated) derivators.

Lemma 6.1. *Let $\mathbb{D}$ be a prederivator, let X, Y be objects of $\mathcal{D}ia$ and let $u : X \longrightarrow Y$ be left adjoint to the map $v : Y \longrightarrow X$. Then $u^* : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$ is left adjoint to $v^* : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)$. Furthermore if u is fully faithful then so is v^*, and if v is fully faithful then so is u^*.*

Proof. Apply the 2-functor $\mathbb{D}$ to the unit $\eta : 1 \Longrightarrow vu$ and counit $\varepsilon : uv \Longrightarrow 1$ of the adjunction, to obtain the counit and unit of the adjunction $u^* \dashv v^*$. □

Proposition 6.2. *Let $\mathbb{D}$ be a prederivator, and let X be a category in $\mathcal{D}ia$ with a terminal object v. Then the functor $p_X^* : \mathbb{D}(e) \longrightarrow \mathbb{D}(X)$ is fully faithful, its right adjoint*

$$\Gamma_*(X, -) : \mathbb{D}(X) \longrightarrow \mathbb{D}(e)$$

exists, and $\Gamma_*(X, -)$ is canonically isomorphic to the evaluation functor at the point $v \in X$

$$v^* : \mathbb{D}(X) \longrightarrow \mathbb{D}(e).$$

Proof. Apply Lemma 6.1 with $u = p_X : X \longrightarrow e$. □

Proposition 6.3. *Let $\mathbb{D}$ be a derivator. Let $u : X \longrightarrow Y$ be a map in $\mathcal{D}ia$. Assume that u has a left adjoint. Then the following assertions are true.*

(i) *For any object F of $\mathbb{D}(e)$, the canonical map*

$$u_!(F_X) = u_! u^*(F_Y) \longrightarrow F_Y$$

is an isomorphism.

(ii) *For any object G in $\mathbb{D}(Y)$, the canonical map*

$$\Gamma_*(Y, G) \longrightarrow \Gamma_*\big(X, u^*(G)\big)$$

is an isomorphism.

Proof. Let us first see that (i) implies (ii). Let $p = p_Y$ be the projection $p : Y \longrightarrow e$. By (i) the natural map $u_! u^* p^* \Longrightarrow p^*$ is an isomorphism. Taking right adjoints we have an isomorphism $p_* \Longrightarrow p_* u_* u^*$, meaning that (ii) follows.

It remains to prove (i). Fix an object $F \in \mathbb{D}(e)$. By Der 2 it suffices to show that, for every point $y \in Y$, the map

$$(*) \qquad u_!(F_X)_y = \big(u_! u^*(F_Y)\big)_y \longrightarrow (F_Y)_y$$

is an isomorphism. Now note that $(F_Y)_y = F$, that $y\backslash F_X = F_{y\backslash X}$, and that Der 4 gives us a canonical identification

$$\Gamma_!(y\backslash X, F_{y\backslash X}) \simeq u_!(F_X)_y.$$

Hence the map $(*)$ identifies with

$$\Gamma_!(y\backslash X, F_{y\backslash X}) \longrightarrow F,$$

which is an isomorphism by the dual of Proposition 6.2 and because $y\backslash X$ has an initial object. □

Remark 6.4. Proposition 6.3 has a dual version (just apply it to $\mathbb{D}^{\mathrm{op}}$): if a map $u : X \longrightarrow Y$ has a right adjoint then, for any object F of $\mathbb{D}(Y)$, the canonical map

$$\Gamma_!\big(X, u^*(F)\big) \longrightarrow \Gamma_!(Y, F)$$

is an isomorphism.

6.5. For a functor $f : X \longrightarrow Y$ in $\mathcal{D}ia$, and an object y of Y, we denote by $X_y = f^{-1}(y)$ the fiber of f over the point y. If f is a fibration in the sense of Grothendieck (see e.g. [7, Exposé VI]), then the functor

$$X_y \longrightarrow y\backslash X, \qquad x \longmapsto (x, 1_y)$$

has a right adjoint. Dually, if a map $f : X \longrightarrow Y$ in $\mathcal{D}ia$ is an opfibration then, for any point y of Y, the canonical functor $X_y \longrightarrow X/y$ has a left adjoint.

Example 6.6. For us, the most important example of an opfibration will be the projection $X \times W \longrightarrow W$; see the proof of Lemma 7.6. The most important example of a fibration will be the map $X/y \longrightarrow X$; see the proof of Proposition 7.8.

6.7. We set up a little more notation. Let $u : X \longrightarrow Y$ be map in $\mathcal{D}ia$, let y be a point of Y and let F be an object of $\mathbb{D}(X)$. If i is the inclusion $i : X_y \longrightarrow X$, we will write $F_y = i^*(F)$ for the object $F_y \in \mathbb{D}(X_y)$. If $j : X_y \longrightarrow X/y$ denotes the canonical functor, one then has $j^*(F/y) = F_y$. We thus have a canonical morphism

$$\Gamma_*(X/y, F/y) \longrightarrow \Gamma_*(X_y, F_y).$$

One can check that the composed map

$$u_*(F)_y \longrightarrow \Gamma_*(X/y, F/y) \longrightarrow \Gamma_*(X_y, F_y)$$

is the base change morphism associated to the pullback square below (see 1.9)

$$\begin{array}{ccc} X_y & \xrightarrow{\ i\ } & X \\ \downarrow & & \downarrow{\scriptstyle u} \\ e & \longrightarrow & Y. \end{array}$$

Now that the notation is in place we make the following observations, about the way derivators behave with respect to fibrations and opfibrations.

Lemma 6.8. *Let $\mathbb{D}$ be a derivator. Let $f : X \longrightarrow Y$ be an opfibration in $\mathcal{D}ia$ and F an object of $\mathbb{D}(X)$. Then for any point y of Y the canonical map*

$$f_*(F)_y \longrightarrow \Gamma_*(X_y, F_y)$$

is an isomorphism.

Proof. In (6.7) we saw that the map of Lemma 6.8 factors as

$$f_*(F)_y \xrightarrow{\alpha} \Gamma_*(X/y, F/y) \xrightarrow{\beta} \Gamma_*(X_y, F_y).$$

The fact that α is an isomorphism is by Der 4. The fact that β is an isomorphism follows from Proposition 6.3(ii), because the canonical functor $X_y \longrightarrow X/y$ has a left adjoint. □

Proposition 6.9. *We consider the following pullback square in $\mathcal{D}ia$*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y. \end{array}$$

If the functor u is a fibration or if the functor f is an opfibration, then the base change map (1.9)

$$u^* f_*(F) \longrightarrow g_* v^*(F)$$

is an isomorphism for any object F of $\mathbb{D}(X)$.

Proof. We first prove the result assuming that f is an opfibration. By Lemma 6.8 we already know this property locally (i.e. when Y' is the point). Moreover, as opfibrations are stable by pullback (see [7, Exposé VI, Corollaire 6.9]), the functor g is also an opfibration. Let y' be a point of Y', and consider the pullback diagram

$$\begin{array}{ccccc} X'_{y'} & \xrightarrow{i} & X' & \xrightarrow{v} & X \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ e & \xrightarrow[y']{} & Y' & \xrightarrow[u]{} & Y. \end{array}$$

We can now apply Lemma 6.8 to the two pullback squares

$$\begin{array}{ccc} X'_{y'} & \xrightarrow{i} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle g} \\ e & \xrightarrow[y']{} & Y' \end{array} \qquad\qquad \begin{array}{ccc} X'_{y'} & \xrightarrow{vi} & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle f} \\ e & \xrightarrow[u(y')]{} & Y \end{array}$$

obtaining isomorphisms, for every object $F \in \mathbb{D}(X)$,

$$y'^*u^*f_*(F) \simeq y'^*g_*v^*(F) \simeq \pi_*i^*v^*(F).$$

That is the base-change map $u^*f_*(F) \longrightarrow g_*v^*(F)$, which is a morphism in $\mathbb{D}(Y')$, induces an isomorphism in $\mathbb{D}(e)$ for any $y' : e \longrightarrow Y'$. By Der 2 we conclude that $u^*f_*(F) \longrightarrow g_*v^*(F)$ is an isomorphism.

The case where u is a fibration is a direct consequence of the opfibration case. If u is a fibration, then u^{op} is an opfibration. But cohomological direct image functors in $\mathbb{D}^{\mathrm{op}}$ correspond to homological direct image functors in $\mathbb{D}$. Thus, for any object F in $\mathbb{D}(Y')$, the base change map of $\mathbb{D}^{\mathrm{op}}$ comes to

$$v_!g^*(F) \longrightarrow f^*u_!(F),$$

and must be an isomorphism. The right adjoint of the isomorphism $v_!g^* \Longrightarrow f^*u_!$ is the base change map $u^*f_* \Longrightarrow g_*v^*$ of $\mathbb{D}$. □

7. Basic properties of global cartesian squares

In this section we study global cartesian squares and their functoriality properties.

Proposition 7.1. *Let $\mathbb{D}$ be a derivator and let $u : X \longrightarrow Y$ be a map in $\mathcal{D}ia$. If u is fully faithful then the two direct image functors*

$$u_! : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y) \quad \textit{and} \quad u_* : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)$$

are both fully faithful.

Proof. The statements being dual, it suffices to prove that the functor u_* is fully faithful. This is equivalent to proving that, for any object F in $\mathbb{D}(X)$, the counit map

$$\varepsilon_F : u^*u_*(F) \longrightarrow F$$

is an isomorphism. Let $F \in \mathbb{D}(X)$ be an object and $x \in X$ a point. We have the following identifications

$$\begin{aligned}\bigl(u^*u_*(F)\bigr)_x &= u_*(F)_{u(x)} \\ &= \Gamma_*\bigl(X/u(x), F/u(x)\bigr).\end{aligned}$$

But as the functor u is fully faithful the canonical map from X/x to $X/u(x)$ is an isomorphism. Hence there is a canonical isomorphism

$$\Gamma_*\bigl(X/u(x), F/u(x)\bigr) \simeq \Gamma_*(X/x, F/x).$$

As the category X/x has a terminal object $(x, 1_x)$, Proposition 6.2 identifies

$$\Gamma_*(X/x, F/x) \simeq (F/x)_{(x,1_x)} = F_x.$$

In conclusion we get a canonical isomorphism from $(u^*u_*(F))_x$ to F_x, and one checks easily that this is the evaluation of the counit map ε_F at the point x. The result now follows from Der 2. □

Proposition 7.2. *As in Proposition* 7.1 *let $\mathbb{D}$ be a derivator, and let $u : X \longrightarrow Y$ be a fully faithful functor in $\mathcal{D}ia$. An object $F \in \mathbb{D}(Y)$ belongs to the essential image of $u_* : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)$ if and only if, for every object $y \in Y - u(X)$, the unit of adjunction $\eta_F : F \longrightarrow u_*u^*F$ induces an isomorphism*

$$(\eta_F)_y : F_y \longrightarrow u_*u^*(F)_y.$$

Dually, the essential image of $u_! : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)$ consists of the objects $F \in \mathbb{D}(Y)$ so that, for all $y \in Y - u(X)$, the map

$$(\varepsilon_F)_y : u_!u^*(F)_y \longrightarrow F_y$$

is an isomorphism.

Proof. The two statements being dual, it suffices to prove the assertion for u_*. Proposition 7.1 tells us that the functor u_* is fully faithful. Formal properties about adjoints of fully faithful functors yield that

(1) The natural transformation $\varepsilon : u^*u_* \longrightarrow 1$ is an isomorphism.
(2) An object $F \in \mathbb{D}(Y)$ belongs to the essential image of u_* if and only if the map $\eta_F : F \longrightarrow u_*u^*F$ is an isomorphism.

The axiom Der 2 (the conservativity axiom) says that the map $\eta_F : F \longrightarrow u_*u^*F$ will be an isomorphism if and only if, for every point $y \in Y$, the induced map

$$(\eta_F)_y : F_y \longrightarrow u_*u^*(F)_y$$

is an isomorphism. What the proposition asserts is that it suffices to check the points $y \in Y - u(X)$; in other words we will show that, for every $y \in u(X)$, the map $(\eta_F)_y$ is automatically an isomorphism.

Formal facts about adjoints tell us that the composite

$$u^*F \xrightarrow{u^*\eta_F} u^*u_*u^*(F) \xrightarrow{\varepsilon_{u^*(F)}} u^*F$$

is the identity. The fact that u_* is fully faithful gives (see fact (1) above) that the map $\varepsilon_{u^*(F)}$ is an isomorphism. It follows that $u^*\eta_F$ must be the 2-sided inverse of the isomorphism $\varepsilon_{u^*(F)}$; it is invertible. But then, for any point $x \in X$, we have that

$$x^*u^*\eta_F : x^*u^*F \longrightarrow x^*u^*u_*u^*(F)$$

is an isomorphism. This precisely asserts that $(\eta_F)_{u(x)} : F_{u(x)} \longrightarrow u_*u^*(F)_{u(x)}$ must be an isomorphism. □

Proposition 7.3. *Let $\mathbb{D}$ be a derivator. We will use the notations of* 1.14. *We have a functor $\sigma : \lrcorner \longrightarrow \square$ from the 'corner category' to the 'square category.' We consider an object C of $\mathbb{D}(\square)$. Then the following conditions are equivalent:*

(i) *The global commutative square C is cartesian.*
(ii) *The unit map $C \longrightarrow \sigma_*\sigma^*(C)$ is an isomorphism.*
(iii) *The object C lies in the essential image of the functor $\sigma_* : \mathbb{D}(\lrcorner) \longrightarrow \mathbb{D}(\square)$.*
(iv) *One has a canonical isomorphism $C_{0,0} \simeq \Gamma_*(\lrcorner, \sigma^*(C))$.*

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is almost by definition; C is cartesian if and only if, for every object $B \in \mathbb{D}(\square)$, the map

$$\mathsf{Hom}_{\mathbb{D}(\square)}(B, C) \longrightarrow \mathsf{Hom}_{\mathbb{D}(\lrcorner)}(\sigma^* B, \sigma^* C) \cong \mathsf{Hom}_{\mathbb{D}(\square)}(B, \sigma_*\sigma^* C)$$

is an isomorphism. This is equivalent to $C \longrightarrow \sigma_*\sigma^*(C)$ being an isomorphism.

Next observe that the inclusion $\sigma : \lrcorner \hookrightarrow \square$ is fully faithful, and Proposition 7.1 tells us that $\sigma_* : \mathbb{D}(\lrcorner) \longrightarrow \mathbb{D}(\square)$ is also fully faithful. The formal properties of adjoints of fully faithful functors say that (ii) is equivalent to (iii). Proposition 7.2 tells us that (iii) happens if and only if $C_y \longrightarrow \sigma_*\sigma^*(C)_y$ is an isomorphism for the one point in $\square - \lrcorner$, that is for the point $y = (0, 0)$. The base change axiom tells us that $\sigma_*\sigma^*(C)_y \simeq \Gamma_*(\lrcorner, \sigma^*(C))$ and hence the equivalence of (iii) and (iv). $\square$

7.4. Given a triangulated derivator $\mathbb{D}$ and an object $W \in \mathcal{D}ia$, it is possible to form a new triangulated derivator $\mathbb{D}_W$. We saw, in Caution 3.14(i), the role this derivator plays in the additivity theorem and in its proof. In the remainder of this section we will define $\mathbb{D}_W$, prove that it satisfies the axioms of a triangulated derivator, and study the relation between cartesian squares in $\mathbb{D}_W(\square)$ and cartesian squares in $\mathbb{D}(\square)$.

Let $\mathbb{D}$ be a derivator and W a category in $\mathcal{D}ia$. One defines a prederivator $\mathbb{D}_W$ as follows. For any category X in $\mathcal{D}ia$,

$$\mathbb{D}_W(X) = \mathbb{D}(X \times W).$$

For a functor $u : X \longrightarrow Y$ in $\mathcal{D}ia$, one has an inverse image functor given by

$$u^* = (u \times 1_W)^* : \mathbb{D}_W(Y) \longrightarrow \mathbb{D}_W(X),$$

and similarly for 2-cells. For any point w of W and any category X in $\mathcal{D}ia$, one defines a functor

$$\mathbb{D}_W(X) \longrightarrow \mathbb{D}(X), \qquad F \longmapsto F_w$$

by the formula $F_w = (1_X \times w)^*(F)$. This notation is compatible with the similar one introduced in 6.7.

Any functor $u : X \longrightarrow Y$ in $\mathcal{D}ia$ has a (co)homological direct image functor in $\mathbb{D}_W$. For example, the cohomological direct image u_* of u in $\mathbb{D}_W$ is defined as the cohomological direct image of $u \times 1_W$ in $\mathbb{D}$.

Lemma 7.5. *Let X be a category in $\mathcal{D}ia$. The family of functors indexed by points w of W*

$$\mathbb{D}_W(X) \longrightarrow \mathbb{D}(X), \qquad F \longmapsto F_w$$

is conservative.

Proof. Immediate, since the family of functors $\mathbb{D}_W(X) \longrightarrow \mathbb{D}(X) \longrightarrow \mathbb{D}(e)$, taking F to F_w to $F_{(x,w)}$, is conservative. $\square$

Lemma 7.6. *Let X be a category in $\mathcal{D}ia$ and F an object of $\mathbb{D}_W(X)$. Then, for any object w of W, the canonical map*

$$\varGamma_*(X, F)_w \longrightarrow \varGamma_*(X, F_w)$$

is an isomorphism in $\mathbb{D}(e)$.

Proof. We consider, for any point w of W, the following cartesian square

$$\begin{array}{ccc} X & \xrightarrow{(1_X, w)} & X \times W \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ e & \xrightarrow[w]{} & W \end{array}$$

where p denotes the canonical projection. If we consider F as an object of $\mathbb{D}_W(X)$, then $\varGamma_*(X, F) = p_*(F)$ in $\mathbb{D}_W(e) = \mathbb{D}(W)$. But p is an opfibration so that, by Lemma 6.8, we have a canonical isomorphism

$$p_*(F)_w \simeq \varGamma_*\big(X, (1_X \times w)^*(F)\big) = \varGamma_*(X, F_w).$$

If we translate, using the notations of 7.4, we obtain a canonical isomorphism

$$\varGamma_*(X, F)_w \simeq \varGamma_*(X, F_w).$$

Hence the result. $\square$

Proposition 7.7. *A global commutative square C in $\mathbb{D}_W$ is cartesian if and only if, for any point w of W, the global commutative square C_w is cartesian in $\mathbb{D}$.*

Proof. Let $\sigma : \lrcorner \longrightarrow \square$ be the canonical inclusion. Applying Lemma 7.6 with $X = \lrcorner$ and $F = \sigma^*(C)$, we obtain a canonical isomorphism

$$\varGamma_*\big(\lrcorner, \sigma^*(C)\big)_w \simeq \varGamma_*\big(\lrcorner, \sigma^*(C)_w\big) = \varGamma_*\big(\lrcorner, \sigma^*(C_w)\big).$$

The result now follows from criterion (iv) of Proposition 7.3 and Lemma 7.5. $\square$

Proposition 7.8. *Let $\mathbb{D}$ be a derivator and let W be a category in $\mathcal{D}ia$. Then $\mathbb{D}_W$ is also a derivator. Moreover, if $\mathbb{D}$ is pointed (respectively triangulated) then so is $\mathbb{D}_W$.*

Proof. It is straightforward to check Der 1, 3 and 5. The conservativity axiom (Der 2) comes from Lemma 7.5. We need to check the base change axiom (Der 4). We proceed as follows. Let $u : X \longrightarrow Y$ be a functor in $\mathcal{D}ia$, y a point of Y, and F an object of $\mathbb{D}_W(X) = \mathbb{D}(X \times W)$. We then have the canonical pullback square below (see 1.1)

$$\begin{array}{ccc} X/y & \xrightarrow{i} & X \\ {\scriptstyle u/y}\downarrow & & \downarrow{\scriptstyle u} \\ Y/y & \xrightarrow[j]{} & Y. \end{array}$$

The product with W thus gives the following pullback square

$$\begin{array}{ccc} X/y \times W & \xrightarrow{i \times 1_W} & X \times W \\ {\scriptstyle u/y \times 1_W}\downarrow & & \downarrow{\scriptstyle u \times 1_W} \\ Y/y \times W & \xrightarrow[j \times 1_W]{} & Y \times W. \end{array}$$

The functor j is a Grothendieck fibration. As Grothendieck fibrations are stable by pullback, the functor $j \times 1_W$ is a fibration as well. From Proposition 6.9 we have a canonical isomorphism

$$(j \times 1_W)^*(u \times 1_W)_*(F) \simeq (u/y \times 1_W)_*(i \times 1_W)^*(F)$$

in $\mathbb{D}(Y/y \times W) = \mathbb{D}_W(Y/y)$. If we consider F as an object of $\mathbb{D}_W(X)$ and use the notations of 1.9, this isomorphism can also be written

$$u_*(F)/y \simeq (u/y)_*(F/y).$$

This leads to an isomorphism

$$\Gamma_*\big(Y/y, u_*(F)/y\big) \simeq \Gamma_*\big(Y/y, (u/y)_*(F/y)\big) \simeq \Gamma_*(X/y, F/y).$$

As the category Y/y has a final object (namely $(y, 1_y)$), Proposition 6.2 (applied to $\mathbb{D}_W$) tells us that one has a canonical isomorphism

$$u_*(F)_y = y^* u_*(F) = (y, 1_y)^*\big(u_*(F)/y\big) \simeq \Gamma_*\big(Y/y, u_*(F)/y\big).$$

Hence we have produced a canonical isomorphism

$$u_*(F)_y \simeq \Gamma_*(X/y, F/y).$$

The base change axiom is thus proved for $\mathbb{D}_W$.

It is easy to see that Der 6 for $\mathbb{D}$ implies Der 6 for $\mathbb{D}_W$. In other words, if $\mathbb{D}$ is pointed then so is $\mathbb{D}_W$.

Now assume Der 7 holds for $\mathbb{D}$. Proposition 7.7 and its dual tell us the following. If C is a global commutative square in $\mathbb{D}_W$ then the following are equivalent

$$\left\{\begin{array}{c} C \text{ is} \\ \text{cartesian} \\ \text{in } \mathbb{D}_W \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} C_w \text{ is} \\ \text{cartesian} \\ \text{in } \mathbb{D},\ \forall w \in W \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} C_w \text{ is} \\ \text{cocartesian} \\ \text{in } \mathbb{D},\ \forall w \in W \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} C \text{ is} \\ \text{cocartesian} \\ \text{in } \mathbb{D}_W \end{array}\right\}$$

which means that Der 7 holds for $\mathbb{D}_W$. In other words if $\mathbb{D}$ is triangulated then so is $\mathbb{D}_W$. □

7.9. Let $\mathbb{D}$ be a triangulated derivator. In Theorem 1.17 we learned that the category $\mathbb{D}(e)$ is triangulated. The proof, which we do not include in this article, teaches us further that the triangles, as well as the suspension functor, can be described explicitly as follows.

Since $\mathbb{D}$ is a pointed derivator, Der 1 and Der 3 ensure that the category $\mathbb{D}(e)$ has finite products and coproducts, and Der 6 implies that $\mathbb{D}(e)$ also has a zero object that we denote by 0 (that is 0 is both an initial and a terminal object in $\mathbb{D}(e)$). Let I be the category associated to the graph

$$0 \longleftarrow 1.$$

We then have two functors

$$s = 0 : e \longrightarrow I \quad \text{and} \quad t = 1 : e \longrightarrow I.$$

The functor s is a closed immersion, and the functor t an open immersion. We define two endofunctors of $\mathbb{D}(e)$, the *suspension functor* $\Sigma = t^? s_*$, and the *loop functor* $\Omega = s^! t_!$. It is clear, by construction, that the suspension functor is left adjoint to the loop functor. Der 7 implies that the suspension functor is an equivalence of categories. We sketch the argument: One can produce in a functorial way, for any object F in $\mathbb{D}(e)$, a cocartesian global commutative square that is locally of the shape

$$\begin{array}{ccc} F & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma(F). \end{array}$$

(We invite the reader to have a look at the situation described in 9.4 to understand the manipulation of the functors $s^!$ and $t^?$.) One can also show that if Σ is invertible, then the category $\mathbb{D}(e)$ is additive. For this one proceeds as in algebraic topology. One can also define a class of *distinguished triangles* in $\mathbb{D}(e)$ as follows. Let $\square_2$ be the category corresponding to the graph

$$\begin{array}{ccccc} (0,0) & \longleftarrow & (0,1) & \longleftarrow & (0,2) \\ \uparrow & & \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,1) & \longleftarrow & (1,2). \end{array}$$

We have three embedding functors from the square category $\square$ (see 1.14) to $\square_2$.

$$a:\square \longrightarrow \square_2, \qquad (i,j) \longmapsto (i,j),$$

$$b:\square \longrightarrow \square_2, \qquad (i,j) \longmapsto (i,j+1),$$

$$c:\square \longrightarrow \square_2, \qquad (i,j) \longmapsto (i,2j).$$

A *global triangle* in $\mathbb{D}$ is an object T of $\mathbb{D}(\square_2)$ such that $T_{1,0} \simeq T_{0,2} \simeq 0$ and such that $a^*(T)$ and $b^*(T)$ are cocartesian. The object T is then locally of shape

$$\begin{array}{ccccc} T_{0,0} & \longrightarrow & T_{0,1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_{1,1} & \longrightarrow & T_{1,2}. \end{array}$$

One can show that for any global triangle T, one has a canonical isomorphism $\Sigma T_{0,0} \simeq T_{1,2}$. Hence we get a triangle in $\mathbb{D}(e)$

$$T_{0,0} \longrightarrow T_{0,1} \longrightarrow T_{1,1} \longrightarrow \Sigma T_{0,0}.$$

We say that a triangle in $\mathbb{D}(e)$ is *distinguished* if it is isomorphic (as a triangle) to a triangle obtained from a global triangle as above. A precise statement of Theorem 1.17 is that, with these definitions of the suspension functor Σ and of distinguished triangles, one has defined a triangulated category structure on $\mathbb{D}(e)$. Note that Der 5 allows us to extend any morphism in $\mathbb{D}(e)$ to a distinguished triangle.

Corollary 7.10 *(Maltsiniotis). Let $\mathbb{D}$ be a triangulated derivator. For any category X in $\mathcal{D}ia$ the category $\mathbb{D}(X)$ is canonically endowed with a structure of a triangulated category. Furthermore for any functor $u : X \longrightarrow Y$ in $\mathcal{D}ia$ the inverse image functor $u^* : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$ is triangulated.*

Proof. Let X be any category in $\mathcal{D}ia$. By Proposition 7.8 the derivator $\mathbb{D}_X$ is triangulated, and by Theorem 1.17 the category $\mathbb{D}_X(e) = \mathbb{D}(X)$ is triangulated. The second assertion, on the exactness of inverse image functors, comes from the definition of distinguished triangles (7.9) and from Proposition 7.7. □

8. Completion by pullback

In the proof of the additivity theorem we twice run into the following situation. We are given a morphism $j : V \longrightarrow X$ in $\mathcal{D}ia$, and we want to know that some (explicit) subcategory $\mathcal{C} \subset \mathbb{D}(X)$ maps fully faithfully, onto its image $\mathcal{C}' \subset \mathbb{D}(V)$, under the functor $j^* : \mathbb{D}(X) \longrightarrow \mathbb{D}(V)$.

Example 8.1. Consider a triangulated derivator $\mathbb{D}$ and let X be the category associated to the following poset

$$\begin{array}{ccccccccc}
 & & (0,1) & \longleftarrow & (0,2) & \longleftarrow & (0,3) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
(1,0) & \longleftarrow & (1,1) & \longleftarrow & (1,2) & \longleftarrow & (1,3) & \longleftarrow & (1,4) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
(2,0) & \longleftarrow & (2,1) & \longleftarrow & (2,2) & \longleftarrow & (2,3) & \longleftarrow & (2,4).
\end{array}$$

Define $\mathcal{C}$ to the full subcategory of $\mathbb{D}(X)$, made up of objects F which are locally of the shape

$$\begin{array}{ccccccccc}
 & & F_{(0,1)} & \longrightarrow & F_{(0,2)} & \longrightarrow & F_{(0,3)} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
F_{(1,0)} & \longrightarrow & F_{(1,1)} & \longrightarrow & F_{(1,2)} & \longrightarrow & F_{(1,3)} & \longrightarrow & F_{(1,4)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_{(2,0)} & \longrightarrow & F_{(2,1)} & \longrightarrow & F_{(2,2)} & \longrightarrow & F_{(2,3)} & \longrightarrow & F_{(2,4)},
\end{array}$$

and furthermore satisfy

(i) The object $F_{(0,3)}$ is isomorphic to zero in $\mathbb{D}(e)$.
(ii) All the commutative squares are homotopy cartesian; this can be reformulated saying that, for any map of regions $i : \square \longrightarrow X$, the pullback $i^*(F)$ is cartesian in $\mathbb{D}(\square)$.

Consider also the subcategory V of X defined by the diagram below

$$\begin{array}{ccccccccc}
(1,0) & \longleftarrow & (1,1) & \longleftarrow & (1,2) & \longleftarrow & (1,3) & \longleftarrow & (1,4) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
(2,0) & \longleftarrow & (2,1) & \longleftarrow & (2,2) & \longleftarrow & (2,3) & \longleftarrow & (2,4).
\end{array}$$

If $j : V \longrightarrow X$ denotes the inclusion (which is an example of open immersion), then the functor

$$\mathcal{C} \longrightarrow \mathbb{D}(V), \qquad F \longmapsto j^*(F)$$

is fully faithful. The image may be identified as the full subcategory $\mathcal{C}' \subset \mathbb{D}(V)$, whose objects satisfy the analog of (ii) above; all commutative squares are homotopy cartesian. The proof is by a combination of Proposition 8.13 and Remark 8.10 below.

Remark 8.2. The above is a very simple example. The examples that come up in the proof of additivity may be found in Cautions 5.1 and 5.2. Note that, in the proof of the additivity theorem, we only need to know that the functor $j^* : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$ takes the isomorphisms in $\mathcal{C} \subset \mathbb{D}(Y)$ bijectively to the isomorphisms in the image $\mathcal{C}' \subset \mathbb{D}(X)$; we care only about the groupoids.

In what follows we formulate some facts, ever so slightly more general than we need. Together with their duals they easily cover what is used in Cautions 5.1 and 5.2.

8.3. Let us fix a derivator $\mathbb{D}$. Let $n \geqslant 1$ be an integer, and let $\square_n$ be the category associated with the graph

$$\begin{array}{ccccccc} (0,0) & \longleftarrow & (0,1) & \longleftarrow & \cdots & \longleftarrow & (0,n) \\ \uparrow & & \uparrow & & & & \uparrow \\ (1,0) & \longleftarrow & (1,1) & \longleftarrow & \cdots & \longleftarrow & (1,n). \end{array}$$

We consider given a region X (see 2.1) and a closed immersion in $\mathcal{D}ia$

$$k : \square_n \longrightarrow X.$$

For an integer $0 \leqslant i \leqslant n$, we define U_i to be the full subcategory of X made of objects $x \neq k(0, j)$ for $0 \leqslant j \leqslant n - i - 1$. In other words U_i contains the points of $X - k(\square_n)$, as well as the subdiagram of $k(\square_n)$ below:

$$\begin{array}{ccccccccccc} & & & & k(0,n-i) & \longleftarrow & \cdots & \longleftarrow & k(0,n) \\ & & & & \uparrow & & & & \uparrow \\ k(1,0) & \longleftarrow & \cdots & \longleftarrow & k(1,n-i) & \longleftarrow & \cdots & \longleftarrow & k(1,n). \end{array}$$

For $1 \leqslant i \leqslant n$, we denote by $j_i : U_{i-1} \longrightarrow U_i$ the inclusion functors, which are open immersions. Remember that $\square = \square_1$ is the square category (see 1.14). We have functors

$$u_i : \square \longrightarrow U_i$$

defined by the formula $u_i(\varepsilon, \eta) = k(\varepsilon, \eta + n - i)$. Let $\lrcorner$ be the corner category, and $\sigma : \lrcorner \longrightarrow \square$ the inclusion functor. Then we have the cartesian square below, which defines the functor $v_i : \lrcorner \longrightarrow U_{i-1}$

$$\begin{array}{ccc} \lrcorner & \xrightarrow{\sigma} & \square \\ {\scriptstyle v_i}\downarrow & & \downarrow{\scriptstyle u_i} \\ U_{i-1} & \xrightarrow[j_i]{} & U_i . \end{array}$$

We know from Proposition 7.1 that the cohomological direct image functor

$$(j_i)_* : \mathbb{D}(U_{i-1}) \longrightarrow \mathbb{D}(U_i)$$

is fully faithful. The next lemma and proposition will describe its essential image.

Lemma 8.4. *Let G be an object of $\mathbb{D}(U_{i-1})$. Then there is a canonical isomorphism*

$$(j_i)_*(G)_{k(0,n-i)} \simeq \Gamma_*\big(\lrcorner, v_i^*(G)\big).$$

Proof. By the base change axiom we know that

$$(j_i)_*(G)_{k(0,n-i)} \simeq \Gamma_*\big(U_{i-1}/k(0,n-i), G/k(0,n-i)\big).$$

It is clear that the functor v_i factors canonically through a functor

$$w : \lrcorner \longrightarrow U_{i-1}/k(0,n-i).$$

Since $v_i^*(G) = w^*(G/k(0,n-i))$ we have first that

$$\Gamma_*\big(\lrcorner, v_i^*(G)\big) = \Gamma_*\big(\lrcorner, w^*\big(G/k(0,n-i)\big)\big).$$

We deduce a canonical map

$$\Gamma_*\big(U_{i-1}/k(0,n-i), G/k(0,n-i)\big) \longrightarrow \Gamma_*\big(\lrcorner, w^*\big(G/k(0,n-i)\big)\big) = \Gamma_*\big(\lrcorner, v_i^*(G)\big),$$

and we have to show that it is an isomorphism. By Proposition 6.3(ii) it suffices to prove that the functor w has a left adjoint, which is easily verified. □

Proposition 8.5. *An object F of $\mathbb{D}(U_i)$ is in the essential image of the functor $(j_i)_*$ if and only if the global commutative square $u_i^*(F)$ is cartesian.*

Proof. By Proposition 7.2 the object F belongs to the essential image of $(j_i)_*$ if and only if, for the unique point $y = k(0,n-i)$ in $U_i - U_{i-1}$, the map $F_{k(0,n-i)} \longrightarrow (j_i)_* j_i^*(F)_{k(0,n-i)}$ is an isomorphism. Lemma 8.4, applied to $G = j_i^*(F)$, gives us an isomorphism $(j_i)_* j_i^*(F)_{k(0,n-i)} \simeq \Gamma_*(\lrcorner, v_i^* j_i^*(F)) = \Gamma_*(\lrcorner, \sigma^* u_i^*(F))$, while $F_{k(0,n-i)}$ is clearly isomorphic to $u_i^*(F)_{0,0}$. That is F belongs to the essential image if and only if the natural map

$$u_i^*(F)_{0,0} \longrightarrow \Gamma_*\big(\lrcorner, \sigma^* u_i^*(F)\big)$$

is an isomorphism. Now Proposition 7.3, applied in the case of $C = u_i^*(F)$, ends the proof. □

8.6. We keep the notation of 8.3. Let $j = j_0 : U_0 \longrightarrow X$ be the inclusion functor. For any integer i, $1 \leqslant i \leqslant n$ we had maps $j_i : \square \longrightarrow U_i$, where the composite $\square \longrightarrow U_i \longrightarrow X$ is the functor defined by the commutative square in X of shape

$$\begin{array}{ccc}
k(0,n-i) & \longleftarrow & k(0,n-i+1) \\
\uparrow & & \uparrow \\
k(1,n-i) & \longleftarrow & k(1,n-i+1).
\end{array}$$

We can now characterize the essential image of the functor j_* as follows.

Proposition 8.7. *An object F of $\mathbb{D}(X)$ is in the essential image of the functor $j_*:\mathbb{D}(U_0)\longrightarrow\mathbb{D}(X)$ if and only if, for any integer i, $1\leqslant i\leqslant n$, the global commutative square $j_i^*(F)$ is cartesian.*

Proof. In the category $\mathcal{D}ia$ we have an identity $j=j_nj_{n-1}\cdots j_2j_1$. Because $\mathbb{D}$ is a strict 2-functor we conclude that $j^*=j_1^*j_2^*\cdots j_{n-1}^*j_n^*$. Taking right adjoints we have that j_* is canonically isomorphic to the composition of the functors

$$\mathbb{D}(U_0)\xrightarrow{(j_0)_*}\mathbb{D}(U_1)\xrightarrow{(j_2)_*}\cdots\xrightarrow{(j_{n-1})_*}\mathbb{D}(U_{n-1})\xrightarrow{(j_n)_*}\mathbb{D}(U_n)=\mathbb{D}(X).$$

Proposition 8.5 characterizes for us the essential image of each of the fully faithful functors $(j_i)_*$; the proposition amounts to concatenating these characterizations. □

Of course there are other possible characterizations of the essential image of the map $j_*:\mathbb{D}(U_0)\longrightarrow\mathbb{D}(X)$. For example one can prove the following:

Proposition 8.8. *For each i, $1\leqslant i\leqslant n$, define a functor $\ell_i:\square\longrightarrow X$ by taking $\square$ isomorphically to the square*

$$\begin{array}{ccc} k(0,n-i) & \longleftarrow & k(0,n)\\ \uparrow & & \uparrow\\ k(1,n-i) & \longleftarrow & k(1,n). \end{array}$$

An object F of $\mathbb{D}(X)$ is in the essential image of the functor $j_*:\mathbb{D}(U_0)\longrightarrow\mathbb{D}(X)$ if and only if, for any integer i, $1\leqslant i\leqslant n$, the global commutative square $\ell_i^*(F)$ is cartesian.

Proof. The proof is a minor variant of what we have seen above so will give only a sketch, leaving the details to the reader. By Proposition 7.2 we know that F is in the essential image of j_* if and only if, for all points $y\in X-U_0$, the map $F_y\longrightarrow j_*j^*(F)_y$ is an isomorphism. The points $y\in X-U_0$ are of the form $k(0,n-i)$ with $1\leqslant i\leqslant n$, which means that each such $y=k(0,n-i)$ can be written as $\ell_i(0,0)$ with $\ell_i:\square\longrightarrow X$ as in the statement of the proposition. It is trivial that $F_y=y^*(F)=(0,0)^*\ell_i^*(F)=\ell_i^*(F)_{0,0}$. As in Lemma 8.4 we identify $j_*j^*(F)_y$ with $\Gamma_*(\lrcorner,\sigma^*\ell_i^*(F))$. Thus F is in the essential image of j_* if and only if all the maps

$$\ell_i^*(F)_{0,0}\longrightarrow\Gamma_*\big(\lrcorner,\sigma^*\ell_i^*(F)\big)$$

are isomorphisms. By Proposition 7.3, applied in the case of $C=\ell_i^*(F)$, the result follows. □

Corollary 8.9. *With the notation as above, let F be an object in $\mathbb{D}(X)$. We have the equivalence of*

(i) *The global commutative squares $j_i^*(F)$ are cartesian, for all $1\leqslant j\leqslant n$.*
(ii) *The global commutative squares $\ell_i^*(F)$ are cartesian, for all $1\leqslant j\leqslant n$.*

Proof. By Propositions 8.7 and 8.8 each of (i) and (ii) is equivalent to F being in the essential image of j_*. $\square$

Remark 8.10. Let us look at the special case where $n = 2$ and $X = \square_2$ is the category

$$\begin{array}{ccccc} (0,0) & \longleftarrow & (0,1) & \longleftarrow & (0,2) \\ \uparrow & & \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,1) & \longleftarrow & (1,2). \end{array}$$

The inclusions j_1 and ℓ_1 are identical; we have three functors $\square \longrightarrow X$, namely $j_1 = \ell_1$, ℓ_2 and j_2. They give subcategories

$$\begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,1) \end{array} \qquad \begin{array}{ccc} (0,0) & \longleftarrow & (0,2) \\ \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,2) \end{array} \qquad \begin{array}{ccc} (0,1) & \longleftarrow & (0,2) \\ \uparrow & & \uparrow \\ (1,1) & \longleftarrow & (1,2). \end{array}$$

Of these the first is the image of j_2, the second of ℓ_2 and the third of $j_1 = \ell_1$. We learn that, for F an object of $\mathbb{D}(X)$, we have

$$\left\{ \begin{array}{c} j_1^*(F) \text{ and } j_2^*(F) \\ \text{are both cartesian} \end{array} \right\} \iff \left\{ \begin{array}{c} j_1^*(F) \text{ and } \ell_2^*(F) \\ \text{are both cartesian} \end{array} \right\}.$$

Dually,

$$\left\{ \begin{array}{c} j_1^*(F) \text{ and } j_2^*(F) \\ \text{are both cocartesian} \end{array} \right\} \iff \left\{ \begin{array}{c} j_2^*(F) \text{ and } \ell_2^*(F) \\ \text{are both cocartesian} \end{array} \right\}.$$

If $\mathbb{D}$ is a triangulated derivator this simplifies to saying that the three global commutative squares $j_1^*(F) = \ell_1^*(F)$, $j_2^*(F)$ and $\ell_2^*(F)$ are cartesian = cocartesian if any two of them are.

Next we make a general observation about pointed derivators. Recall that, for any pointed derivator $\mathbb{D}$, the categories $\mathbb{D}(X)$ all have zero objects 0.

Proposition 8.11. *Let $\mathbb{D}$ be a pointed derivator and $v : V \longrightarrow U$ an open immersion in $\mathcal{D}ia$. Then the functor*

$$v_! : \mathbb{D}(V) \longrightarrow \mathbb{D}(U)$$

is fully faithful, and its essential image consists of the objects F of $\mathbb{D}(U)$ such that $F_u \simeq 0$ for any point u of U that is not in the image of v.

Proof. The fact that $v_!$ is fully faithful comes from Proposition 7.1. By Proposition 7.2, F belongs to the essential image of $v_!$ if and only if, for every point $u \in U - v(V)$, the natural map

$(v)_!v^*(F)_u \longrightarrow F_u$ is an isomorphism. To prove the proposition it suffices to show that $(v)_!(G)_u$ vanishes for every object G of $\mathbb{D}(V)$.

But now the facts that v is an open immersion, and that $u \in U$ does not lie in the image of v, mean that $u\backslash V = \emptyset$. Der 1 says that $\mathbb{D}(u\backslash V) = \mathbb{D}(\emptyset) = \{0\}$. Der 4 gives the formula

$$(v)_!(G)_u = \Gamma_!(u\backslash V, u\backslash G) = \Gamma_!(\emptyset, 0) = 0.$$

Hence the result. □

8.12. We return to the assumptions and notations of 8.3 and 8.6. We assume furthermore that the derivator $\mathbb{D}$ is pointed. We denote by V the full subcategory of X made of objects x of X such that $x \neq k(0,i)$ for $0 \leqslant i \leqslant n$. Let $v_0 : V \longrightarrow U_0$ be the inclusion functor. One sees easily that it is an open immersion. We will write $v : V \longrightarrow X$ for the inclusion functor.

Proposition 8.13. *Let $\mathbb{D}(X)'_0$ be the full subcategory of $\mathbb{D}(X)$ whose objects are the F's such that, for any integer i, $0 \leqslant i \leqslant n-1$, $j_i^*(F)$ is cartesian, and furthermore $F_{k(0,n)} \simeq 0$. Then the functor*

$$\mathbb{D}(X)'_0 \longrightarrow \mathbb{D}(V), \qquad F \longmapsto v^*(F)$$

is an equivalence of categories.

Proof. The inclusion $v_0 : V \longrightarrow U_0$ is an open immersion, hence Proposition 8.11 applies. The essential image of $(v_0)_! : \mathbb{D}(V) \longrightarrow \mathbb{D}(U_0)$ is the subcategory $\mathbb{D}(U_0)_0 \subset \mathbb{D}(U_0)$, whose objects are the F's such that $F_{k(0,n)} \simeq 0$. The adjoint $v_0^* : \mathbb{D}(U_0) \longrightarrow \mathbb{D}(V)$ maps the essential image of $(v_0)_!$ by an equivalence of categories onto $\mathbb{D}(V)$. Hence $v_0^* : D(U_0)_0 \longrightarrow D(V)$ is an equivalence of categories.

Next let $\mathbb{D}(X)'$ be the full subcategory of $\mathbb{D}(X)$ whose objects are the F's such that, for any integer i with $1 \leqslant i \leqslant n$, the global commutative square $j_i^*(F)$ is cartesian. Proposition 8.7 tells us that the functor

$$\mathbb{D}(X)' \longrightarrow \mathbb{D}(U_0), \qquad F \longmapsto j^*(F)$$

is an equivalence of categories. The combination of the two equivalences of categories above implies the result. □

9. Split gluing

It may happen that the category $\mathbb{D}(X)$ has a natural subcategory which splits as a product. In Caution 5.3 we met such a situation; we made an assertion, which this section will prove.

Proposition 9.1. *Let $\mathbb{D}$ be a pointed derivator and let $i : Z \longrightarrow X$ be a closed immersion in $\mathcal{D}ia$. Then the functor*

$$i_* : \mathbb{D}(Z) \longrightarrow \mathbb{D}(X)$$

is fully faithful, and its essential image consists of the objects F of $\mathbb{D}(X)$ such that $F_x \simeq 0$ for any point x of X that is not in the image of i.

Proof. Apply Proposition 8.11 to $\mathbb{D}^{\mathrm{op}}$. $\square$

9.2. In the remainder of this section we consider the following situation. Let

$$U \xrightarrow{j} X \xleftarrow{i} Z$$

be two maps in $\mathcal{D}ia$ such that j is an open immersion, i is a closed immersion and $j(U) = X - i(Z)$. First we prove

Lemma 9.3. *Let $\mathbb{D}$ be a pointed derivator. With the notation as in* (9.2) *and $F \in \mathbb{D}(X)$ any object, we have the equivalence*

$$\{j^*(F) \simeq 0\} \iff \{F \simeq i_*(G) \text{ for some } G \in \mathbb{D}(Z)\}.$$

Proof. Proposition 9.1 asserts that F lies in the essential image of i_* if and only if $F_x \simeq 0$ for all $x \in X - i(Z)$. By assumption $X - i(Z) = j(U)$, so F is in the essential image of i_* is and only if $F_{j(u)} = j^*(F)_u \simeq 0$ for all $u \in U$. Der 2 implies that this is equivalent to $j^*(F) \simeq 0$. $\square$

9.4. With $j : U \longrightarrow X$, $i : Z \longrightarrow X$ as in 9.2, assume further that $\mathbb{D}$ be a triangulated derivator. The fully faithful functor $j_! : \mathbb{D}(U) \longrightarrow \mathbb{D}(X)$ has a right adjoint $j^* : \mathbb{D}(X) \longrightarrow \mathbb{D}(U)$. Corollary 7.10 tells us that these are exact functors of triangulated categories. Lemma 9.3 tells us that the kernel of j^* is the image of the fully faithful functor $i_* : \mathbb{D}(Z) \longrightarrow \mathbb{D}(X)$. It follows that j^* is naturally identified as the map from $\mathbb{D}(X)$ to the Verdier quotient $\mathbb{D}(X)/i_*\mathbb{D}(Z)$; the reader can find this, for example, in the statement and proof of [29, Proposition 9.1.18]. We have the following triangulated functors between triangulated categories

$$\mathbb{D}(U) \xrightarrow{j_!} \mathbb{D}(X) \xrightarrow{i^*} \mathbb{D}(Z),$$
$$\mathbb{D}(U) \xleftarrow{j^*} \mathbb{D}(X) \xleftarrow{i_*} \mathbb{D}(Z),$$
$$\mathbb{D}(U) \xrightarrow{j_*} \mathbb{D}(X) \xrightarrow{i^!} \mathbb{D}(Z).$$

The middle row is naturally identified as $i_*\mathbb{D}(Z) \longrightarrow \mathbb{D}(X) \longrightarrow \mathbb{D}(X)/i_*\mathbb{D}(Z)$, the top row is obtained by taking left adjoints and the bottom row by taking right adjoints. We have the following vanishing formulæ

$$i^* j_! \simeq 0, \qquad j^* i_* \simeq 0, \qquad i^! j_* \simeq 0,$$

and the functors $j_!$, j_* and i_* are fully faithful (Proposition 7.1). In other words, we are in the situation of the six gluing functors as defined in [1] (see also [29, §9.2]). The functor $j_!$ can be thought of as the functor that extends a 'sheaf' on U by 0, and dually, the functor i_* can be regarded as the functor extending a 'sheaf' on Z by 0.

Let $\mathbb{E}$ be a full triangulated subcategory of $\mathbb{D}(X)$ stable by the functors $j_!j^*$ and i_*i^*. Denote by $\mathbb{E}_U$ (respectively by $\mathbb{E}_Z$) the full subcategory of $\mathbb{E}$ whose objects are the F's in $\mathbb{E}$ such that $i^*(F) \simeq 0$ (respectively such that $j^*(F) \simeq 0$). We then obtain a functor

$$S : \mathbb{E} \longrightarrow \mathbb{E}_U \times \mathbb{E}_Z, \qquad F \longmapsto \big(j_!j^*(F), i_*i^*(F)\big).$$

We want to find sufficient conditions for S to be an equivalence of categories.

Example 9.5. Let $X = \square_2$ (see 8.3 for the notation). Define U to be the full subcategory of X whose objects are the pairs (i, j) for $i = 0, 1$ and $j = 1, 2$, and set $Z = X - U$. Then the inclusion of U in X is an open immersion. Define $\mathbb{E}$ to be the full subcategory of $\mathbb{D}(X)$ whose objects are the objects F which are locally of shape

$$\begin{array}{ccccc} F_{(0,0)} & \longrightarrow & 0 & \longrightarrow & F_{(0,2)} \\ \downarrow & & \downarrow & & \downarrow \\ F_{(1,0)} & \longrightarrow & 0 & \longrightarrow & F_{(1,2)}. \end{array}$$

Then $\mathbb{E}_U$ (respectively $\mathbb{E}_Z$) is simply the full subcategory of $\mathbb{D}(X)$ whose objects are the objects F such that $F_{(i,j)} \simeq 0$ for $j \neq 2$ (respectively for $j \neq 0$). Proposition 9.7 will show that the map $S : \mathbb{E} \longrightarrow \mathbb{E}_U \times \mathbb{E}_Z$ is an equivalence of categories. The technical condition, that needs to be verified in applying Proposition 9.7, comes down to saying that for any $(i, j) \in X - U$ (this just means that $j = 0$), $(i, 1)$ is a terminal object of the poset $U/(i, j)$, and that the objects of $\mathbb{E}$ vanishes on the points $(i, 1)$.

Remark 9.6. Example 9.5 is very simple. In the proof of additivity we apply Proposition 9.7 to more complicated regions; see Caution 5.3.

Proposition 9.7. *With the hypothesis given in* 9.4*, suppose that, for any point z of Z, the category $U/i(z)$ has a terminal object. Suppose further that, for any point z of Z and for any object F of $\mathbb{E}$, we have $F_{j(u)} \simeq 0$, where (u, f) is any terminal object of $U/i(z)$.*[2] *Then the functor*

$$S : \mathbb{E} \longrightarrow \mathbb{E}_U \times \mathbb{E}_Z, \qquad F \longmapsto \big(j_! j^*(F), i_* i^*(F)\big)$$

is an equivalence of categories.

Proof. We define a functor

$$T : \mathbb{E}_U \times \mathbb{E}_Z \longrightarrow \mathbb{E}$$

by the formula $T(F, G) = F \oplus G$. The proof will consist in showing that the functor T is an equivalence of categories. Since $ST : \mathbb{E}_U \times \mathbb{E}_Z \longrightarrow \mathbb{E}_U \times \mathbb{E}_Z$ is clearly naturally isomorphic to the identity, it would then follow that S is a quasi-inverse of T, in particular an equivalence.

Formal facts, about fully faithful triangulated functors with adjoints, tell us that, for any object F of $\mathbb{D}(X)$, we have the following canonical distinguished triangle:

$$j_! j^*(F) \xrightarrow{\varepsilon} F \xrightarrow{\eta} i_* i^*(F) \longrightarrow \Sigma j_! j^*(F). \tag{1}$$

Here $\varepsilon : j_! j^* \Longrightarrow 1$ is the counit of adjunction, while $\eta : 1 \Longrightarrow i_* i^*$ is the unit. The reader can find this, for example, in the proof of [29, Proposition 9.1.18]. We will show that, for any F and G in $\mathbb{E}$, one has

$$\mathsf{Hom}\big(i_* i^*(F), j_! j^*(G)\big) = 0. \tag{2}$$

[2] Recall: u is an object of U, and f a map from $j(u)$ to $i(z)$.

If we let $G = \Sigma F$ in (2) above, we find that the triangle (1) above splits whenever F is in $\mathbb{E}$. That is $F \simeq j_! j^*(F) \oplus i_* i^*(F) = TS(F)$. It follows that the functor T is essentially surjective. Since we always have

$$\mathsf{Hom}\big(j_! j^*(G), i_* i^*(F)\big) = \mathsf{Hom}\big(j^*(G), j^* i_* i^*(F)\big)$$

and $j^* i_* = 0$, the assertion (2) also implies that any map $G \oplus F \longrightarrow G' \oplus F'$, with $G, G' \in \mathbb{E}_U$ and $F, F' \in \mathbb{E}_Z$, must be of the form $g \oplus f$ for some $g : G \longrightarrow G'$ and $f : F \longrightarrow F'$; that is (2) implies that the functor T is fully faithful. It remains to prove (2).

The formula

$$\mathsf{Hom}\big(i_* i^*(F), j_! j^*(G)\big) = \mathsf{Hom}\big(i^*(F), i^! j_! j^*(G)\big)$$

shows that it suffices to prove that, for any object F of $\mathbb{E}$, one has

$$i^! j_! j^*(F) \simeq 0.$$

For any object G of $\mathbb{D}(X)$ one has a canonical distinguished triangle

$$i_* i^!(G) \longrightarrow G \longrightarrow j_* j^*(G) \longrightarrow \Sigma i_* i^!(G).$$

Putting $G = j_! j^*(F)$ we obtain a distinguished triangle

$$i_* i^! j_! j^*(F) \longrightarrow j_! j^*(F) \longrightarrow j_* j^* j_! j^*(F) \longrightarrow \Sigma i_* i^! j_! j^*(F).$$

Remembering that $j^* j_!$ is naturally isomorphic to the identity this simplifies slightly to

$$i_* i^! j_! j^*(F) \longrightarrow j_! j^*(F) \longrightarrow j_* j^*(F) \longrightarrow \Sigma i_* i^! j_! j^*(F).$$

Applying the functor i^* we obtain a distinguished triangle

$$i^* i_* i^! j_! j^*(F) \longrightarrow i^* j_! j^*(F) \longrightarrow i^* j_* j^*(F) \longrightarrow \Sigma i^* i_* i^! j_! j^*(F).$$

Recalling that $i^* i_*$ is naturally isomorphic to the identity and $i^* j_! \simeq 0$, this simplifies to an isomorphism

$$i^* j_* j^*(F) \simeq \Sigma i^! j_! j^*(F);$$

we want to show the vanishing of $i^! j_! j^*(F)$, which is equivalent to the vanishing of $i^* j_* j^*(F)$. By Der 2 this reduces further to showing that, for all points $z \in Z$, we have $(i^* j_* j^*(F))_z \simeq 0$.

Let z be a point of Z. We have the canonical identifications

$$\big(i^* j_* j^*(F)\big)_z = \big(j_* j^*(F)\big)_{i(z)} \simeq \Gamma_*\big(U/i(z), j^*(F)/i(z)\big).$$

By assumption the category $U/i(z)$ has a terminal object (u, f), so Proposition 6.2 tells us that

$$\Gamma_*\big(U/i(z), j^* F/i(z)\big) \simeq \big(j^* F/i(z)\big)_{(u,f)} = F_{j(u)} \simeq 0.$$

Hence the result. □

10. Exact morphisms of derivators and strictification

It is time to worry about the functoriality of derivator K-theory. In this section we define morphisms of derivators. It is clear from the definitions that any strict, exact morphism of derivators $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ will induce a map in K-theory. Unfortunately this is not enough for us; in Section 11 we will want to look at maps induced by non-strict, exact morphisms of derivators. In this section we will learn how to construct such morphisms, and why they induce maps in K-theory.

10.1. We define morphisms of prederivators as morphisms of 2-functors. More explicitly, a morphism of prederivators $(\Phi, \beta) : \mathbb{D}(X) \longrightarrow \mathbb{D}'(X)$ consists of two items of data

(i) For any object $X \in \mathcal{D}ia$ there is given a functor $\Phi_X : \mathbb{D}(X) \longrightarrow \mathbb{D}'(X)$.
(ii) For any map $u : X \longrightarrow Y$ in $\mathcal{D}ia$ there is given a natural isomorphism $\beta_u : u^*\Phi_Y \longrightarrow \Phi_X u^*$.

These must satisfy the following properties:

(1) If $u : X \longrightarrow Y$ and $v : Y \longrightarrow Z$ are composable maps in $\mathcal{D}ia$, then the composite of the two 2-cells

$$\begin{array}{ccccc} \mathbb{D}(Z) & \xrightarrow{v^*} & \mathbb{D}(Y) & \xrightarrow{u^*} & \mathbb{D}(X) \\ \downarrow{\scriptstyle \Phi_Z} & \beta_v \nearrow & \downarrow{\scriptstyle \Phi_Y} & \beta_u \nearrow & \downarrow{\scriptstyle \Phi_X} \\ \mathbb{D}'(Z) & \xrightarrow[v^*]{} & \mathbb{D}'(Y) & \xrightarrow[u^*]{} & \mathbb{D}'(X) \end{array}$$

equals the 2-cell

$$\begin{array}{ccc} \mathbb{D}(Z) & \xrightarrow{u^*v^*} & \mathbb{D}(X) \\ \downarrow{\scriptstyle \Phi_Z} & \beta_{vu} \nearrow & \downarrow{\scriptstyle \Phi_X} \\ \mathbb{D}'(Z) & \xrightarrow[u^*v^*]{} & \mathbb{D}'(X). \end{array}$$

(2) For any 2-cell in $\mathcal{D}ia$

$$X \underset{v}{\overset{u}{\rightrightarrows}} Y \quad (\Downarrow \alpha)$$

one has $\alpha^*\beta_v = \beta_u\alpha^*$.

A morphism of prederivators $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is *strict* if, for any map $u : X \longrightarrow Y$ in $\mathcal{D}ia$, the isomorphism β_u is an identity (so that we then have $u^*\Phi_Y = \Phi_X u^*$ in a coherent way). Prederivators form a 2-category; the composition of morphisms of prederivators is hopefully obvious, and the 2-morphisms are the natural isomorphisms.

Let $\mathbb{D}$ and $\mathbb{D}'$ be two derivators and $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators. Then, for any map $u : X \longrightarrow Y$ in $\mathcal{D}ia$, we have the canonical 2-cells

$$r_u : \Phi_Y u_* \longrightarrow u_* \Phi_X, \tag{3}$$

$$l_u : u_! \Phi_X \longrightarrow \Phi_Y u_!. \tag{4}$$

The map r_u is defined as follows. It corresponds by adjunction to the map

$$u^* \Phi_Y u_* \simeq \Phi_X u^* u_* \longrightarrow \Phi_X$$

obtained by composing the counit $u^* u_* \longrightarrow 1_{\mathbb{D}(X)}$ with Φ_X and then composing with $\beta_u u_*$. The map l_u is defined in the dual way, using $u_!$ instead of u_* and β_u^{-1} instead of β_u.

A morphism of derivators $\Phi : \mathbb{D} \longrightarrow \mathbb{D}$ is *right exact* (respectively *left exact*) if, for any increasing map $u : X \longrightarrow Y$ between finite partially ordered sets, the 2-cell r_u (respectively l_u) is an isomorphism. A morphism of derivators is *exact* if it is both right exact and left exact.

Remark 10.2. A morphism of derivators $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is right exact if and only if the induced morphism $\Phi^{\mathrm{op}} : \mathbb{D}^{\mathrm{op}} \longrightarrow \mathbb{D}'^{\mathrm{op}}$ is left exact.

Example 10.3. For any exact functor $\mathcal{E} \longrightarrow \mathcal{E}'$ between exact categories, the induced morphism of derivators $\mathbb{D}^b_{\mathcal{E}} \longrightarrow \mathbb{D}^b_{\mathcal{E}'}$ is exact (see for example [2, 4.11 and 4.12]).

Proposition 10.4. *A morphism of derivators $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is right exact if and only if, for any finite partially ordered set X and any object F of $\mathbb{D}(X)$, the canonical map*

$$\Phi_e \Gamma_*(X, F) \longrightarrow \Gamma_*\big(X, \Phi_X(F)\big)$$

is an isomorphism in $\mathbb{D}'(e)$.

Proof. This is obviously a necessary condition. To prove that it is sufficient, let $u : X \to Y$ be a map between finite partially ordered sets in $\mathcal{D}ia$ and F be an object of $\mathbb{D}(X)$. We want to show that the map

$$r_u(F) : \Phi_Y u_*(F) \longrightarrow u_* \Phi_X(F)$$

is an isomorphism in $\mathbb{D}'(Y)$. But using Der 2 in $\mathbb{D}'$ it is sufficient to check that, for any point y of Y, the map

$$\big(\Phi_Y u_*(F)\big)_y \simeq \Phi_e\big(u_*(F)_y\big) \longrightarrow \big(u_* \Phi_X(F)\big)_y$$

is an isomorphism in $\mathbb{D}'(e)$. The base change axiom in $\mathbb{D}$ gives

$$u_*(F)_y \simeq \Gamma_*(X/y, F/y)$$

and the base change axiom in $\mathbb{D}'$ gives

$$\big(u_*\Phi_X(F)\big)_y \simeq \Gamma_*\big(X/y, \Phi_X(F)/y\big) \simeq \Gamma_*\big(X/y, \Phi_{X/y}(F/y)\big).$$

We are thus reduced to proving that the canonical map

$$\Phi_e \Gamma_*(X/y, F/y) \longrightarrow \Gamma_*\big(X/y, \Phi_{X/y}(F/y)\big)$$

is an isomorphism in $\mathbb{D}'(e)$, which is true by assumption. □

Example 10.5. Let $\mathbb{D}$ be a derivator and let $a : V \longrightarrow W$ be a map in $\mathcal{D}ia$. We have two derivators $\mathbb{D}_V$ and $\mathbb{D}_W$ (see 7.4). There is a strict morphism of derivators

$$a^* : \mathbb{D}_W \longrightarrow \mathbb{D}_V;$$

the formula is that, for every object $X \in \mathcal{D}ia$,

$$a^*_X = (1_X \times a)^* : \mathbb{D}_W(X) = \mathbb{D}(X \times W) \longrightarrow \mathbb{D}(X \times V) = \mathbb{D}_V(X).$$

We assert that the morphism a^* is exact.

To prove this, first note that it suffices to show the right exactness: this will imply that

$$\big(a^{\mathrm{op}}\big)^* : \mathbb{D}^{\mathrm{op}}_{W^{\mathrm{op}}} \longrightarrow \mathbb{D}^{\mathrm{op}}_{V^{\mathrm{op}}}$$

is right exact too, which implies that a^* is left exact. Next we use Proposition 10.4. Let F be an object of $\mathbb{D}_W(X)$. We want to prove that the canonical map

$$a^* \Gamma_*(X, F) \longrightarrow \Gamma_*\big(X, a^*(F)\big)$$

is an isomorphism in $\mathbb{D}_V(e) = \mathbb{D}(V)$. By virtue of Lemma 7.6 we have, for any point v of V,

$$\begin{aligned}\big(a^*\Gamma_*(X, F)\big)_v &= \Gamma_*(X, F)_{a(v)} \\ &\simeq \Gamma_*(X, F_{a(v)}) \\ &\simeq \Gamma_*\big(X, a^*(F)_v\big) \\ &\simeq \Gamma_*\big(X, a^*(F)\big)_v.\end{aligned}$$

Hence our claim is proved using Der 2.

Example 10.6. Let $\mathbb{D}$ be a derivator and $a : V \longrightarrow W$ be a map between finite partially ordered sets in $\mathcal{D}ia$. We can then define a (non-strict) morphism of derivators

$$a_* : \mathbb{D}_V \longrightarrow \mathbb{D}_W$$

by

$$(a_*)_X = (1_X \times a)_* : \mathbb{D}_V(X) = \mathbb{D}(X \times V) \longrightarrow \mathbb{D}(X \times W) = \mathbb{D}_W(X)$$

for any category X in $\mathcal{D}ia$. For any map $u : X \longrightarrow Y$ in $\mathcal{D}ia$, we still have to define an isomorphism

$$\beta_u : (a_*)_Y u^* = (1_Y \times a)_*(u \times 1_V)^* \longrightarrow (u \times 1_W)^*(1_X \times a)_* = u^*(a_*)_X.$$

The base change map (1.9) associated to the commutative square

$$\begin{array}{ccc} X \times V & \xrightarrow{u \times 1_V} & Y \times V \\ {\scriptstyle 1_X \times a}\downarrow & & \downarrow{\scriptstyle 1_Y \times a} \\ X \times W & \xrightarrow[u \times 1_W]{} & Y \times W \end{array}$$

gives a map

$$\alpha_u : (u \times 1_W)^*(1_X \times a)_* = u^*(a_*)_X \longrightarrow (1_Y \times a)_*(u \times 1_V)^* = (a_*)_Y u^*.$$

The map α_u is an isomorphism: this is a reformulation of the fact that the morphism of derivators $u^* : \mathbb{D}_Y \longrightarrow \mathbb{D}_X$ is right exact (see 10.5). We define $\beta_u = \alpha_u^{-1}$. It is obvious that the morphism of derivators a_* is right exact: for any map $u : X \longrightarrow Y$ in $\mathcal{D}ia$ one has, by functoriality,

$$(1_X \times a)^*(u \times 1_W)^* = (u \times 1_V)^*(1_Y \times a)^*.$$

This implies that

$$(1_Y \times a)_*(u \times 1_V)_* \simeq (u \times 1_W)_*(1_X \times a)_*$$

which can be rewritten $(a_*)_Y u_* \simeq u_*(a_*)_X$. Dually, we also have a left exact morphism

$$a_! : \mathbb{D}_V \longrightarrow \mathbb{D}_W$$

defined by the formula

$$(a_!)_X = (1_X \times a)_! : \mathbb{D}_V(X) = \mathbb{D}(X \times V) \longrightarrow \mathbb{D}(X \times W) = \mathbb{D}_W(X)$$

for any category X in $\mathcal{D}ia$.

Proposition 10.7. *Let $\mathbb{D}$ be a triangulated derivator, and let X be a finite partially ordered set. Then the category $\mathbb{D}(X)$ is generated as a triangulated category by the objects $x_*(M)$, where x runs over the set of points of X, and M over the objects of $\mathbb{D}(e)$. In other words, the smallest full triangulated subcategory of $\mathbb{D}(X)$ that contains the $x_*(M)$'s is the category $\mathbb{D}(X)$ itself.*

Proof. We proceed by induction on the number n of elements of X. If $n \leqslant 1$ the assertion is obvious, so that we can suppose that $n \geqslant 2$. Let U be the set of minimal elements of X, and let $j : U \longrightarrow X$ be the inclusion. It is clear that j is an open immersion. Set $Z = X - U$, and $i : Z \longrightarrow X$ the inclusion. If Z is empty then $X = U$ has to be discrete, and once again our assertion is trivial. Hence we may suppose that Z is not empty, which implies that the number of elements in U and in Z is strictly smaller than n. Moreover, according to 9.4, we are in the six gluing functors situation. In particular, we have a natural distinguished triangle in $\mathbb{D}(X)$

$$(*)\qquad j_!j^*(F) \to F \longrightarrow i_*i^*(F) \longrightarrow \Sigma j_!j^*(F)$$

for any object F of $\mathbb{D}(X)$. It therefore suffices to show that:

(i) For any object $F' \in \mathbb{D}(U)$ the object $j_!(F')$ lies in the category generated by $x_*(P)$, where x is a point of X and P an object of $\mathbb{D}(e)$.
(ii) For any object $F'' \in \mathbb{D}(Z)$ the object $i_*(F'')$ lies in the category generated by $x_*(P)$, where x is a point of X and P an object of $\mathbb{D}(e)$.

To prove (i) it helps to take the distinguished triangle $(*)$ and put $F = j_*G$ for an object $G \in \mathbb{D}(U)$. We deduce a distinguished triangle

$$j_!j^*j_*(G) \to j_*(G) \longrightarrow i_*i^*j_*(G) \longrightarrow \Sigma j_!j^*j_*(G).$$

If we recall that j^*j_* is naturally isomorphic to the identity, this becomes

$$j_!(G) \to j_*(G) \longrightarrow i_*i^*j_*(G) \longrightarrow \Sigma j_!(G).$$

In other words $j_!(G)$ is in the triangulated category generated by $j_*(G)$ and $i_*(F'')$, for suitable F''. We conclude that, in the presence of (ii), proving (i) reduces to showing that every object $j_*(G)$ lies in the category generated by $x_*(P)$, where x is a point of X and P an object of $\mathbb{D}(e)$.

But now induction applies. Every object $F'' \in \mathbb{D}(Z)$ is in the subcategory generated by $z_*(N)$, with z a point in Z and N an object of $\mathbb{D}(e)$. This means that $i_*(F'')$ is generated by the objects $i_*z_*(N)$, which are isomorphic to $i(z)_*(N)$ with $i(z)$ a point in X. Similarly every object $G \in \mathbb{D}(U)$ is generated by the objects $u_*(M)$, with u a point in U and M an object of $\mathbb{D}(e)$. This means that $j_*(G)$ is generated by the objects $j_*u_*(M) \simeq j(u)_*(M)$. □

10.8. Let $\mathbb{D}$ be a triangulated derivator. Remember that $\square$ is the following category:

$$\square \quad = \quad \begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,1). \end{array}$$

Proposition 10.9. *Let $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of triangulated derivators. The following conditions are equivalent.*

(i) *The morphism $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is exact.*
(ii) *The morphism $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is right exact.*
(iii) *The morphism $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is left exact.*
(iv) *The functor $\Phi_e : \mathbb{D}(e) \longrightarrow \mathbb{D}'(e)$ sends the zero object to the zero object, and the functor $\Phi_\square : \mathbb{D}(\square) \longrightarrow \mathbb{D}'(\square)$ sends global cartesian squares (see* 1.14*) to global cartesian squares.*

(v) *For any X in $\mathcal{D}ia$, the functor $\Phi_X : \mathbb{D}(X) \longrightarrow \mathbb{D}'(X)$ is triangulated, and for any map $u : X \longrightarrow Y$ in $\mathcal{D}ia$, the 2-cell $\beta_u : u^*\Phi_Y \longrightarrow \Phi_X u^*$ (see 10.1) is an isomorphism of triangulated functors.*

Proof. It is obvious that (i) implies (ii) and (iii). Using Proposition 7.3, it is also clear that either (ii) or (iii) implies (iv). If condition (iv) is true for $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ then, by Proposition 7.7, it is also true for the morphisms

$$\Phi_X : \mathbb{D}_X \longrightarrow \mathbb{D}'_X$$

defined by the formulas

$$(\Phi_X)_Y = \Phi_{Y\times X} : \mathbb{D}_X(Y) = \mathbb{D}(Y \times X) \longrightarrow \mathbb{D}'(Y \times X) = \mathbb{D}'_X(Y).$$

Using the definition of distinguished triangles (see 7.9), we conclude that (iv) implies (v). To finish the proof it suffices to prove that (v) implies (i). We will prove that (v) implies (ii). Dually we will have that (v) implies (iii), and (ii) and (iii) together are clearly equivalent to (i).

From now we assume that condition (v) is true, and will prove (ii). Let X be a finite poset and x a point of X. If $x : e \longrightarrow X$ is the inclusion then the category e/y is either empty or the terminal category, depending on whether $x \leqslant y$. The base change axiom for the derivator $\mathbb{D}$ gives the formula, for any object $M \in \mathbb{D}(e)$,

$$x_*(M)_y \simeq \begin{cases} M & \text{if } x \leqslant y, \\ 0 & \text{otherwise.} \end{cases}$$

We compute therefore that

$$\begin{aligned} \big(\Phi_X x_*(M)\big)_y &= y^*\Phi_X x_*(M) \\ &\simeq \Phi_e y^* x_*(M) \quad \text{since } y^*\Phi_X \simeq \Phi_e y^* \\ &= \Phi_e\big(x_*(M)_y\big) \\ &\simeq \big(x_*\Phi_e(M)\big)_y, \end{aligned}$$

where the isomorphism $\Phi_e(x_*(M)_y) \simeq (x_*\Phi_e(M))_y$ is because both sides are $\Phi_e(M)$ when $x \leqslant y$ and 0 otherwise. By Der 2 the canonical map must be an isomorphism

$$\Phi_X x_*(M) \longrightarrow x_*\Phi_e(M).$$

Let $u : X \longrightarrow Y$ be a map in $\mathcal{D}ia$ between two finite posets. We want to prove (ii); we want to show that, for any object F of $\mathbb{D}(X)$, the map

$$\Phi_Y u_*(F) \longrightarrow u_*\Phi_X(F)$$

is an isomorphism. Now the functors Φ_X, Φ_Y and u_* are triangulated by (v), and Proposition 10.7 implies that it is sufficient to consider the case where $F = x_*(M)$, with x a point of X and M an object of $\mathbb{D}(e)$. In this case, we have the following canonical isomorphisms

$$\begin{aligned}\Phi_Y u_* x_*(M) &\simeq \Phi_Y u(x)_*(M)\\ &\simeq u(x)_*\Phi_e(M)\\ &\simeq u_* x_* \Phi_e(M)\\ &\simeq u_* \Phi_X x_*(M).\end{aligned}$$

This ends the proof. □

Example 10.10. Let $\mathbb{D}$ be a triangulated derivator and let $i : Z \longrightarrow W$ be a closed immersion of finite partially ordered sets. We define an exact morphism of triangulated derivators

$$i^! : \mathbb{D}_W \longrightarrow \mathbb{D}_Z$$

by the formula

$$\left(i^!\right)_X = (1_X \times i)^! : \mathbb{D}_W(X) = \mathbb{D}(X \times W) \longrightarrow \mathbb{D}(X \times Z) = \mathbb{D}_Z(X)$$

for any category X in $\mathcal{D}ia$. As in Example 10.6, we still have to define the coherent isomorphisms β_u for any map u in $\mathcal{D}ia$ (with the notation of 10.1). Let $u : X \longrightarrow Y$ be a map in $\mathcal{D}ia$. By Example 10.6 the morphism $u_! : \mathbb{D}_X \longrightarrow \mathbb{D}_Y$ is left exact, and the equivalence of (ii) and (iii) in Proposition 10.9 allows us to deduce that $u_! : \mathbb{D}_X \longrightarrow \mathbb{D}_Y$ is also right exact. Hence $u_!$ commutes with the operators i_*. In other words we have a canonical isomorphism of functors

$$(u \times 1_W)_!(1_X \times i)_* \longrightarrow (1_Y \times i)_*(u \times 1_Z)_!.$$

By adjunction this provides an isomorphism of functors

$$\beta_u : u^*\left(i^!\right)_Y = (u \times 1_Z)^*(1_Y \times i)^! \longrightarrow (1_X \times i)^!(u \times 1_W)^* = \left(i^!\right)_X u^*.$$

It remains to show that the morphism $i^!$ is exact. By Proposition 10.9 it suffices to prove that the functors $(i^!)_X = (1_X \times i)^!$ are triangulated for any category X in $\mathcal{D}ia$. But this follows from the fact that $(i^!)_X$ is a right adjoint of the triangulated functor

$$(1_X \times i)_* : \mathbb{D}(X \times Z) \longrightarrow \mathbb{D}(X \times W).$$

10.11. A morphism of prederivators $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is an *equivalence* if, for any category X in $\mathcal{D}ia$, the functor

$$\Phi_X : \mathbb{D}(X) \longrightarrow \mathbb{D}'(X)$$

is an equivalence of categories. It is a standard fact about 2-functors that a morphism of prederivators $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is an equivalence if and only if there exists a morphism $\Psi : \mathbb{D}' \longrightarrow \mathbb{D}$ such that $\Phi\Psi$ is isomorphic to the identity of $\mathbb{D}'$ and $\Psi\Phi$ is isomorphic to the identity of $\mathbb{D}$. Such a morphism Ψ is called a *quasi-inverse* of Φ.

Remark 10.12. Let $\mathbb{D}$ and $\mathbb{D}'$ be two prederivators and let $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ be an equivalence. Then $\mathbb{D}$ is a derivator (respectively a pointed derivator, respectively a triangulated derivator) if and only if $\mathbb{D}'$ is a derivator (respectively a pointed derivator, respectively a triangulated derivator). If $\mathbb{D}$ and $\mathbb{D}'$ are derivators, then any equivalence Φ is exact.

Remark 10.13. If $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is a strict morphism of prederivators which is an equivalence, then there is no reason in general for there to exist a strict quasi-inverse of Φ. This is one of the reasons we are not only dealing with strict morphisms of derivators. Nevertheless, we can always strictify morphisms of (pre)derivators as follows.

Proposition 10.14. *Let $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of prederivators. Then there are two strict morphisms of prederivators*

$$\Phi' : \mathbb{D}'' \longrightarrow \mathbb{D} \quad \textit{and} \quad \Phi'' : \mathbb{D}'' \longrightarrow \mathbb{D}',$$

with Φ' an equivalence, and an isomorphism $\Phi\Phi' \simeq \Phi''$. In other words, there is an essentially commutative diagram of prederivators

$$\begin{array}{ccc} & \mathbb{D}'' & \\ {\scriptstyle \Phi'} \swarrow & & \searrow {\scriptstyle \Phi''} \\ \mathbb{D} & \xrightarrow[\Phi]{} & \mathbb{D}' \end{array}$$

where the morphisms Φ' and Φ'' are strict, and Φ' is an equivalence.

Proof. We first define the prederivator $\mathbb{D}''$. For a category X in $\mathcal{D}ia$, the objects of the category $\mathbb{D}''(X)$ are the triples (F', f, F) where F' is an object of $\mathbb{D}'(X)$, F is an object of $\mathbb{D}(X)$, and f is a given isomorphism $f : F' \longrightarrow \Phi_X(F)$ in the category $\mathbb{D}'(X)$. A map $(F', f, F) \longrightarrow (G', g, G)$ in $\mathbb{D}''(X)$ is a couple (a', a), where a' is a map from F' to G' in $\mathbb{D}'(X)$, where a is a map from F to G in $\mathbb{D}(X)$, and the square

$$\begin{array}{ccc} F' & \xrightarrow{f} & \Phi_X(F) \\ {\scriptstyle a'} \downarrow & & \downarrow {\scriptstyle \Phi_X(a)} \\ G' & \xrightarrow[g]{} & \Phi_X(G) \end{array}$$

commutes in $\mathbb{D}'(X)$. For a functor $u : X \longrightarrow Y$ in $\mathcal{D}ia$, we define an inverse image functor

$$u^* : \mathbb{D}''(Y) \longrightarrow \mathbb{D}''(X)$$

by the formulas:

(i) On objects: The functor $u^* : \mathbb{D}''(Y) \longrightarrow \mathbb{D}''(X)$ takes an object (F', f, F) of $\mathbb{D}''(Y)$ to the object $(u^*(F'), \beta_{u,F} \circ u^*(f), u^*(F))$ of $\mathbb{D}''(X)$, where $\beta_{u,F}$ is the structural isomorphism $u^*\Phi_Y(F) \longrightarrow \Phi_X u^*(F)$ of Φ (see 10.1).

(ii) On morphisms: The functor $u^* : \mathbb{D}''(Y) \longrightarrow \mathbb{D}''(X)$ takes a morphism (a', a) to the morphism $(u^*(a'), u^*(a))$.

We leave it to the reader to check that, with this definition, $u^* : \mathbb{D}''(Y) \longrightarrow \mathbb{D}''(X)$ is a functor. We also leave it to the reader to define what $\mathbb{D}''$ does to 2-cells in $\mathcal{D}ia$. And then the reader should check that with these definitions $\mathbb{D}''$ is a prederivator.

The functor

$$\mathbb{D}''(X) \longrightarrow \mathbb{D}(X), \qquad (F', f, F) \longmapsto F$$

defines a strict morphism of prederivators $\Phi' : \mathbb{D}'' \longrightarrow \mathbb{D}$, and the functor

$$\mathbb{D}''(X) \longrightarrow \mathbb{D}'(X), \qquad (F', f, F) \longmapsto F'$$

defines a strict morphism of prederivators $\Phi'' : \mathbb{D}'' \longrightarrow \mathbb{D}'$. It is obvious that Φ' is an equivalence, and that $\Phi\Phi'$ is isomorphic to Φ''. □

Remark 10.15. If, in Proposition 10.14, we furthermore assume that $\mathbb{D}$ and $\mathbb{D}'$ are triangulated derivators and $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ is an exact morphism, then $\mathbb{D}''$ is also a triangulated derivator and Φ' and Φ'' are exact morphisms. The fact that $\mathbb{D}''$ is a triangulated derivator and the morphism Φ' is exact is just because the Φ' is an equivalence; see Remark 10.12. The fact that Φ'' is exact is also immediate.

10.16. A prederivator $\mathbb{D}$ is *small* if, for any category X in $\mathcal{D}ia$, the category $\mathbb{D}(X)$ is small. Let $\mathcal{TD}_{str}$ be the category with objects the small triangulated derivators, and morphisms the *strict* exact morphisms of derivators. One defines the category $\pi\mathcal{TD}_{str}$ (respectively $\pi\mathcal{TD}$) as the category whose objects are the small triangulated derivators, and whose morphisms are the isomorphism classes of *strict* exact morphisms (respectively of exact morphisms) of derivators. There are obvious canonical functors

$$\gamma_{str} : \mathcal{TD}_{str} \longrightarrow \pi\mathcal{TD}_{str} \quad \text{and} \quad i : \pi\mathcal{TD}_{str} \longrightarrow \pi\mathcal{TD}$$

which are the identity on objects. One defines the functor

$$\gamma : \mathcal{TD}_{str} \longrightarrow \pi\mathcal{TD}$$

as the composition $\gamma = i\gamma_{str}$.

Proposition 10.17. *The class of equivalences of derivators admits a right calculus of fractions in the category $\pi\mathcal{TD}_{str}$; see* [10, Chapter I, 2.3]. *Moreover, the canonical functor*

$$i : \pi\mathcal{TD}_{str} \longrightarrow \pi\mathcal{TD}$$

sends equivalences of derivators to isomorphisms, and furthermore it is universal with this property; the functor i induces an isomorphism of categories between the localization of $\pi\mathcal{TD}_{str}$, with respect to the equivalences of derivators, and the category $\pi\mathcal{TD}$.

Proof. First we prove that the class of equivalences of derivators admits a right calculus of fractions. It is clear that any identity is an equivalence of derivators and that equivalences of derivators are stable by composition. Consider now a diagram in $\pi\mathcal{TD}_{str}$

$$\begin{array}{ccc} & & \mathbb{D}' \\ & & \big\downarrow{\scriptstyle S} \\ \mathbb{E} & \xrightarrow[\Phi]{} & \mathbb{D} \end{array}$$

where S is an equivalence. Then S has a quasi-inverse S^{-1}. Proposition 10.14 and Remark 10.15, applied to the morphism $S^{-1}\Phi$, tell us that there exists a commutative diagram in $\pi\mathcal{TD}_{str}$

$$\begin{array}{ccc} \mathbb{E}' & \xrightarrow{\Phi'} & \mathbb{D}' \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ \mathbb{E} & \xrightarrow[\Phi]{} & \mathbb{D} \end{array}$$

where T is an equivalence. Next we have to show that, given two strict morphisms Φ and Φ' from $\mathbb{D}$ to $\mathbb{D}'$ and a strict equivalence $S:\mathbb{D}' \longrightarrow \mathbb{E}'$ such that $S\Phi \simeq S\Phi'$, there is a strict equivalence $T:\mathbb{E} \longrightarrow \mathbb{D}$ such that $\Phi T \simeq \Phi' T$. The point is that, if S^{-1} is a quasi-inverse of S, we have

$$\Phi \simeq S^{-1}S\Phi \simeq S^{-1}S\Phi' \simeq \Phi',$$

so $T = 1_{\mathbb{D}}$ will do. So far, this establishes that there is a right calculus of fractions.

The fact that any equivalence of derivators induces an isomorphism in $\pi\mathcal{TD}$ is by the existence of a quasi-inverse; see 10.13. The universality of the map i follows from Proposition 10.14, Remark 10.15 and the calculus of fractions. □

Remark 10.18. Next we state a corollary that will tell us that K-theory is functorial in non-strict morphisms of derivators. In reading Corollary 10.19, the reader should put $L = K$, and $\mathcal{C}$ should be the homotopy category of spaces.

Corollary 10.19. *Let $L_{str}:\mathcal{TD}_{str} \longrightarrow \mathcal{C}$ be a functor that sends strict equivalences of derivators to isomorphisms. There is a unique functor $L:\pi\mathcal{TD} \longrightarrow \mathcal{C}$ such that the triangle below commutes*

$$\begin{array}{ccc} \mathcal{TD}_{str} & \xrightarrow{L_{str}} & \mathcal{C} \\ {\scriptstyle \gamma}\big\downarrow & \nearrow_{L} & \\ \pi\mathcal{TD}. & & \end{array}$$

In other words, the functor γ identifies the category $\pi\mathcal{TD}$ with the localization of $\mathcal{TD}_{str}$ by the class of strict equivalences of triangulated derivators.

Proof. By Proposition 10.17 it suffices to prove that, for any isomorphic strict morphisms of triangulated derivators Φ and Φ', one has $L_{str}(\Phi) = L_{str}(\Phi')$. We will do this by constructing an auxiliary derivator.

Let $\mathbb{D}'$ be any prederivator. Define a new prederivator $\mathbb{D}''$ as follows. For a category X in $\mathcal{D}ia$, $\mathbb{D}''(X)$ is the full subcategory of the category of arrows of $\mathbb{D}'(X)$ whose objects are the isomorphisms. It is clear that this defines a prederivator $\mathbb{D}''$. Moreover, we have three strict morphisms of prederivators

$$i : \mathbb{D}' \longrightarrow \mathbb{D}'' \quad \text{and} \quad s, t : \mathbb{D}'' \longrightarrow \mathbb{D}',$$

where i sends an object to its identity, and s (respectively t) sends an isomorphism to its source (respectively to its target). The morphisms i, s and t are equivalences of prederivators. If $\mathbb{D}'$ is a triangulated derivator, it follows that so is $\mathbb{D}''$. We also have that $si = ti = 1_{\mathbb{D}'}$, so that $L_{str}(s) = L_{str}(t)$.

Now let $\Phi : \mathbb{D} \longrightarrow \mathbb{D}'$ and $\Phi' : \mathbb{D} \longrightarrow \mathbb{D}'$ be strict, exact morphisms of triangulated derivators, and $\alpha : \Phi \Longrightarrow \Phi'$ an isomorphism. Then α defines a strict, exact morphism $\Phi'' : \mathbb{D} \longrightarrow \mathbb{D}''$, so that $s\Phi'' = \Phi$ and $t\Phi'' = \Phi'$. Hence

$$L_{str}(\Phi) = L_{str}(s)L_{str}(\Phi'') = L_{str}(t)L_{str}(\Phi'') = L_{str}(\Phi'). \qquad \square$$

Remark 10.20. The category $\mathcal{TD}_{str}$ has finite products. Its terminal object is 0, the *zero derivator*, defined by $0(X) = e$ for any X in $\mathcal{D}ia$. The binary product of $\mathbb{D}$ and $\mathbb{D}'$ is given by

$$(\mathbb{D} \times \mathbb{D}')(X) = \mathbb{D}(X) \times \mathbb{D}'(X) \quad \text{for any } X \text{ in } \mathcal{D}ia.$$

It is easy to see that the category $\pi\mathcal{TD}_{str}$ has finite products and that the functor from $\mathcal{TD}_{str}$ to $\pi\mathcal{TD}_{str}$ commutes with finite products. Using Proposition 10.17 and [10, Chapter I, Proposition 3.1] one deduces that the category $\pi\mathcal{TD}$ has finite products, and that the canonical functor from $\mathcal{TD}_{str}$ to $\pi\mathcal{TD}$ commutes with finite products (the reader might also wish to prove this directly). Under the assumptions of Corollary 10.19, if the category $\mathcal{C}$ has finite products and if the functor L_{str} commutes with finite products, we conclude that the functor L also commutes with finite products.

11. Equivalent versions of additivity

The additivity theorem is a statement about the derivator $\mathbb{E}\mathrm{xact}(\mathbb{D})$, and about two maps out of it. In this section we give the precise formulation of the theorem, and prove the equivalence with the assertion in Remark 3.19; see also Caution 3.18. The reader should note that, even though the statement of the key Proposition 11.12 can be made purely in terms of strict morphisms of derivators, the proof appeals to three non-strict morphisms. This explains why we need Section 10.

11.1. Before we go much further we want a few little lemmas about cheap ways to construct global cartesian squares. First some notation: let I be the category defined by the graph

$$0 \longleftarrow 1.$$

We wish to consider the following three functors

$$s=0:e\longrightarrow I,\quad t=1:e\longrightarrow I\quad \text{and}\quad p:I\longrightarrow e.$$

Now $\square$ is isomorphic to $I\times I$, and we get a wealth of functors on $\square=I\times I$. There are the functors

$$s\times 1_I,\quad t\times 1_I,\quad 1_I\times s\quad \text{and}\quad 1_I\times t$$

all of which are functors $I\longrightarrow I\times I=\square$. There are also the two functors

$$1_I\times p\quad \text{and}\quad p\times 1_I$$

which are functors $\square=I\times I\longrightarrow I$. And finally we also wish to consider the four functors

$$s\times s,\quad s\times t,\quad t\times s\quad \text{and}\quad t\times t$$

which are all functors $e\longrightarrow I\times I=\square$. We adopt the notation $(s\times s)(e)=(0,0)$, $(s\times t)(e)=(0,1)$, $(t\times s)(e)=(1,0)$ and $(t\times t)(e)=(1,1)$.

Lemma 11.2. *Let $\mathbb{D}$ be a derivator and let F be any object in $\mathbb{D}(I)$. Then the global commutative squares $(p\times 1_I)^*(F)$ and $(1_I\times p)^*(F)$ are cartesian in $\mathbb{D}(I\times I)=\mathbb{D}(\square)$.*

Proof. By symmetry it suffices to prove the assertion for $1_I\times p$. Now observe that $1_I\times p:I\times I\longrightarrow I$ has a left adjoint $1_I\times t:I\longrightarrow I\times I$. The map $1_I\times t$ factors as

$$I\xrightarrow{\rho}\lrcorner\xrightarrow{\sigma}\square.$$

By Lemma 6.1 we know that the functor $(1_I\times p)^*$ has a left adjoint $(\sigma\rho)^*=\rho^*\sigma^*$. In other words, $(1_I\times p)^*$ is naturally isomorphic to $\sigma_*\rho_*$. Any object in the essential image of $(1_I\times p)^*\simeq\sigma_*\rho_*$ lies in the essential image of σ_*. By Proposition 7.3 the essential image of the functor σ_* consists precisely of the cartesian global commutative squares. $\square$

Lemma 11.3. *Let $\mathbb{D}$ be a derivator. Then the composites*

$$\mathbb{D}(e)\xrightarrow{t_!}\mathbb{D}(I)\xrightarrow{s^*}\mathbb{D}(e),\qquad \mathbb{D}(e)\xrightarrow{t_!}\mathbb{D}(I)\xrightarrow{t^*}\mathbb{D}(e)$$

are naturally isomorphic, respectively, to $0:\mathbb{D}(e)\longrightarrow\mathbb{D}(e)$ and $1:\mathbb{D}(e)\longrightarrow\mathbb{D}(e)$. Dually the composites

$$\mathbb{D}(e)\xrightarrow{s_*}\mathbb{D}(I)\xrightarrow{s^*}\mathbb{D}(e),\qquad \mathbb{D}(e)\xrightarrow{s_*}\mathbb{D}(I)\xrightarrow{t^*}\mathbb{D}(e)$$

are naturally isomorphic, respectively, to $1:\mathbb{D}(e)\longrightarrow\mathbb{D}(e)$ and $0:\mathbb{D}(e)\longrightarrow\mathbb{D}(e)$.

Proof. The assertions being dual it suffices to prove the statements for $t_!$. The fact that $t^*t_!\simeq 1$ is because $t_!$ is fully faithful; see Proposition 7.1. For the assertion that $s^*t_!\simeq 0$ we proceed as follows. We know that the functor $t:e\longrightarrow I$ embeds e as the initial object. The functor t is an open immersion. Proposition 8.11 establishes that any object $t_!(F)$ will have the property that

$t_!(F)_y$ vanishes when $y \in I - t(e)$. The only point y with this property is $y = 0 \in I$. We conclude that $0 \simeq t_!(F)_0 = s^*t_!(F)$. □

Definition 11.4. Let $\mathbb{D}$ be a triangulated derivator. We define two exact morphisms

$$\theta : \mathbb{D} \longrightarrow \mathbb{D}_\Box, \qquad \rho : \mathbb{D} \longrightarrow \mathbb{D}_\Box .$$

In the notation of Examples 10.5 and 10.6 they are given as the composites

$$\mathbb{D} \xrightarrow{t_!} \mathbb{D}_I \xrightarrow{(1_I \times p)^*} \mathbb{D}_\Box,$$
$$\mathbb{D} \xrightarrow{s_*} \mathbb{D}_I \xrightarrow{(p \times 1_I)^*} \mathbb{D}_\Box .$$

Next we compute what these maps do.

Lemma 11.5. *Let $\mathbb{D}$ be a triangulated derivator. For any object F in $\mathbb{D}(e)$ we have isomorphisms, natural in F,*

$$\theta(F)_{0,0} \simeq 0, \qquad \theta(F)_{0,1} \simeq 0, \qquad \theta(F)_{1,0} \simeq F, \qquad \theta(F)_{1,1} \simeq F.$$

The computation for ρ yields

$$\rho(F)_{0,0} \simeq F, \qquad \rho(F)_{0,1} \simeq 0, \qquad \rho(F)_{1,0} \simeq F, \qquad \rho(F)_{1,1} \simeq 0.$$

Proof. The statements are dual, so we prove only the assertion about θ. The point is that the computation is easy; for example

$$\theta(F)_{0,1} = (s \times t)^*\theta(F) = (s \times t)^*(1_I \times p)^*t_!(F) = s^*t_!(F) \simeq 0,$$

where the last isomorphism is by Lemma 11.3. The remaining computations are similar. □

Remark 11.6. Let $\mathbb{D}$ be a triangulated derivator. If X is an object of $\mathcal{D}ia$, then Proposition 7.8 asserts that $\mathbb{D}_X$ is also a triangulated derivator. For any object $F \in \mathbb{D}_X(e)$ the objects $\theta(F), \rho(F) \in \mathbb{D}_X(\Box)$ are

(1) Cartesian by Lemma 11.2.
(2) Satisfy $\theta(F)_{0,1} \simeq \rho(F)_{0,1} \simeq 0$ by Lemma 11.5.

Next we consider the category of all $G \in \mathbb{D}_X(\Box)$ satisfying this.

Definition 11.7. Let $\mathbb{D}$ be a triangulated derivator and let X be a category in $\mathcal{D}ia$. One defines the category $\mathbb{E}\mathrm{xact}(\mathbb{D})(X)$ of *short exact sequences* in $\mathbb{D}_X$ as a full subcategory of $\mathbb{D}(X \times \Box) = \mathbb{D}_X(\Box)$. The objects of $\mathbb{E}\mathrm{xact}(\mathbb{D})(X)$ are defined to be the objects $G \in \mathbb{D}_X(\Box)$ such that

(1) G is cartesian.
(2) $G_{0,1} \simeq 0$.

11.8. Suppose $\mathbb{D}$ is a triangulated derivator. Proposition 7.7 says that, if $j : X \longrightarrow Y$ is any morphism in $\mathcal{D}ia$, then the functor $(j \times 1_\Box)^* : \mathbb{D}(Y \times \Box) \longrightarrow \mathbb{D}(X \times \Box)$ takes $\mathbb{E}\mathrm{xact}(\mathbb{D})(Y) \subset \mathbb{D}(Y \times \Box)$ to $\mathbb{E}\mathrm{xact}(\mathbb{D})(X) \subset \mathbb{D}(X \times \Box)$. This means that $\mathbb{E}\mathrm{xact}(\mathbb{D})$ can be given, uniquely, the structure of a prederivator, so that the natural inclusion $\mathbb{E}\mathrm{xact}(\mathbb{D}) \hookrightarrow \mathbb{D}_\Box$ is a strict morphism of prederivators.

In 11.1 we produced many functors, among them $(1_I \times s) : I \longrightarrow I \times I = \Box$ and $(t \times 1_I) : I \longrightarrow I \times I = \Box$. In concrete terms these are, respectively, the inclusions of

$$\begin{array}{c} (0,0) \\ \uparrow \\ (1,0) \end{array} \qquad \text{and} \qquad (1,0) \longrightarrow (1,1)$$

$$\text{as subcategories of} \qquad \begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \uparrow \\ (1,0) & \longrightarrow & (1,1). \end{array}$$

In the notation of Example 10.5 we have strict morphisms of derivators

$$(1_I \times s)^* : \mathbb{D}_\Box \longrightarrow \mathbb{D}_I,$$
$$(t \times 1_I)^* : \mathbb{D}_\Box \longrightarrow \mathbb{D}_I\,.$$

The composites

$$\mathbb{E}\mathrm{xact}(\mathbb{D}) \xrightarrow{\text{inclusion}} \mathbb{D}_\Box \xrightarrow{(1_I \times s)^*} \mathbb{D}_I,$$
$$\mathbb{E}\mathrm{xact}(\mathbb{D}) \xrightarrow{\text{inclusion}} \mathbb{D}_\Box \xrightarrow{(t \times 1_I)^*} \mathbb{D}_I$$

are strict morphisms of prederivators, and Proposition 8.13 and its dual tell us that both composites are equivalences. We conclude that the prederivator $\mathbb{E}\mathrm{xact}(\mathbb{D})$ is a triangulated derivator; we have two equivalences with the triangulated derivator $\mathbb{D}_I$.

Finally we have to prove that the inclusion $\mathbb{E}\mathrm{xact}(\mathbb{D}) \longrightarrow \mathbb{D}_\Box$ is exact. This means we must prove, for every morphism $j : X \longrightarrow Y$ in $\mathcal{D}ia$, that the morphisms $j_!$, j_*, take $\mathbb{E}\mathrm{xact}(\mathbb{D})(X) \subset \mathbb{D}_\Box(X)$ to $\mathbb{E}\mathrm{xact}(\mathbb{D})(Y) \subset \mathbb{D}_\Box(Y)$. But this is immediate from Example 10.6; we know that the maps $j_!$ and j_*, viewed as morphisms of derivators $\mathbb{D}_X \longrightarrow \mathbb{D}_Y$, are, respectively, left exact and right exact.

Lemma 11.9. *Definition* 11.4 *gives us two maps $\theta, \rho : \mathbb{D} \longrightarrow \mathbb{D}_\Box$. We assert that the maps θ, ρ factor as*

$$\mathbb{D} \underset{\rho'}{\overset{\theta'}{\rightrightarrows}} \mathbb{E}\mathrm{xact}(\mathbb{D}) \xrightarrow{\text{inclusion}} \mathbb{D}_\Box$$

with $\theta', \rho' : \mathbb{D} \longrightarrow \mathbb{E}\mathrm{xact}(\mathbb{D})$ both exact morphisms of triangulated derivators.

Proof. The fact that the images of θ and ρ lie in $\mathbb{E}\mathrm{xact}(\mathbb{D}) \subset \mathbb{D}_\square$ was seen in Remark 11.6. The factorization exists as maps of derivators. To check that the morphisms θ', ρ' are exact we observe that the two composites

$$\mathbb{D} \underset{\rho'}{\overset{\theta'}{\rightrightarrows}} \mathbb{E}\mathrm{xact}(\mathbb{D}) \xrightarrow{\text{inclusion}} \mathbb{D}_\square \xrightarrow{(1_I \times s)^*} \mathbb{D}_I$$

are exact; they equal, respectively, $(1_I \times s)^*\theta$ and $(1_I \times s)^*\rho$. Since the map

$$\mathbb{E}\mathrm{xact}(\mathbb{D}) \xrightarrow{\text{inclusion}} \mathbb{D}_\square \xrightarrow{(1_I \times s)^*} \mathbb{D}_I$$

is an equivalence, it follows that θ' and ρ' are exact. □

11.10. Recall the category $\mathcal{TD}_{str}$ of 10.16; the objects are small triangulated derivators, and the morphisms are the strict, exact morphisms. A functor L_{str}, from the category $\mathcal{TD}_{str}$ to the category of spaces, will be called an *additive space functor* providing

(a) For any equivalence of triangulated derivators $\mathbb{D} \longrightarrow \mathbb{D}'$, the map

$$L_{str}(\mathbb{D}) \longrightarrow L_{str}(\mathbb{D}')$$

is a weak homotopy equivalence.

(b) The functor L_{str} sends the zero derivator to a weakly contractible space and, for any triangulated derivators $\mathbb{D}$ and $\mathbb{D}'$, the canonical map

$$L_{str}(\mathbb{D} \times \mathbb{D}') \longrightarrow L_{str}(\mathbb{D}) \times L_{str}(\mathbb{D}')$$

is a weak homotopy equivalence.

Corollary 10.19 allow us to extend L_{str} to a functor $L : \pi\mathcal{TD} \longrightarrow \mathsf{Hot}$, where Hot is the homotopy category of spaces. Remark 10.20 guarantees that L commutes with finite products.

The example we care about is where L_{str} is K-theory.

11.11. We wish to consider in $\mathcal{D}ia$ the two morphisms $(s \times s)$, $(t \times t)$, which (see 11.1) are both functors $e = e \times e \longrightarrow I \times I = \square$. They induce strict exact morphisms of triangulated derivators $(s \times s)^*, (t \times t)^* : \mathbb{D}_\square \longrightarrow \mathbb{D}$. They define two strict exact morphisms of triangulated derivators by restriction to $\mathbb{E}\mathrm{xact}(\mathbb{D})$

$$(s \times s)^*, (t \times t)^* : \mathbb{E}\mathrm{xact}(\mathbb{D}) \longrightarrow \mathbb{D}\,.$$

We also have strict exact morphisms of triangulated derivators

$$\mathbb{D}_I \underset{t^*}{\overset{s^*}{\rightrightarrows}} \mathbb{D}\,.$$

We prove:

Proposition 11.12. *Let L_{str} be an additive space functor. Then the following conditions are equivalent:*

(i) *The map*

$$\big(L_{str}\big((s\times s)^*\big), L_{str}\big((t\times t)^*\big)\big) : L_{str}\big(\mathbb{E}\mathrm{xact}(\mathbb{D})\big) \longrightarrow L_{str}(\mathbb{D})\times L_{str}(\mathbb{D})$$

is a weak homotopy equivalence.

(ii) *The map*

$$\big(L_{str}(s^*), L_{str}(t^*)\big) : L_{str}(\mathbb{D}_I) \longrightarrow L_{str}(\mathbb{D})\times L_{str}(\mathbb{D})$$

is a weak homotopy equivalence.

Proof. In the notation of 11.10, we need to prove that

$$\big(L\big((s\times s)^*\big), L\big((t\times t)^*\big)\big) : L\big(\mathbb{E}\mathrm{xact}(\mathbb{D})\big) \longrightarrow L(\mathbb{D})\times L(\mathbb{D})$$

is an isomorphism in Hot if and only if

$$\big(L(s^*), L(t^*)\big) : L(\mathbb{D}_I) \longrightarrow L(\mathbb{D})\times L(\mathbb{D})$$

is. In Lemma 11.9 we produced two exact (non-strict) morphisms $\theta', \rho' : \mathbb{D}\longrightarrow \mathbb{E}\mathrm{xact}(\mathbb{D})$. They induce two maps $L(\theta'), L(\rho')$ of the form $L(\mathbb{D})\longrightarrow L(\mathbb{E}\mathrm{xact}(\mathbb{D}))$. From 11.10 we know that, for any triangulated derivator $\mathbb{D}$, the (non-strict) direct sum operation $\mu : \mathbb{D}\times\mathbb{D}\longrightarrow\mathbb{D}$ induces a commutative H-space structure[3] on the space $L(\mathbb{D})$. We will prove that each of (i) and (ii) above is equivalent to the following assertion

(iii) *The composite*

$$L(\mathbb{D})\times L(\mathbb{D}) \xrightarrow{L(\theta')\times L(\rho')} L\big(\mathbb{E}\mathrm{xact}(\mathbb{D})\big)\times L\big(\mathbb{E}\mathrm{xact}(\mathbb{D})\big) \xrightarrow{L(\mu)} L\big(\mathbb{E}\mathrm{xact}(\mathbb{D})\big)$$

is an isomorphism in Hot.

The assertion (i) says that some map $L(\mathbb{E}\mathrm{xact}(\mathbb{D}))\longrightarrow L(\mathbb{D})\times L(\mathbb{D})$ is an isomorphism in Hot, while the assertion (iii) claims that some map $L(\mathbb{D})\times L(\mathbb{D})\longrightarrow L(\mathbb{E}\mathrm{xact}(\mathbb{D}))$ is an isomorphism in Hot. To prove the equivalence (i) $\Longleftrightarrow$ (iii) we will show that the composite

$$L(\mathbb{D})\times L(\mathbb{D})\longrightarrow L\big(\mathbb{E}\mathrm{xact}(\mathbb{D})\big)\longrightarrow L(\mathbb{D})\times L(\mathbb{D})$$

is an isomorphism in Hot. In fact we will study the composite at the level before we apply the functor L. On the level of derivators the map is induced by

$$\mathbb{D}\times\mathbb{D}\xrightarrow{\theta'\times\rho'}\mathbb{E}\mathrm{xact}(\mathbb{D})\times\mathbb{E}\mathrm{xact}(\mathbb{D})\xrightarrow{\mu}\mathbb{E}\mathrm{xact}(\mathbb{D})\xrightarrow{((s\times s)^*,(t\times t)^*)}\mathbb{D}\times\mathbb{D}\,.$$

[3] We even have an infinite loop space structure; see Corollary A.8.

The composite above equals the composite below

$$\mathbb{D}\times\mathbb{D}\xrightarrow{\theta\times\rho}\mathbb{D}_\square\times\mathbb{D}_\square\xrightarrow{\mu}\mathbb{D}_\square\xrightarrow{((s\times s)^*,(t\times t)^*)}\mathbb{D}\times\mathbb{D}$$

which is nothing other than the map

$$\begin{pmatrix}(s\times s)^*\theta & (s\times s)^*\rho\\ (t\times t)^*\theta & (t\times t)^*\rho\end{pmatrix}:\mathbb{D}\times\mathbb{D}\longrightarrow\mathbb{D}\times\mathbb{D};$$

in Lemma 11.5 we computed that this comes to

$$\begin{pmatrix}(s\times s)^*\theta & (s\times s)^*\rho\\ (t\times t)^*\theta & (t\times t)^*\rho\end{pmatrix}\simeq\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

which is an equivalence of derivators.

Now we need to show (ii) $\Longleftrightarrow$ (iii). The assertion (iii) is a statement that some map $L(\mathbb{D})\times L(\mathbb{D})\longrightarrow L(\mathbb{E}\mathrm{xact}(\mathbb{D}))$ is an isomorphism in Hot, and the assertion of (ii) claims that some map $L(\mathbb{D}_I)\longrightarrow L(\mathbb{D})\times L(\mathbb{D})$ is an isomorphism in Hot. In (11.8) we found that the composite

$$\mathbb{E}\mathrm{xact}(\mathbb{D})\xrightarrow{\text{inclusion}}\mathbb{D}_\square\xrightarrow{(t\times 1_I)^*}\mathbb{D}_I$$

is an equivalence of derivators; applying L to it we have an isomorphism Φ from $L(\mathbb{E}\mathrm{xact}(\mathbb{D}))$ to $L(\mathbb{D}_I)$. Our proof that (ii) $\Longleftrightarrow$ (iii) will consist of showing that the composite

$$L(\mathbb{D})\times L(\mathbb{D})\longrightarrow L\big(\mathbb{E}\mathrm{xact}(\mathbb{D})\big)\xrightarrow{\Phi}L(\mathbb{D}_I)\longrightarrow L(\mathbb{D})\times L(\mathbb{D})$$

is an isomorphism in Hot.

Once again it is convenient to compute this map before applying L. In computing the composite we first replace $\mathbb{E}\mathrm{xact}(\mathbb{D})$ by the larger derivator $\mathbb{D}_\square$; all the maps were defined as restrictions to $\mathbb{E}\mathrm{xact}(\mathbb{D})$ of maps on $\mathbb{D}_\square$. Our composite comes to

$$\mathbb{D}\times\mathbb{D}\xrightarrow{\theta\times\rho}\mathbb{D}_\square\times\mathbb{D}_\square\xrightarrow{\mu}\mathbb{D}_\square\xrightarrow{(t\times 1_I)^*}\mathbb{D}_I\xrightarrow{(s^*,t^*)}\mathbb{D}\times\mathbb{D}.$$

This composite is nothing other than the map

$$\begin{pmatrix}(t\times s)^*\theta & (t\times s)^*\rho\\ (t\times t)^*\theta & (t\times t)^*\rho\end{pmatrix}:\mathbb{D}\times\mathbb{D}\longrightarrow\mathbb{D}\times\mathbb{D}.$$

The useful Lemma 11.5 computes for us that this is

$$\begin{pmatrix}(t\times s)^*\theta & (t\times s)^*\rho\\ (t\times t)^*\theta & (t\times t)^*\rho\end{pmatrix}\simeq\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}.$$

This, of course, is not an equivalence of derivators. But on applying the functor L we deduce the map

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : L(\mathbb{D}) \times L(\mathbb{D}) \longrightarrow L(\mathbb{D}) \times L(\mathbb{D}),$$

which is an isomorphism in Hot. □

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Appendix A. Gamma derivators

A.1. Let $\mathbb{D}$ be a triangulated derivator. Let S be a finite set considered as a discrete category. We denote by $P(S)$ the set of subsets of S ordered by inclusion. We have a canonical, fully faithful functor

$$i : S \longrightarrow P(S), \qquad s \longmapsto \{s\}.$$

For any category X in $\mathcal{D}ia$ we define $\Gamma\,\mathbb{D}(S)(X)$ to be the essential image of the functor $(1_X \times i)_*$. That is, $\Gamma\,\mathbb{D}(S)(X) \subset \mathbb{D}(X \times P(S))$ is the full subcategory whose objects are

$$\mathrm{Ob}\bigl[\Gamma\,\mathbb{D}(S)(X)\bigr] = \left\{ \begin{array}{c|c} \text{Objects} & \text{there exists an object} \\ F \in \mathbb{D}(X \times P(S)) & G \in \mathbb{D}(X \times S),\ \text{and an} \\ & \text{isomorphism } F \simeq (1_X \times i)_* G \end{array} \right\}.$$

This defines a map, so far only on objects, $\Gamma\,\mathbb{D}(S) : \mathcal{D}ia^{\mathrm{op}} \longrightarrow \mathcal{C}at$, sending X to $\Gamma\,\mathbb{D}(S)(X)$. The functor $(i_*)_X : \mathbb{D}_S(X) \longrightarrow \mathbb{D}_{P(S)}(X)$ factors as

$$\mathbb{D}_S(X) = \mathbb{D}(X \times S) \xrightarrow{\alpha_X} \Gamma\,\mathbb{D}(S)(X) \xrightarrow{\beta_X} \mathbb{D}_{P(S)}(X) = \mathbb{D}\bigl(X \times P(S)\bigr).$$

Next we extend $\Gamma\,\mathbb{D}(S)$ to be a 2-functor of 2-categories.

Lemma A.2. *With the notation as above $\Gamma\,\mathbb{D}(S)$ extends, uniquely, to a triangulated derivator, in such a way that the maps*

$$\mathbb{D}_S \xrightarrow{\alpha} \Gamma\,\mathbb{D}(S) \xrightarrow{\beta} \mathbb{D}_{P(S)}$$

are both morphisms of derivators, with α an equivalence and β a strict, exact morphism.

Proof. Example 10.6, applied to the functor $i_* : \mathbb{D}_S \longrightarrow \mathbb{D}_{P(S)}$, tells us that it is a right exact morphism of derivators. Proposition 10.9 allows us to conclude that it is exact.

The fact that i_* is a morphism of derivators means that, for any morphism $u : X \longrightarrow Y$ in $\mathcal{D}ia$, the map

$$u^* : \mathbb{D}_{P(S)}(Y) \longrightarrow \mathbb{D}_{P(S)}(X)$$

takes objects in the essential image of i_* to objects in the essential image of i_*. We conclude that u^* takes $\Gamma\,\mathbb{D}(S)(Y) \subset \mathbb{D}_{P(S)}(Y)$ to $\Gamma\,\mathbb{D}(S)(X) \subset \mathbb{D}_{P(S)}(X)$. This means that $\Gamma\,\mathbb{D}(S)$ is a subprederivator of $\mathbb{D}_{P(S)}$, that is the inclusion β is a strict morphism. Furthermore, α is a (non-strict) morphism of prederivators.

Now Proposition 7.1 says that the functor $i_*:\mathbb{D}_S(X) \longrightarrow \mathbb{D}_{P(S)}(X)$ is fully faithful, which means that the map $\alpha_X:\mathbb{D}_S(X) \longrightarrow \Gamma\,\mathbb{D}(S)(X)$ is an equivalence of categories. That is, α is an equivalence of prederivators. Since $\mathbb{D}_S$ is a triangulated derivator, so is $\Gamma\,\mathbb{D}(S)$. Also, since $\beta\alpha = i_*$ is exact and α is an equivalence, it follows that β is exact. □

Lemma A.3. *For any finite set S, the canonical strict exact morphism*

$$\Gamma\,\mathbb{D}(S) \longrightarrow \prod_{s\in S} \Gamma\,\mathbb{D}\big(\{s\}\big)$$

is an equivalence of derivators.

Proof. Note that we have an equivalence $\mathbb{D}_S \simeq \Gamma\,\mathbb{D}(S)$, and for $\mathbb{D}_S$ the assertion is an immediate consequence of the non-triviality axiom. □

Lemma A.4. *Let $\mathbb{D}$ be any triangulated derivator. The essential image of the functor $i_*:\mathbb{D}(S) \longrightarrow \mathbb{D}(P(S))$ consists of the objects F of $\mathbb{D}(P(S))$ such that, for any subsets U and V of S satisfying the condition $U \cap V = \emptyset$, the canonical map*

$$F_{U\cup V} \longrightarrow F_U \oplus F_V$$

is an isomorphism.

Proof. Let F be an object of $\mathbb{D}(S)$, and U a subset of S. By the base change axiom one has the following identification.

$$i_*(F)_U = \Gamma_*(S/U, F/U).$$

Here $S/U = U$, where on the left U is an element of $P(S)$, and on the right U is a set seen as a discrete category. Furthermore, F/U is the restriction $F|_U$ of F at U. The non-triviality axiom gives canonical isomorphisms

$$i_*(F)_U \simeq \Gamma_*(U, F|_U) \simeq \bigoplus_{u\in U} F_u.$$

This description of $i_*(F)$ implies that, for any subsets U and V of S satisfying the condition $U \cap V = \emptyset$, the canonical map

$$i_*(F)_{U\cup V} \longrightarrow F_U \oplus F_V$$

is an isomorphism.

Next we need to prove the converse. Let F be any object in $\mathbb{D}(P(S))$. By Proposition 7.1 the functor $i_*:\mathbb{D}(S) \longrightarrow \mathbb{D}(P(S))$ is fully faithful. If η is the unit of adjunction $\eta:1 \longrightarrow i_*i^*$, the full faithfulness means that the map

$$i^*\eta : i^*F \longrightarrow i^*i_*i^*F$$

is an isomorphism. If $s \in S$ is any point, we deduce that the map $s^*i^*F \longrightarrow s^*i^*i_*i^*F$ most certainly must also be an isomorphism, that is

$$\eta_{\{s\}} : F_{\{s\}} \longrightarrow i_*i^*F_{\{s\}}$$

is an isomorphism.

Now suppose the object $F \in \mathbb{D}(P(S))$ satisfies the hypothesis of the lemma. Then, if G is either F and i_*i^*F, the natural map

$$G_U \longrightarrow \bigoplus_{u \in U} G_{\{u\}}$$

is an isomorphism. For $G = F$ this is by assumption, while for $G = i_*i^*F$ it is by the first part of the proof. In the commutative square

$$\begin{array}{ccc} F_U & \xrightarrow{\eta_U} & i_*i^*F_U \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \bigoplus_{u\in U} F_{\{u\}} & \xrightarrow[\oplus\eta_{\{u\}}]{} & \bigoplus_{u\in U} i_*i^*F_{\{u\}} \end{array}$$

we now know that the maps β, γ and $\eta_{\{u\}}$ are isomorphisms. Hence so is η_U. Der 2 allows us to deduce that $\eta : F \longrightarrow i_*i^*F$ is an isomorphism. □

A.5. We recall here the definition of Segal's category Γ. The objects of Γ are finite sets, and morphisms in Γ from S to T are maps $a : S \longrightarrow P(T)$ such that, given any elements s and s' of S, if $s \neq s'$ then $a(s) \cap a(s') = \emptyset$. The composition of

$$a : S \longrightarrow P(T) \quad \text{and} \quad b : T \longrightarrow P(U)$$

is defined by

$$b \circ a : S \longrightarrow P(U), \qquad s \longmapsto \bigcup_{t \in a(s)} b(t).$$

For any integer $n \geqslant 0$, one denotes by $\mathbf{n}$ the set of integers $\{1, \ldots, n\}$. In particular, $\mathbf{0}$ is the empty set.

There is a canonical functor π from Γ to $\mathcal{D}ia$, defined by $\pi(S) = P(S)$ on objects, and which associates to a map $a : S \longrightarrow P(T)$ the functor

$$\pi(a) : P(S) \longrightarrow P(T), \qquad U \longmapsto \bigcup_{u \in U} a(u).$$

Given a triangulated derivator $\mathbb{D}$, we obtain a presheaf on Γ with values in the category of triangulated derivators

$$S \longmapsto \mathbb{D}_{\pi(S)} = \mathbb{D}_{P(S)}\,.$$

One checks easily that one gets a subpresheaf $\Gamma\,\mathbb{D}$ of the latter defined by

$$S \longmapsto \Gamma\,\mathbb{D}(S).$$

Remark A.6. Let $p:\mathbf{2} \longrightarrow \mathbf{1} \simeq e$ be the canonical map. Under the canonical strict equivalence $\mathbb{D}_{\mathbf{2}} \simeq \mathbb{D} \times \mathbb{D}$, the exact morphism

$$p_* : \mathbb{D}_{\mathbf{2}} \longrightarrow \mathbb{D}$$

corresponds to the exact morphism

$$\mu : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}, \qquad (F, G) \longrightarrow F \oplus G.$$

Let $m:\mathbf{1} \longrightarrow P(\mathbf{2})$ be the map in Γ which sends the unique element of $\mathbf{1}$ to $\mathbf{2} = \{1, 2\} \in P(\mathbf{2})$. This defines a strict exact morphism

$$m^* : \Gamma\,\mathbb{D}(\mathbf{2}) \longrightarrow \Gamma\,\mathbb{D}(\mathbf{1}),$$

and under the identifications $\Gamma\,\mathbb{D}(\mathbf{2}) \simeq \mathbb{D}_{\mathbf{2}}$ and $\Gamma\,\mathbb{D}(\mathbf{1}) \simeq \mathbb{D}$, m^* corresponds to the morphism p_* above. In other words, the operator m^* can be seen as a strictification of the binary direct sum operation. More generally, the category Γ acts on the $\Gamma\,\mathbb{D}(S)$'s as a way to strictify the associativity and commutativity of the symmetric monoidal structure defined on $\mathbb{D}$ by the direct sum operation.

Proposition A.7. *Let L_{str} be an additive space functor, as in* 11.10. *Then $L_{str}(\Gamma\,\mathbb{D})$ is a Γ-space as defined by Segal* [31]. *In other words, for any integer $n \geqslant 0$, the Segal map*

$$L_{str}(\Gamma\,\mathbb{D})(\mathbf{n}) \longrightarrow L_{str}(\Gamma\,\mathbb{D})(\mathbf{1})^n = \underbrace{L_{str}(\Gamma\,\mathbb{D})(\mathbf{1}) \times \cdots \times L_{str}(\Gamma\,\mathbb{D})(\mathbf{1})}_{n\ times}$$

is a weak homotopy equivalence.

Proof. This is a direct consequence of 11.10(a) and (b), coupled with Lemma A.3. □

Corollary A.8. *The space $L_{str}(\mathbb{D})$ is canonically (in particular functorially) endowed with the structure of an infinite loop space.*

Proof. We have a canonical equivalence of derivators

$$\Gamma\,\mathbb{D}(\mathbf{1}) \longrightarrow \mathbb{D}.$$

By property (a) of 11.10, the map $L_{str}(\Gamma\,\mathbb{D}(\mathbf{1})) \longrightarrow L_{str}(\mathbb{D})$ must be a weak homotopy equivalence. The corollary now follows by applying Segal's delooping machine [31] to the special Γ-space $L_{str}(\Gamma\,\mathbb{D})$. □

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