A New Look at a Fekete-Szegö Theorem

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Abstract. Let E be compact in $\mathbb{C}$. The main purpose is to give specific estimates on the cardinal of the union of the sets $p^{-1}(0)$ where p runs over all the monic irreducible polynomials in $\mathbf{Z}[i][X]$ having all their roots in E. We make use of tools from complex approximation theory.

§1. Introduction

Let E be compact in $\mathbb{C}$. We are interested in the problem of estimating the number of *irreducible monic* polynomials $p(X)$ in the ring $\mathbf{Z}[i][X]$, having all their roots in E. Generalising earlier works of I. Schur and M. Fekete, this last author and G. Szegö have established (actually in a more general setting, see below) in 1955, [1], the following basic result.

Theorem A. *If the capacity* $\text{cap}(E)$ *of* E *is* < 1, *then there are only finitely many polynomials as above. Conversely, if* $\text{cap}(E) > 1$ *and if* Ω *is any neighborhood of* E, *then there exist infinitely many such polynomials once we have replaced* E *by* Ω *in the condition on the roots.*

Let $K(E) := \cup p^{-1}(0)$ where p runs over the set of all the monic irreducible polynomials in $\mathbf{Z}[i][X]$ having all their roots in E. $K(E)$ is called the *algebraic kernel* of E with respect to $\mathbf{Z}[i]$. Thus the first part of the Fekete-Szegö theorem states that $K(E)$ is a finite set whenever $\text{cap}(E) < 1$. Furthermore, obviously, the cardinal $|K(E)|$ of $K(E)$ is given by the sum of the degrees of all the polynomials appearing there. The corresponding set obtained on replacing $\mathbf{Z}[i]$ by $\mathbf{Z}$ is denoted by $K_0(E)$.

The clever proof of Fekete and Szegö consists (for the first part) in establishing, elementarily, the existence of a polynomials $p(X) \in \mathbf{Z}[i][X]$with $||p||_E < 1$ and then using a resultant theoretic argument (see Section 3). However, the existence of p is proved via a non effective way which does not permit to say more on $K(E)$ than its finiteness.

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