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On the multivariate transfinite diameter

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Abstract. We prove several new results on the multivariate transfinite diameter and its connection with pluripotential theory: a formula for the transfinite diameter of a general product set, a comparison theorem and a new expression involving Robin's functions. We also study the transfinite diameter of the pre-image under certain proper polynomial mappings.

1. Introduction. We present several new results on the multivariate transfinite diameter which, we believe, should clarify its rather close connection with well-known objects of pluripotential theory. We shall first recall the definition, fix the notation and provide the necessary background. An outline of the paper appears at the end of this introductory section.

The space of all polynomials in $\mathbb{C}^n$ is denoted by $\mathcal{P}(\mathbb{C}^n)$ and the subspace of polynomials of degree at most d by $\mathcal{P}_d(\mathbb{C}^n)$. The dimension of the latter is $N := N_d(n) := \binom{d+n}{d}$, which is also the number of multi-indices whose length does not exceed d. We arrange the multi-indices in a sequence (α_i), $i = 1, 2, \ldots$, such that $\alpha_i \prec \alpha_{i+1}$ for every i where $\prec$ is the usual graded lexicographic order. Recall that this order is defined by $\alpha \prec \beta$ if either $|\alpha| \le |\beta|$ or $|\alpha| = |\beta|$ but the first (starting from the left) non-zero entry of $\alpha - \beta$ is negative. Thus, for example, $\alpha_1 = (0, \ldots, 0, 0)$ and $\alpha_N = (d, 0, \ldots, 0)$ and the z^{α_i}, $i = 1, \ldots, N_d(n)$, form the usual monomial basis of $\mathcal{P}_d(\mathbb{C}^n)$.

The *Vandermonde determinant* of a collection of $N_d(n)$ points $z_i = (z_{i1}, \ldots, z_{in}) \in \mathbb{C}^n$ is the $N \times N$ determinant defined by

$$\mathrm{VDM}(z_1, \ldots, z_N) = \det(z_i^{\alpha_j})_{i,j=1}^N. \tag{1.1}$$

As a function of the $N \times n$ complex variables z_{ij}, VDM is a polynomial of

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