

# TOTALLY REAL POINTS IN THE MANDELBROT SET

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ABSTRACT. Recently, Noytaptim and Petsche proved that the only totally real parameters  $c \in \overline{\mathbb{Q}}$  for which  $f_c(z) := z^2 + c$  is postcritically finite are 0,  $-1$  and  $-2$  [NP]. In this note, we show that the only totally real parameters  $c \in \overline{\mathbb{Q}}$  for which  $f_c$  has a parabolic cycle are  $\frac{1}{4}$ ,  $-\frac{3}{4}$ ,  $-\frac{5}{4}$  and  $-\frac{7}{4}$ .

## INTRODUCTION

Consider the family of quadratic polynomials  $f_c : \mathbb{C} \rightarrow \mathbb{C}$  defined by

$$f_c(z) := z^2 + c, \quad c \in \mathbb{C}.$$

The Mandelbrot set  $M$  is the set of parameters  $c \in \mathbb{C}$  for which the orbit of the critical point 0 under iteration of  $f_c$  remains bounded:

$$M := \{c \in \mathbb{C} \mid \forall n \geq 1, f_c^{on}(0) \in \overline{D}(0, 2)\}.$$

**Definition 1.** A parameter  $c \in \mathbb{C}$  is *postcritically finite* if the orbit of 0 under iteration of  $f_c$  is finite.

**Definition 2.** A parameter  $c \in \mathbb{C}$  is *parabolic* if  $f_c$  has a periodic cycle with multiplier a root of unity.

Postcritically finite parameters and parabolic parameters are algebraic numbers contained in  $M$ . More precisely,  $c \in \mathbb{C}$  is a postcritically finite parameter if and only if  $c$  is an algebraic integer whose Galois conjugates all belong to  $M$  (see [M] and [Bu]). In addition, if  $c \in \mathbb{C}$  is a parabolic parameter, then  $4c$  is an algebraic integer (see [Bo]); moreover,  $4c$  is an algebraic unit in  $\overline{\mathbb{Z}}/2\overline{\mathbb{Z}}$  (see [M, Remark 3.2]).

**Definition 3.** An algebraic number  $c \in \overline{\mathbb{Q}}$  is *totally real* if its Galois conjugates are all in  $\overline{\mathbb{Q}} \cap \mathbb{R}$ .

Recently, Noytaptim and Petsche [NP] completely determined the totally real postcritically finite parameters.

**Proposition 1** (Noytaptim-Petsche). *The only totally real parameters  $c \in \overline{\mathbb{Q}}$  for which  $z \mapsto z^2 + c$  is postcritically finite are  $-2$ ,  $-1$  and  $0$ .*

Their proof relies on the fact that the Galois conjugates of a postcritically finite parameter are also postcritically finite parameters, thus contained in  $M$ , and on the fact that  $M \cap \mathbb{R} = [-2, \frac{1}{4}]$  has small arithmetic capacity. In this note, we revisit their proof. We then determine the totally real parabolic parameters.

**Proposition 2.** *The only totally real parameters  $c \in \overline{\mathbb{Q}}$  for which  $z \mapsto z^2 + c$  has a parabolic cycle are  $\frac{1}{4}$ ,  $-\frac{3}{4}$ ,  $-\frac{5}{4}$  and  $-\frac{7}{4}$ .*

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## 1. POSTCRITICALLY FINITE PARAMETERS

We first revisit the proof of Noytaptim and Petsche [NP].

*Proof of Proposition 1.* Assume that  $c$  is a totally real postcritically finite parameter. Then,  $c$  and all of its Galois conjugates are real postcritically finite parameters, thus lie in the interval  $[-2, 0]$ . Indeed,

- for  $c \in (0, \frac{1}{4})$ , the orbit of 0 under iteration  $f_c$  is infinite, converging to an attracting fixed point of  $f_c$ ;
- for  $c = \frac{1}{4}$  the orbit of 0 under iteration  $f_c$  is infinite, converging to a parabolic fixed point of  $f_c$  at  $z = \frac{1}{2}$ ;
- for  $c \in (-\infty, -2) \cup (\frac{1}{4}, +\infty)$  the orbit of 0 under iteration  $f_c$  is infinite, converging to  $\infty$ .

Let  $a$  be a solution of  $a + 1/a = c$ ; that is,  $a^2 - ca + 1 = 0$ . Then  $a$  is an algebraic integer of modulus 1 with nonpositive real part, and all of its Galois conjugates also have modulus 1 and nonpositive real part.

By Kronecker's theorem,  $a$  is a root of unity. And since the Galois conjugates of  $a$  all have nonpositive real part, the only possibilities are the following:

- $a = -1$ , which is mapped to  $c = -2$ ;
- $a = e^{\pm i2\pi/3}$ , which is mapped to  $c = -1$ ;
- $a = \pm i$ , which is mapped to  $c = 0$ .

Therefore, the only postcritically finite parameters that are totally real are  $-2$ ,  $-1$ , and  $0$ .  $\square$

## 2. PARABOLIC PARAMETERS

We now present the proof of Proposition 2. Note that  $c = \frac{1}{4}$ ,  $c = -\frac{3}{4}$ ,  $c = -\frac{5}{4}$  and  $c = -\frac{7}{4}$  are indeed parabolic parameters. Indeed,

- $f_{\frac{1}{4}}$  has a fixed point with multiplier 1 at  $z = \frac{1}{2}$ ;
- $f_{-\frac{3}{4}}$  has a fixed point with multiplier  $-1$  at  $z = -\frac{1}{2}$ ;
- $f_{-\frac{5}{4}}$  has a cycle of period 2 with multiplier  $-1$  consisting of the two roots of  $4z^2 + 4z - 1$ ;
- $f_{-\frac{7}{4}}$  has a cycle of period 3 with multiplier 1 consisting of the three roots of  $8z^3 + 4z^2 - 18z - 1$ .

*Proof of Proposition 2.* Assume that  $c$  is a totally real parabolic parameter. Then, the Galois conjugates of  $c$  also are parabolic parameters. Either  $c = \frac{1}{4}$ ,  $c = -\frac{3}{4}$ ,  $c = -\frac{5}{4}$ , or  $c$  and all of its Galois conjugates lie in the interval  $[-2, -\frac{5}{4})$ . Indeed, a parabolic cycle must attract the orbit of 0 under iteration of  $f_c$ . However,

- for  $c \in (-\frac{3}{4}, \frac{1}{4})$ , the orbit of 0 under iteration  $f_c$  converges to an attracting fixed point of  $f_c$ ;
- for  $c \in (-\frac{5}{4}, -\frac{3}{4})$  the orbit of 0 under iteration  $f_c$  converges to an attracting cycle of period 2 of  $f_c$ ;
- for  $c \in (-\infty, -2) \cup (\frac{1}{4}, +\infty)$  the orbit of 0 under iteration  $f_c$  converges to  $\infty$ .

Let us assume that  $c \in [-2, -\frac{5}{4})$ . Then,  $b := 4c + 6$  and all of its Galois conjugates lie in the interval  $[-2, 1)$ . Let  $a$  be a solution of  $a + 1/a = b$ ; that is,  $a^2 - ba + 1 = 0$ .

Then  $a$  is an algebraic integer of modulus 1 with real part less than  $\frac{1}{2}$ , and all of its Galois conjugates also have modulus 1 and real part less than  $\frac{1}{2}$ .

By Kronecker's theorem,  $a$  is a root of unity. And since the Galois conjugates of  $a$  all have real part less than  $\frac{1}{2}$ , the only possibilities are the following:

- $a = -1$ ,  $b = -2$  and  $c = -2$ ; this is not a parabolic parameter;
- $a = e^{\pm i2\pi/3}$ ,  $b = -1$  and  $c = -\frac{7}{4}$ ; this is indeed a parabolic parameter;
- $a = \pm i$ ,  $b = 0$  and  $c = -\frac{3}{2}$ ; in that case  $4c = -6$  is not an algebraic unit in  $\overline{\mathbb{Z}}/2\overline{\mathbb{Z}}$  and so,  $c$  is not a parabolic parameter;
- $a = e^{\pm i2\pi/5}$ ,  $b = 2\cos(\frac{2\pi}{5})$  and  $c = \frac{\sqrt{5}-13}{8}$ ; in this case,  $f_c$  has an attracting cycle of period 4 and so,  $c$  is not a parabolic parameter;
- $a = e^{\pm i4\pi/5}$ ,  $b = 2\cos(\frac{4\pi}{5})$  and  $c = \frac{-\sqrt{5}-13}{8}$ ; then the Galois conjugate  $\frac{\sqrt{5}-13}{8}$  is not a parabolic parameter and so,  $c$  is not a parabolic parameter.

This completes the proof of the proposition.  $\square$

Remark: the following proof that  $-\frac{3}{2}$  is not a parabolic parameter was explained to us by Valentin Huguin. It follows from [Bo] that for all  $n \geq 1$ ,

$$\text{discriminant}(f_c^{\circ n}(z) - z, z) = P_n(4c) \quad \text{with} \quad P_n(b) \in \mathbb{Z}[b] \quad \text{and} \quad \pm P_n \text{ monic.}$$

As an example,

$$P_1(b) = -b + 1, \quad P_2(b) = (b-1)(b+3)^3, \quad P_3(z) = (b-1)(b+7)^3(b^2+b+7)^4,$$

and

$$P_4(z) = (b-1)(b+3)^3(b+5)^6(b^3+9b^2+27b+135)^4(b^2-2b+5)^5.$$

Note that this yields an alternate proof that  $c = \frac{1}{4}$ ,  $c = -\frac{3}{4}$ ,  $c = -\frac{5}{4}$  and  $c = -\frac{7}{4}$  are parabolic parameters. In addition,

$$P_n(0) = \text{discriminant}(z^{2^n} - z, z) \equiv 1 \pmod{2}.$$

As a consequence

$$P_n(-6) \equiv 1 \pmod{2}.$$

Thus, for all  $n \geq 1$ , the roots of  $f_{-\frac{3}{2}}^{\circ n}(z) - z$  are simple, which shows that  $f_{-\frac{3}{2}}$  has no parabolic cycle.

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