A NEW PROOF OF A CONJECTURE OF YOCCOZ

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Abstract. We give a new proof of the following conjecture of Yoccoz:

$$(\exists C \in \mathbb{R})\ (\forall \theta \in \mathbb{R}) \quad \log R(P_\theta) \le -Y(\theta) + C,$$

where $P_\theta(z) = e^{2i\pi\theta} z + z^2$, $R(P_\theta)$ is the conformal radius of the Siegel disk of P_θ (or 0 if there is none) and $Y(\theta)$ is Yoccoz's Brjuno function.

In a former article we obtained a first proof based on the control of parabolic explosion. Here, we present a more elementary proof based on Yoccoz's initial methods.

We then extend this result to some new families of polynomials such as $z^d + c$ with $d > 2$. We also show that the conjecture does not hold for $e^{2i\pi\theta} z + z^d$ with $d > 2$.

Introduction and statements

Definition 1. *Assume $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a holomorphic germ of the form*

$$f(z) = \mathrm{e}^{i2\pi\theta}\, z + \mathcal{O}(z^2).$$

Then, there is a unique formal series

$$\phi_f(Z) = Z + \sum_{n=2}^{+\infty} b_n Z^n$$

in $\mathbb{C}[[Z]]$ such that

$$\phi_f(\mathrm{e}^{i2\pi\theta}\, Z) = f \circ \phi_f(Z).$$

We let $R(f) \in [0, +\infty]$ be the radius of convergence of the linearizing series ϕ_f.

Definition 2. *Let $P_\theta : \mathbb{C} \to \mathbb{C}$ be the quadratic polynomial defined by:*

$$P_\theta(z) = \mathrm{e}^{i2\pi\theta}\, z + z^2.$$

In [Y], Yoccoz used a technique of Il'Yashenko and the polynomial-like map theory of Douady and Hubbard to prove the following result.

Theorem A (Yoccoz)**.** *For all $\varepsilon > 0$, there exists a constant $c(\varepsilon) > 0$ such that the following holds. If θ is such that $R(P_\theta) > 0$ and if $f : \mathbb{D} \to \mathbb{C}$ is a univalent map such that $f(0) = 0$ and $f'(0) = \mathrm{e}^{i2\pi\theta}$, then:*

$$R(f) \ge c(\varepsilon) \cdot \big(R(P_\theta)\big)^{1+\varepsilon}.$$

Our first result, whose proof takes its roots in the one of Yoccoz, asserts that one can choose a constant $c(\varepsilon)$ which does not depend on ε. This follows from the results obtained in [BC1] but there, the techniques are much more elaborate than the ones we present here.

Theorem 1. *Assume $f : \mathbb{D} \to \mathbb{C}$ is a univalent map such that $f(0) = 0$ and $f'(0) = \mathrm{e}^{i2\pi\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then*

$$R(f) \ge \frac{1}{10} R(P_\theta).$$

We shall prove this theorem in section 1. The constant $1/10$ is not optimal.

Definition 3. *Yoccoz's arithmetic Brjuno function is defined by*

$$Y(\theta) = \sum_{n=0}^{+\infty} \theta_0 \cdots \theta_{n-1} \log \frac{1}{\theta_n}$$

where $\theta_0 = \mathrm{Frac}(\theta) = \theta - \lfloor \theta \rfloor$ *and* $\theta_{n+1} = \mathrm{Frac}(1/\theta_n)$ *when* θ *is irrational, and* $Y(\theta) = +\infty$ *if* θ *is rational.*

Definition 4. *A Brjuno number is an irrational real number* θ *satisfying Brjuno's condition:*

$$\theta \in \mathcal{B} \iff Y(\theta) < +\infty.$$

Theorem B (Yoccoz)**.** *There exists a constant* $C \in \mathbb{R}$ *such that for all* $\theta \in \mathcal{B}$*, for all univalent function* $f : \mathbb{D} \to \mathbb{C}$ *fixing the origin with multiplier* $\mathrm{e}^{i2\pi\theta}$*,*

$$\log R(f) \geq -Y(\theta) - C.$$

Corollary (Yoccoz)**.** *If* θ *is a Brjuno number,* $R(P_\theta) > 0$ *and*

$$\log R(P_\theta) \geq -Y(\theta) - C - \log 2.$$

The term $-\log 2$ comes from the fact P_θ is univalent only on the disk $D(0, 1/2)$.

Theorem C (Yoccoz)**.** *There exists a constant* $C \in \mathbb{R}$ *such that for all* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *there exists a univalent function* $f : \mathbb{D} \to \mathbb{C}$ *fixing the origin with multiplier* $\mathrm{e}^{i2\pi\theta}$*, such that*

$$\log R(f) \leq -Y(\theta) + C.$$

This includes the case $\theta \notin \mathcal{B}$ *(i.e.* $Y(\theta) = +\infty$*) if we interpret the above inequality as* $R(f) = 0$*.*

Combining theorems A and C, Yoccoz obtained the following corollaries.

Corollary (Yoccoz)**.** *For all* $\varepsilon > 0$*, there exists* $C_\varepsilon \in \mathbb{R}$ *(that a priori may tend to* $+\infty$ *as* $\varepsilon \longrightarrow 0$*) such that for all* $\theta \in \mathbb{R} \setminus \mathbb{Q}$*,*

$$\log R(P_\theta) \leq -(1 - \varepsilon) Y(\theta) + C_\varepsilon.$$

In particular, if θ *is not a Brjuno number, then* $R(Q_\theta) = 0$*.*

Corollary (Yoccoz)**.** $R(P_\theta) > 0$ *if and only if* θ *is a Brjuno number.*

The second author found an independent proof of the optimality of Brjuno's condition in [C], working directly in the family P_θ. He looked at how parabolic points explode into cycles and how these cycles hinder each others. The control on parabolic explosion uses the combinatorics of quadratic polynomials, and the Yoccoz inequality on the limbs of the Mandelbrot set. The *relative Schwarz lemma* of the first author then enabled us to have a good enough control on conformal radii to prove the following result, conjectured by Yoccoz [Y]. This result is an immediate corollary of Yoccoz's theorem C and our theorem 1, which provides a new proof.

Theorem 2. *There exists a constant* $C \in \mathbb{R}$ *such that for all* $\theta \in \mathbb{R}$*,*

$$\log R(P_\theta) \leq -Y(\theta) + C.$$

Note that the function

$$\Upsilon : \mathcal{B} \to \mathbb{R} \quad \text{defined by} \quad \Upsilon(\theta) = \log R(P_\theta) + Y(\theta)$$

is therefore uniformly bounded. In [BC2], we prove that this function is uniformly continuous, and thus, has a continuous extension to $\mathbb{R}$.

In section 2, we show that our techniques extend to other families of polynomials. Let us define a class of well-behaved polynomials that was studied by Lukas Geyer in [G]. Given a polynomial $f : \mathbb{C} \to \mathbb{C}$, a critical orbit tail is an equivalence class in the set of forward critical orbits[1], with the relation $z \equiv z' \iff \exists m, n \in \mathbb{N}$ such that $f^{\circ n}(z) = f^{\circ m}(z')$. We say it is infinite if a point of the class (and therefore every point in the class) has infinite forward orbit.

Definition 5. *We will say that a polynomial has property (G) if the number of infinite critical orbit tails is equal to the number of indifferent cycles.*

Remark. If f has property (G), its iterates do not necessarily.

By the Fatou-Shishikura inequality, the number of non-repelling cycles of a polynomial is bounded from above by the number of infinite critical orbit tails and the number of irrationally indifferent cycles is bounded from above by the number of infinite critical orbit tails outside the basins of attraction of (super)-attracing or parabolic cycles. Thus, if f has property (G), the non-repelling cycles are parabolic or indifferent cycles and the basin of attraction of each parabolic cycle contains at most one infinite critical orbit tail. It follows that each parabolic cycle has exactly one cycle of petals and is virtually repelling (see [BE]).

Lukas Geyer proved optimality of Brjuno's condition for polynomials having property (G) (and even for a bigger class[2]), by using the same method as Yoccoz. It is therefore natural that our new observation adapts in this setting.

Definition 6. *The critical orbits of a polynomial $f : \mathbb{C} \to \mathbb{C}$ are the sets $\left\{f^{\circ k}(c)\right\}_{k\geq 0}$ where c is a critical point of f. A point z in a critical orbit is said to be free[3] if for all critical point c', $\forall k \in \mathbb{N}$, $\forall \ell \in \mathbb{N}^*$,*

$$\big(f^{\circ k}(c') = f^{\circ \ell}(z)\big) \quad \implies \big(k \geq \ell \text{ and } f^{\circ (k-\ell)}(c') = z\big).$$

We denote by Z_f the set of non-free points of critical orbits.

When a critical point has a finite forward orbit, then the critical point and its iterates are non-free points.

Definition 7. *A polynomial has "property (G) with bound N" if in addition to having property (G), the cardinal of the union of all indifferent cycles and the set Z_f is at most N.*[4]

Theorem 3. *Let $N \in \mathbb{N}$ and $\mathcal{C}$ be a compact set of degree d polynomials f fixing 0 with indifferent multiplier $\mathrm{e}^{i2\pi\theta(f)}$, having property (G) with bound N. Then $\exists C \in \mathbb{R}$ such that $\forall f \in \mathcal{C}$,*

$$\log R(f) \leq -Y(\theta(f)) + C.$$

Remark. We did not try to get the most general result possible. For instance, it is possible that the hypothesis that 0 has period 1 is not required.[5]

The following result is a corollary of Yoccoz's theorem B and our theorem 3.

Corollary 1. *Under the same assumptions as in theorem 3 the function*

$$\Upsilon : \left\{f \in \mathcal{C} \,\middle|\, \theta(f) \in \mathcal{B}\right\} \to \mathbb{R} \quad \text{defined by} \quad \Upsilon(f) = \log R(f) + Y\big(\theta(f)\big)$$

is uniformly bounded.

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[1] or, equivalently, in the set of critical *points*

[2] for the class of *saturated* polynomials, i.e. polynomials such that the number of infinite critical orbit tails *in the Julia set* is equal to the number of indifferent cycles

[3] this is not a standard terminology

[4] in a family of polynomials with bounded degrees, it is equivalent to bound the cardinal of Z_f and to bound the sum of local degrees at points in Z_f

[5] But in this case, there is one more condition: that the indifferent cycle 0 belongs to, does not collapse on itself. This condition is not implied by the others.

Corollary 2. *For each integer $d \geq 2$, this holds if $\mathcal{C}$ is the boundary of the central hyperbolic component of the family of unicritical polynomials $z^d + c$, i.e. the set of $c \in \mathbb{C}$ for which the polynomial $z^d + c$ has an indifferent fixed point.*

Corollary 3. *For each integer $d \geq 2$, this holds also for the family*

$$\big\{\mathrm{e}^{i2\pi\theta}\, z(1-z)^{d-1}\big\}_{\theta\in\mathbb{R}}.$$

Proof. The critical points are $z = 1/d$ and $z = 1$ (with multiplicity $d-2$). The second critical point is mapped in one step on $z = 0$. Thus we may apply theorem 3 with $N = 2$. □

Corollary 4. *For the family*

$$f_\theta(z) = \mathrm{e}^{i2\pi\theta}(z + z^d),$$

the following holds: $\exists C > 0$ such that $\forall\theta \in \mathbb{R}$,

$$-\frac{Y\big((d-1)\theta\big)}{d-1} - C \leq \log R(f_\theta) \leq -\frac{Y\big((d-1)\theta\big)}{d-1} + C.$$

Proof. The family f_θ is semi-conjugated to the previous family: more precisely let $\phi(z) = -z^{d-1}$, and $g_\theta(z) = \mathrm{e}^{i2\pi\theta}\, z(1-z)^{d-1}$. Then[6]

$$\phi \circ f_\theta = g_{(d-1)\theta} \circ \phi.$$

The first claim follows at once. □

In appendix A, lemma 2, we will prove that for any integer $m \geq 2$, the function

$$\theta \in \mathcal{B} \mapsto Y(\theta) - \frac{Y(m\theta)}{m}$$

is unbounded on any interval. It follows that the function

$$\theta \in \mathcal{B} \mapsto \log R(f_\theta) + Y(\theta)$$

is unbounded on any interval.

In appendix A, lemma 3, we will prove that

$$(\exists C > 0)\ (\forall m \in \mathbb{N}^*)\ (\forall\theta \in \mathbb{R}) \quad Y(\theta) \leq Y(m\theta) + C\log m.$$

It follows[7] that

$$\log R(f_\theta) \leq -\frac{Y(\theta)}{d-1} + C'.$$

This suggests the following conjecture.

Conjecture 1. *There exists a constant $C = C(d) \in \mathbb{R}$ such that for all polynomial f of degree d with an indifferent fixed point at the origin,*

$$\log R(f) \leq -\frac{Y(\theta)}{d-1} + \log\min|c_i| + C$$

where the c_i are the critical points of f and θ is the rotation number at the origin.

There are possible refinements according to how many recurrent critical points are associated to the indifferent fixed point.

[6]Note how the rotation number changed.

[7]for this, the following statement would have been enough: $\forall m \in \mathbb{N}^*$, $\exists C_m \in \mathbb{R}$, $\forall\theta \in \mathbb{R}$, $Y(\theta) \leq Y(m\theta) + C_m$

1. Optimality of the quadratic polynomial

In this section, we prove theorem 1. Our proof follows closely Yoccoz's proof of theorem A.

Let us assume that $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $R(P_\theta) > 0$, since otherwise, there is nothing to prove. Consider a univalent function $f : \mathbb{D} \to \mathbb{C}$ fixing 0 with multiplier $\mathrm{e}^{i2\pi\theta}$. Following Il'Yashenko and Yoccoz, consider the one-parameter families of maps

$$\big\{f_a : D(0,1) \to \mathbb{C}\big\}_{a\in\mathbb{C}} \quad \text{and} \quad \big\{g_b : D\big(0,1/|b|\big) \to \mathbb{C}\big\}_{b\in\mathbb{C}}$$

defined by:

$$f_a(z) = f(z) + az^2 \quad \text{and} \quad g_b(w) = \frac{1}{b} f_{1/b}(bw) = \frac{1}{b} f(bw) + w^2.$$

The family g_b extends analytically at $b = 0$ by $g_0 = P_\theta$. We have:

$$(\forall b \in \mathbb{C}^*) \quad R(g_b) = \frac{1}{|b|} R(f_{1/b}).$$

Definition 8. *A quadratic-like map is a ramified covering $f : U \to V$ of degree 2, between two simply connected domains $U \Subset V$.*

The following observation is due to Yoccoz [Y].

Lemma 1. *If $|b| \le 1/10$, the map g_b has a quadratic-like restriction $g_b : U_b \to V$ with*

$$U_b = \big\{z \in D(0,4)\,\big|\, g_b(z) \in D(0,44/9)\big\} \quad \textit{and} \quad V = D(0,44/9).$$

Proof. Since f is univalent, we have, for all $z \in \mathbb{D}$:

$$\big|f(z)\big| \le \frac{|z|}{\big(1-|z|\big)^2}.$$

It follows that when $|b| \le 1/10$, $|w| = 4$ and $|\zeta| \le 44/9$,

$$\left|\frac{1}{b} f(bw)\right| \le \frac{100}{9} = 16 - \frac{44}{9} < |w^2 - \zeta|.$$

By Rouché's theorem, every $\zeta \in D(0,44/9)$ has exactly two preimages by g_b in $D(0,4)$. Thus, $g_b : U_b \to V$ is proper of degree 2. If U_b were not connected, the component of U_b containing 0 would be mapped biholomorphically to V, which, by Schwarz's lemma, is not possible since $\big|g_b'(0)\big| = 1$. □

We now introduce in the argument the following two facts (they both are valid in a much more general setting). The following result is a corollary of lemma 1.

Proposition 1. *The map $b \mapsto \log R(g_b)$ is harmonic in a neighborhood of $\overline{D}(0,1/10)$.*

Proof. Choose $r_0 > 1/10$ so that $g_b : U_b \to V$ is quadratic-like for all $b \in D(0,r_0)$. Those quadratic-like maps all have an indifferent fixed point. This is the only non repelling cycle of the quadratic-like map. Therefore, the Julia set of the quadratic-like map undergoes a holomorphic motion as b varies in $D(0,r_0)$. The radius of convergence of ϕ_{g_b} coincides with the conformal radius of the Siegel disk Δ_b of the quadratic-like restriction.

Now, when a Siegel disk has a boundary which undergoes a holomorphic motion, its conformal radius has a logarithm $\log \operatorname{rad} \Delta_b$ that varies harmonically. The following proof of this was communicated to us by Saeed Zakeri. First, note that the conformal radius varies continuously. Then, consider an extension[8] of the holomorphic motion to a holomorphic motion of all the plane, but which does not necessarily commute with the dynamics. Let b_0 be any parameter and w_n be any sequence in

[8] Slodkowsky's theorem provides one, but we can also use the Bers-Royden or the Sullivan-Thurston version since this argument is local in terms of the parameter.

the Siegel disk of parameter b_0, converging to a point in the boundary of Δ_{b_0}. For b close to b_0 let $w_n(b)$ be the point that the motion transports w_n to. Now look at

$$u_n(b) = \phi_{g_b}^{-1}\big(w_n(b)\big).$$

For each b, the sequence $\big(w_n(b)\big)$ converges to a point in the boundary of the Siegel disk Δ_b. Thus, $\big|u_n(b)\big|$ converges to $\operatorname{rad}\Delta_b$. As the map

$$(b, w) \mapsto \big(\phi_{g_b}(w), w\big)$$

is bi-analytic, $b \mapsto u_n(b)$ is analytic. Therefore, $\log|u_n(b)|$ is harmonic (it does not vanish). Now, the map $b \mapsto \log \operatorname{rad}\Delta_b$ is the limit of these harmonic functions. □

Definition 9. *Let* $\operatorname*{avg}_{|a|=r} m(a)$ *denote the average of the function* $m(a)$ *on the circle* $|a| = r$ *(with respect to the Lebesgue measure on the circle).*

As an immediate consequence of proposition 1, we have the following equality:

(1) $$\log R(P_\theta) = \operatorname*{avg}_{|b|=1/10} \log R(g_b) = \log 10 + \operatorname*{avg}_{|a|=10} \log R(f_a).$$

Proposition 2. *We have* $\log R(f) \geq \operatorname*{avg}_{|a|=10} \log R(f_a)$

Proof. Look at the formal linearizing power series of f_a:

$$\phi_{f_a}(Z) = Z + \sum_{n=2}^{+\infty} b_n(a) Z^n.$$

By Hadamard's theorem,

$$\frac{1}{R(f_a)} = \limsup_{n\to+\infty} \sqrt[n]{|b_n(a)|}.$$

The coefficients $b_n(a)$ are polynomials in a. Thus

$$\frac{1}{n}\log\big|b_n(0)\big| \leq \operatorname*{avg}_{|a|=10} \frac{1}{n}\log\big|b_n(a)\big|.$$

By lemma 1, for $|a| = 10$, the map f_a has a quadratic-like restriction. In that case, the linearizing map ϕ_{f_a} takes its values in $\mathbb{D}$ and it follows from Cauchy inequalities that

$$\big|b_n(a)\big| \leq \frac{1}{\big(R(f_a)\big)^n}.$$

We have seen that $R(g_b)$ is a continuous non vanishing function on the circle $|b| = 1/10$. Thus, when $|a| = 10$, $R(f_a) = R(g_b)/10$ reaches a minimum $c > 0$ and

$$\frac{1}{n}\log\big|b_n(a)\big| \leq \log\frac{1}{R(f_a)} \leq \log\frac{1}{c}.$$

This uniform upper bound allows us to apply Fatou's lemma:

$$-\log R(f) = \limsup_{n\to+\infty} \frac{1}{n}\log|b_n(0)| \leq \operatorname*{avg}_{|a|=10} \limsup_{n\to+\infty} \frac{1}{n}\log|b_n(a)| = -\operatorname*{avg}_{|a|=10} \log R(f_a).$$

□

Equality 1 and proposition 2 yield:

$$\log R(f) \geq \operatorname*{avg}_{|a|=10} \log R(f_a) = \log R(P_\theta) - \log 10,$$

whence $R(f) \geq \frac{1}{10} R(P_\theta)$. This completes the proof of theorem 1. Q.E.D.

2. Other families of polynomials

In this section, we shall first prove theorem 3. Assume $N \in \mathbb{N}$ and $\mathcal{C}$ is a compact set of degree d polynomials f fixing 0 with indifferent multiplier $\mathrm{e}^{i2\pi\theta(f)}$, having property (G) with bound N. Let Z_f be the set of non-free points of critical orbits (see definition 6). Set

$$G_f(z) = \prod_i (z - w_i)^{n_i} \prod_j (z - u_j)^2$$

where $\{w_i\} = Z_f \setminus \{0\}$, $n_i =$ the local degree of f at w_i, and u_j are the indifferent periodic points of f (including 0). Let

$$g_{f,a} = f + aG_f.$$

First, by compactness of $\mathcal{C}$, by the bound N, and by the definition of G_f, we see that G_f is a bounded family over $\mathcal{C}$. Therefore, there exists $r > 0$ and $R > 0$, independent of f, such that $|a| < r \implies g_{f,a}(z)$ is a polynomial-like map of degree d from the component of $g_{f,a}^{-1}\big(D(0,R)\big)$ contained in $D(0,R)$ to $D(0,R)$. For any fixed $f \in \mathcal{C}$, as a varies in $D(0,r)$, this polynomial-like map cannot undergo a parabolic bifurcation. Therefore its Julia set undergoes a holomorphic motion. So, the same analysis as in section 1 holds and we can thus write:

$$\forall f \in \mathcal{C}, \quad \log R(f) = \operatorname*{avg}_{|a|=r/2} \log R(g_{f,a}).$$

Second, we claim that the indifferent cycles of f stay bounded away from 0 when f varies in $\mathcal{C}$. Otherwise, there would be a map $f_0 \in \mathcal{C}$ having a parabolic fixed point at 0, which could be approximated by maps in $\mathcal{C}$ having an indifferent fixed point at 0 and at least one indifferent cycle close to 0. Then either f_0 would have at least two cycles of petals at 0, or the parabolic fixed point at 0 would be virtually indifferent (see [B] for a definition). In both cases, the basin of attraction of 0 would contain at least two critical points (see [BE] for a proof), contradicting the fact that f_0 has property (G). Thus, $G_f''(0)$ is bounded away from 0.

Let $\widetilde{g}_{f,a}(z) = a g_{f,a}(a^{-1}z)$. Then, as $a \longrightarrow \infty$, $\widetilde{g}_{f,a}$ tends (pointwise) to the degree 2 polynomial

$$P(z) = \mathrm{e}^{i2\pi\theta(f)}\, z + \frac{G_f''(0)}{2} z^2$$

The same analysis as in section 1 also holds and yields

$$\log R(P) \geq \operatorname*{avg}_{|a|=r/2} R(\widetilde{g}_{f,a}).$$

Now

$$\log R(P) = \log R(\mathrm{e}^{i2\pi\theta(f)}\, z + z^2) - \log \frac{|G_f''(0)|}{2}$$

and

$$\log R(\widetilde{g}_{f,a}) = \log R(g_{f,a}) + \log|a|$$

and we proved that

$$\log R(\mathrm{e}^{i2\pi\theta(f)}\, z + z^2) \leq -Y(\theta(f)) + C.$$

Putting it altogether, we get

$$\log R(f) \leq -\log r/2 - \log \frac{|G_f''(0)|}{2} - Y(\theta(f)) + C.$$

Since $G_f''(0)$ is bounded away from 0, we get the upper bound of the theorem.

To apply then Yoccoz's theorem B, there remains to remark that the maps $f \in \mathcal{C}$ are all univalent on a common disk $D(0, r')$, since they have bounded degree and critical points are necessarily bounded away from 0. This completes the proof of theorem 3. Q.E.D.

Appendix A. Estimates on Yoccoz's Brjuno function

Lemma 2. *For any integer $m \geq 2$, the function*

$$\theta \in \mathcal{B} \mapsto Y(\theta) - \frac{Y(m\theta)}{m}$$

is unbounded on any interval.

Proof. The proof relies on the following fact. For any rational number p/q with p and q coprime, any integer $k \geq 1$, and any Brjuno number θ,

$$Y\left(\frac{p}{q} + \frac{k}{N+\theta}\right) \underset{N\to+\infty}{=} \frac{\log N}{q} + \mathcal{O}(1). \tag{2}$$

Let us first show how this enables us to conclude. Assume p/q is a rational number with p and q coprime and assume q and m are coprime. Choose a Brjuno number θ and set

$$\theta_N = \frac{p}{q} + \frac{1}{N+\theta}.$$

Note that

$$m\theta_N = \frac{mp}{q} + \frac{m}{N+\theta} \quad \text{with } mp \text{ and } q \text{ coprime.}$$

Then,

$$Y(\theta_N) \underset{N\to+\infty}{=} \frac{\log N}{q} + \mathcal{O}(1) \quad \text{and} \quad Y(m\theta_N) \underset{N\to+\infty}{=} \frac{\log N}{q} + \mathcal{O}(1).$$

Thus,

$$Y(\theta_N) - \frac{Y(m\theta_N)}{m} \underset{N\to+\infty}{=} \frac{m-1}{m} \cdot \frac{\log N}{q} + \mathcal{O}(1) \underset{N\to+\infty}{\longrightarrow} +\infty.$$

It follows that the function

$$\theta \in \mathcal{B} \mapsto Y(\theta) - \frac{Y(m\theta)}{m}$$

is unbounded in any neighborhood of p/q. This implies our lemma since the set of rational numbers p/q with q and m coprime is dense in $\mathbb{R}$.

Let us now prove estimate (2). We will use the continued fraction notation :

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

Set $\theta_N = \dfrac{p}{q} + \dfrac{k}{N+\theta}$. If N is large enough, p/q is an approximant of θ_N :

$$\frac{p}{q} = [a_0, a_1, \ldots, a_n] \quad \text{and} \quad \theta_N = [a_0, a_1, \ldots, a_n + \alpha_n] \quad \text{with} \quad \alpha_n \in \,]0,1[\,.$$

Set

$$\frac{p'}{q'} = [a_0, a_1, \ldots, a_{n-1}]$$

and for $k < n$, set

$$\alpha_k = [0, a_1, \ldots, a_k].$$

Then, $p'q - q'p = 1$,[9]

$$\alpha_0\alpha_1\cdots\alpha_{n-1} = |q'\theta_N - p'| = \left|\frac{q'p-p'q}{q} + \frac{kq'}{N+\theta}\right| \underset{N\to+\infty}{\longrightarrow} \frac{1}{q},$$

$$\alpha_0\alpha_1\cdots\alpha_n = |q\theta_N - p| = \frac{kq}{N+\theta} \underset{N\to+\infty}{\longrightarrow} 0$$

and

$$\frac{1}{\alpha_n} = -\frac{q'\theta_N - p'}{q\theta_N - p} = \frac{N+\theta}{k} - \frac{q'}{q} = \frac{\theta}{k} \mod \frac{1}{kq}.$$

In particular, α_{n+1}, which is the fractional part of $1/\alpha_n$, can take only kq values which all are Brjuno numbers. It follows that

$$Y(\theta_N) = \underbrace{\log\frac{1}{\alpha_0} + \ldots + \alpha_0\alpha_1\cdots\alpha_{n-2}\log\frac{1}{\alpha_{n-1}}}_{\mathcal{O}(1)}$$

$$+\underbrace{\alpha_0\alpha_1\cdots\alpha_{n-1}\log\frac{1}{\alpha_n}}_{q^{-1}\log N+\mathcal{O}(1)} + \underbrace{\alpha_0\alpha_1\cdots\alpha_n Y(\alpha_{n+1})}_{o(1)} = \frac{\log N}{q} + \mathcal{O}(1).$$

□

Lemma 3. *$\exists C > 0$, $\forall m \in \mathbb{N}^*$, $\forall\theta\in\mathbb{R}$,*

$$Y(\theta) \le Y(m\theta) + C\log m.$$

Proof. We will use the Brjuno sum:

$$B(\theta) = \sum_{n\in\mathbb{N}} \frac{\log q_{n+1}}{q_n}$$

where p_n/q_n are the approximants of θ. We have the following arithmetical property (c.f. [Y], page 14):

$$\big|B(\theta) - Y(\theta)\big| \text{ is bounded.}$$

We recall that

(a) if p_n/q_n are the approximants of α then

$$\frac{1}{2q_nq_{n+1}} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_nq_{n+1}},$$

and also that

(b) $q_n \ge F_n$ where F_n is the n-th Fibonacci number. Last,

(c) if $|\alpha - p/q| < 1/2q^2$, then p/q is an approximant of α.

Now, for every approximant p_n/q_n of θ, note $mp_n/q_n = p'/q'$ with $q' = q_n/(m\wedge q_n)$. Either p'/q' is itself an approximant of $m\theta$ in which case if we note p''/q'' the next approximant of $m\theta$, then

$$\frac{1}{2q'q''} < \left|m\theta - \frac{p'}{q'}\right| = m\left|\theta - \frac{p_n}{q_n}\right| < \frac{m}{q_nq_{n+1}}$$

whence

$$q'' > \frac{q_{n+1}q_n}{2mq'} > \frac{q_{n+1}}{2m}$$

and thus

$$\frac{\log q''}{q'} \ge \frac{\log q_{n+1}}{q'} - \frac{\log 2m}{q'} \ge \frac{\log q_{n+1}}{q_n} - \frac{\log 2m}{q'}.$$

[9] since $\theta_N > p/q$, n is even

Or $mp_n/q_n = p'/q'$ is not an approximant of $m\theta$, which means that

$$\left| m\theta - \frac{p'}{q'} \right| \geq \frac{1}{2q'^2},$$

and thus

$$\frac{1}{q_n q_{n+1}} \geq \left| \theta - \frac{p_n}{q_n} \right| \geq \frac{1}{2mq'^2}$$

whence

$$q_{n+1} \leq \frac{2mq'^2}{q_n} \leq 2mq_n$$

and thus

$$\frac{\log q_{n+1}}{q_n} \leq \frac{\log q_n}{q_n} + \frac{\log 2m}{q_n}.$$

Finally,

$$\begin{aligned} B(\theta) &= \sum_{\text{case 1}} \frac{\log q_{n+1}}{q_n} + \sum_{\text{case 2}} \frac{\log q_{n+1}}{q_n} \\ &\leq \sum \frac{\log q''}{q'} + \log(2m) \sum \frac{1}{q'} + {\sum}' \frac{\log F_n}{F_n} + \log(2m) \sum \frac{1}{F_n}. \end{aligned}$$

The prime in the sum means the summand needs to be replaced by the smallest non increasing sequence greater or equal to the sequence $\log F_n/F_n$. For different values of n, the approximants p'/q' of $m\theta$ are different since $p'/q' = mp_n/q_n$ and thus

$$B(\theta) \leq B(m\theta) + \log(2m) \sum \frac{1}{F_n} + {\sum}' \frac{\log F_n}{F_n} + \log(2m) \sum \frac{1}{F_n}$$

Since F_n is exponentially increasing, the sums (independent of θ) they are involved in are finite. □

Appendix B. Remarks

This section does not claim to bring new results. It is just a discussion of probably known and hopefully useful facts.

First, remember that for a germ $f(z) = \mathrm{e}^{i2\pi\theta}\, z + \mathcal{O}(z^2)$, the radius of convergence $R(f)$ of its linearizing formal power series $\phi_f \in \mathbb{C}[[Z]]$ is not necessarily equal to the conformal radius of the maximal linearization domain $\Delta(f)$ of f. An obvious possibility would be that f has an extension to a bigger domain, which has a bigger maximal linearization domain. But this is not the only thing that can happen, since ϕ_f is not necessarily injective on its disk of convergence. In fact ϕ_f can be *any convergent power series* of the form $z + \mathcal{O}(z^2)$. Indeed, for such a ϕ, we can set $f(z) = \phi\big(\mathrm{e}^{i2\pi\theta}\, \phi^{-1}(z)\big)$ near 0...

For instance, $\phi(z) = \mathrm{e}^z - 1 = z + \dots$ has infinite radius of convergence, and is not injective on $\mathbb{C}$. The map ϕ can also have critical points.

B.1. **Subharmonicity.** If θ is a Brjuno number, then for all analytic family $f_a(z) = \mathrm{e}^{i2\pi\theta}\, z + \mathcal{O}(z^2)$ of analytic germs, the function $a \mapsto -\log R(f_a)$ is the lim sup of subharmonic functions:

$$u : a \mapsto -\log R(f_a) = \limsup_{n \to +\infty} \frac{1}{n} \log |b_n(a)|.$$

By Yoccoz's theorem C, these functions are locally uniformly bounded above. Therefore, by Fatou's lemma, the function $a \mapsto -\log R(f_a)$ is (everywhere) below its average on circles. But we can say more: by the Brelot-Cartan theorem (see [Ra]), if we note $u^*(a) = \limsup_{a' \to a} u(a')$, then u^* is subharmonic and $u = u^*$ except on a polar set.

We however cannot say that u itself is subharmonic (iff $u = u^*$) because it is not necessarily upper semicontinuous, as the following counterexample shows. Let $f_0 = \mathrm{e}^{i2\pi\theta}\, z + \mathcal{O}(z^2)$ be the restriction to $\mathbb{D}$ of a map $\widetilde{f}$ defined on an open set Ω containing $\overline{\mathbb{D}}$, and such that its Siegel disk $\Delta(\widetilde{f})$ in Ω goes over the edge of $\mathbb{D}$. For instance $\widetilde{f}(z) = \mathrm{e}^{i2\pi\theta}\, z$ on $\Omega = \mathbb{C}$. Let $f_a = f_0 + az^2 g(z)$ for $a \in \mathbb{C}$, where $g(z)$ is any analytic function on $\mathbb{D}$ that is singular on all of $\partial\mathbb{D}$. Then for $a \neq 0$, the linearizing map ϕ_{f_a} must map its disk of convergence in $\mathbb{D}$ (as in [Y]). Also, and as a corollary, it is injective on its disk of convergence and maps it to the Siegel disk of f_a. This implies that

$$\limsup_{\substack{a \to 0 \\ \neq}} R(f_a) \leq \mathrm{rad}(U) < R(f_0)$$

where U is the biggest f_0-invariant subdisk of $\Delta(\widetilde{f})$ that is contained in $\mathbb{D}$, and $\mathrm{rad}(U)$ is its conformal radius with respect to 0.

Now, the upper semicontinuity holds if, instead of considering $a \mapsto -\log R(f_a)$ we consider $a \mapsto -\log \mathrm{rad}(\Delta(f_a\big|_{\mathbb{D}}))$ and if $f_a \longrightarrow f_0$ for the compact open topology on $\mathbb{D}$. This is a corollary of the work of Risler [Ri]. Lower semicontinuity also holds, this time for an elementary reason: if $\mathrm{rad}\,\Delta(f_{a_n})$ tends to some real r, the conformal maps $\phi_n : \mathbb{D} \to \Delta(f_{a_n})$ sending 0 to 0 with derivative $\mathrm{rad}\,\Delta(f_{a_n})$ make a normal family. They are also known to linearize f_{a_n}. Therefore, any limit of the ϕ_n must linearize f_a. Thus $\mathrm{rad}\,\Delta(f_a) \geq r$. Analytic dependence on the parameter is not needed.

Proposition 3. *Given $\theta \in \mathcal{B}$, let $H_\theta(\mathbb{D})$ be the set of analytic functions $f : \mathbb{D} \to \mathbb{C}$ fixing 0 with multiplier $\mathrm{e}^{i2\pi\theta}$, equipped with the compact open topology.*[10]

The map $\begin{pmatrix} H_\theta(\mathbb{D}) & \to &]0,1] \\ f & \mapsto & \mathrm{rad}\,\Delta(f) \end{pmatrix}$ *is continuous.*

Proposition 4. *If U is a one complex dimensional parameter space and*

$$(a, z) \in U \times \mathbb{D} \mapsto f_a(z) = \mathrm{e}^{i2\pi\theta}\, z + \mathcal{O}(z^2)$$

is analytic, then the map

$$a \mapsto -\log \mathrm{rad}\,\Delta(f_a)$$

is continuous and subharmonic.

Proof. We already mentioned the continuity.
Now, here is a trick[11] that yields subharmonicity with little effort: consider a function g as in the discussion above, i.e. holomorphic on $\mathbb{D}$ and with singularities at all points of $\partial\mathbb{D}$. Consider the sequence of families

$$(a, z) \mapsto \tau_n^{-1} f_a(\tau_n z) + \frac{1}{n} z^2 g(z) \quad \text{with} \quad \tau_n = 1 - \frac{1}{n}.$$

They all satisfy $-\log R = -\log \mathrm{rad}\,\Delta$ (as in [Y]), whence all these are (continuous) subharmonic functions of a. By the previous proposition, these functions tend (locally uniformly) to $-\log \mathrm{rad}\,\Delta(f_a)$.

B.2. **Holomorphic motions.**

Proposition 5. *Let (U_a) be simply connected open subsets of $\mathbb{C}$ whose boundaries move holomorphically with respect to a. Let c_a be a holomorphically varying point in U_a and $r(a)$ be the conformal radius of U_a with respect to c_a. Then, $a \mapsto -\log r(a)$ is a subharmonic function.*

[10] uniform convergence on compact subsets of $\mathbb{D}$

[11] It would be nice to have a more satisfactory (no power series) proof. Also, it could be true that subharmonicity still holds if the domain of definition of f undergoes a holomorphic motion.

Proof. Let V_a be the image of U_a by the inversion $z \mapsto 1/(z - c_a)$. The set V_a is unbounded and undergoes a holomorphic motion of its boundary. The conformal radius of U_a is the inverse of the capacity radius of $\mathbb{C} \setminus V_a$, which is itself expressable by an energy minimization as follows:[12]

$$-\log r(a) = \log \text{capacity radius} = -\inf_{\mu} E(\mu)$$

where μ varies in the set of probability measures on ∂V_a and $E(\mu)$ (the energy) is defined by

$$E(\mu) = \iint_{\partial V_a \times \partial V_a} -\log \big|u - v\big| d\mu(u) d\mu(v)$$

where the integrand is understood to be $+\infty$ when $u = v$. Choose a basepoint a_0 and let $\xi_a(z) : \partial V_{a_0} \to \partial V_a$ be the holomorphic motion. Then, for all probability measure μ on ∂V_{a_0},

$$E\big((\xi_a)_* \mu\big) = \iint_{\partial V_{a_0} \times \partial V_{a_0}} -\log \big|\xi_a(u) - \xi_a(v)\big| d\mu(u) d\mu(v).$$

This is a harmonic function of a. Now, the supremum $\sup -E = -\inf E$ in the energetic definition of $-\log r(a)$ yields a subharmonic function.

B.3. Harmonicity.

Proposition 6. *If $f_a : U_a \to \mathbb{C}$ is an analytic family of maps of the form $f(z) = \mathrm{e}^{i2\pi\theta} z + \mathcal{O}(z^2)$ and if the boundary of the Siegel disk $\Delta(f_a)$ undergoes a holomorphic motion (we do not require $\Delta(f_a) \Subset U_a$), then the function $a \mapsto -\log \operatorname{rad} \Delta(f_a)$ is harmonic.*

Proof. Same as in the second proof of proposition 1 (courtesy of S. Zakeri). □

This is kind of surprising: let A denote the fact that a simply connected domain undergoes a holomorphic motion (of its boundary), and B denote the fact that this domain is a Siegel disk of an analytically varying family of analytic maps (with fixed rotation number) in $\mathbb{D}$. Then

$$A \implies -\log \operatorname{rad} \text{ is subharmonic},$$
$$B \implies -\log \operatorname{rad} \text{ is subharmonic},$$
$$(A \text{ and } B) \implies -\log \operatorname{rad} \text{ is harmonic}\ldots$$

Is it fair that when a number has two reasons to be negative, then it is null?

B.4. Other radii of interest.

We have

$$R(f) = \text{ the radius of convergence of } \phi_f$$

and

$$\operatorname{rad} \Delta(f) = \text{ the biggest radius } \leq R \text{ below which } \phi_f \text{ maps in } \Delta(f).$$

Here are a few other "natural" radii that one could study

A = the biggest radius $\leq R$ on which ϕ_f is injective,

B = the biggest radius $\leq R$ on which ϕ_f has no critical point,

C = the biggest radius $\geq R$ on which ϕ_f has a meromorphic extension $\widetilde{\phi}_f$,

D = the biggest radius $\leq C$ on which $\widetilde{\phi}_f$ is injective,

E = the biggest radius $\leq C$ on which $\widetilde{\phi}_f$ has no critical point.

[12] As a variant of this, one could instead use the transfinite diameter.

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