

# QUESTIONS ABOUT POLYNOMIAL MATINGS

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ABSTRACT. We survey known results about polynomial mating, and pose some open problems.

The operation of *polynomial mating* is a procedure for producing a dynamical system  $F : \Sigma \rightarrow \Sigma$  on a topological 2-sphere from a pair of complex polynomial dynamical systems  $P : \mathbb{C} \rightarrow \mathbb{C}$  and  $Q : \mathbb{C} \rightarrow \mathbb{C}$ . The concept was introduced by Douady and Hubbard after computer investigations by Hubbard in 1982 showed that the Julia sets of certain quadratic rational maps appeared to be closely related to the Julia sets of certain pairs of quadratic polynomials [Hu2]. A picture of the Julia set for the mating of the basilica and rabbit quadratic polynomials now adorns the cover pages of the widely circulated IMS Stony Brook preprints. In this article, we give precise definitions, discuss foundational issues, survey known results, pose numerous specific open problems, and conclude with a (hopefully) comprehensive bibliography to date.

We focus on the case of mating two polynomials. We remark, however, that there are similar operations in two other closely related areas. A pair of Fuchsian groups can be mated via Bers' simultaneous uniformization theorem [Be]. Investigations of the limiting behavior of such matings feature prominently in the work culminating in the resolution of the Ending Lamination Conjecture for surface groups; see [Mi1] for a gentle but non-up-to-date introduction to the subject, [BCM] for the recent full resolution in the case of surface groups, and [Hu1] in this volume for more details. There is also a related notion for mating a polynomial with a Fuchsian group, surveyed in [Bu].

## 1. NOTATION AND DEFINITIONS

### 1.1. Sets.

- $\mathbb{R}$  is the real line,  $\mathbb{R}^+ = (0, \infty)$ ,  $\mathbb{R}^- = (-\infty, 0)$ .
- $\mathbb{C}$  is the complex plane,  $\mathbb{D} := \{z \in \mathbb{C} \mid 1 > |z|\}$ ,  $\mathbb{U} := \{z \in \mathbb{C} \mid 1 = |z|\}$ ,
- $\mathbb{S}$  is the unit sphere in  $\mathbb{C} \times \mathbb{R} \approx \mathbb{R}^3$ ,
- the upper hemisphere of  $\mathbb{S}$  is  $\mathbb{H}^+ := \mathbb{S} \cap (\mathbb{C} \times \mathbb{R}^+)$ , the lower hemisphere of  $\mathbb{S}$  is  $\mathbb{H}^- := \mathbb{S} \cap (\mathbb{C} \times \mathbb{R}^-)$ , the equator of  $\mathbb{S}$  is  $\mathbb{S} \cap (\mathbb{C} \times \{0\})$ .

1.2. **Thurston maps.** Let  $\Sigma$  be an oriented topological 2-sphere. An orientation-preserving ramified self-covering map  $f : \Sigma \rightarrow \Sigma$  of degree  $d \geq 2$  is a *Thurston map*

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The research of the first author was supported in part by the IUF, the ICERM and the Clay Mathematics Institute.

The research of the third author was supported in part by the NSF.

if it is *postcritically finite*, that is, all the points in the critical set  $\mathcal{C}_f$  have finite forward orbits, or equivalently, the *postcritical set*  $\mathcal{P}_f$  is finite, where

$$\mathcal{P}_f := \bigcup_{n \geq 1} f^{\circ n}(\mathcal{C}_f).$$

Thurston [DH] gives necessary and sufficient homotopy-theoretic combinatorial conditions for a Thurston map  $f$  to be conjugate-up-to-isotopy to a rational map. If this condition fails, we say that  $f$  is *obstructed*.

**1.3. Rational maps.** Let  $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a rational map of degree  $d \geq 2$ . The *Julia set*  $\mathcal{J}_F$  is the closure of the set of repelling cycles. The *Fatou set*  $\mathcal{F}_F$  is the complement of the Julia set. The *forward orbit* of a point  $z \in \mathbb{P}^1$  is

$$\mathcal{O}_F(z) := \{F^{\circ n}(z) \mid n \geq 1\}.$$

The rational map is

- *hyperbolic* if the orbit of every critical point is attracted by some attracting cycle;
- *subhyperbolic* if all critical points are either preperiodic or converge to attracting cycles;
- *geometrically finite* if the closure of its postcritical set meets the Julia set in finitely many points;
- *semihyperbolic* if it has neither parabolic points nor recurrent critical points.

**1.4. Polynomials.** Let  $P : \mathbb{C} \rightarrow \mathbb{C}$  be a monic polynomial of degree  $d \geq 2$ . The *filled-in Julia set* is

$$\mathcal{K}_P := \{z \in \mathbb{C} \mid n \mapsto P^{\circ n}(z) \text{ is bounded}\}.$$

The Julia set  $\mathcal{J}_P$  is the boundary of  $\mathcal{K}_P$ . The *Green's function*  $h_P : \mathbb{C} \rightarrow [0, +\infty)$  is defined by

$$h_P(z) := \lim_{n \rightarrow +\infty} \frac{1}{d^n} \max(0, \log |P^{\circ n}(z)|).$$

When  $\mathcal{K}_P$  is connected, we denote by  $\text{böt}_P : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - \mathcal{K}_P$  the (unique) *Böttcher coordinate* which conjugates  $z \mapsto z^d$  to  $P$  and which is tangent to the identity at infinity. The *external ray* of angle  $\theta \in \mathbb{R}/\mathbb{Z}$  is

$$\mathcal{R}_P(\theta) := \{\text{böt}_P(re^{2\pi i\theta}) \text{ with } r > 1\}.$$

If  $\mathcal{K}_P$  is locally connected, the Böttcher coordinate extends to a continuous map  $\text{böt}_P : \mathbb{C} - \mathring{\mathbb{D}} \rightarrow \mathbb{C} - \mathring{\mathcal{K}}_P$  and we denote by  $\gamma_P : \mathbb{R}/\mathbb{Z} \rightarrow \mathcal{J}_P$  the *Carathéodory loop* defined as

$$\gamma_P(\theta) := \text{böt}_P(e^{2\pi i\theta}).$$

If  $\mathcal{K}_P$  is not locally connected, we may still define  $\gamma_P : \mathbb{Q}/\mathbb{Z} \rightarrow \mathcal{J}_P$  by

$$\gamma_P(\theta) := \lim_{r \rightarrow 1^+} \text{böt}_P(re^{2\pi i\theta}).$$

**1.5. Formal mating.** Let  $P : \mathbb{C} \rightarrow \mathbb{C}$  and  $Q : \mathbb{C} \rightarrow \mathbb{C}$  be two monic polynomials of the same degree  $d \geq 2$ . The *formal mating* of  $P$  and  $Q$  is the ramified covering  $f = P \uplus Q : \mathbb{S} \rightarrow \mathbb{S}$  obtained as follows.

We identify the dynamical plane of  $P$  to the upper hemisphere  $\mathbb{H}^+$  of  $\mathbb{S}$  and the dynamical plane of  $Q$  to the lower hemisphere  $\mathbb{H}^-$  of  $\mathbb{S}$  via the gnomonic projections:

$$\nu_P : \mathbb{C} \rightarrow \mathbb{H}^+ \quad \text{and} \quad \nu_Q : \mathbb{C} \rightarrow \mathbb{H}^-$$

given by

$$\nu_P(z) = \frac{(z, 1)}{\|(z, 1)\|} = \frac{(z, 1)}{\sqrt{|z|^2 + 1}} \quad \text{and} \quad \nu_Q(z) = \frac{(\bar{z}, -1)}{\|(\bar{z}, -1)\|} = \frac{(\bar{z}, -1)}{\sqrt{|z|^2 + 1}}.$$

The map  $\nu_P \circ P \circ \nu_P^{-1}$  defined on the upper hemisphere and  $\nu_Q \circ Q \circ \nu_Q^{-1}$  defined in the lower hemisphere extend continuously to the equator of  $\mathbb{S}$  by

$$(e^{2i\pi\theta}, 0) \mapsto (e^{2i\pi d\theta}, 0).$$

The two maps fit together so as to yield a ramified covering map

$$P \uplus Q : \mathbb{S} \rightarrow \mathbb{S}$$

which is called the *formal mating* of  $P$  and  $Q$ .

**1.6. Hausdorff and topological mating.** Let us now consider the smallest equivalence relation  $\sim_{\text{ray}}$  on  $\mathbb{S}$  such that for all  $\theta \in \mathbb{R}/\mathbb{Z}$ ,

- points in the closure of  $\nu_P(\mathcal{R}_P(\theta))$  are in the same equivalence class, and
- points in the closure of  $\nu_Q(\mathcal{R}_Q(\theta))$  are in the same equivalence class.

The equivalence relation  $\sim_{\text{ray}}$  is *closed* if

$$\{(x, y) \in \mathbb{S} \times \mathbb{S} \mid x \sim_{\text{ray}} y\}$$

forms a closed subset of  $\mathbb{S} \times \mathbb{S}$ . This is the case if and only if the quotient space  $S_{P \perp Q}$  is Hausdorff. In this case, we say that  $P, Q$  are *Hausdorff mateable*.

Assume further now that each equivalence class is closed and connected but is not the entire sphere. In this situation, a theorem of Moore asserts that  $S_{P \perp Q}$  is a topological 2-sphere if and only if no equivalence class separates  $\mathbb{S}$  in two or more connected components. Further, when  $S_{P \perp Q}$  is a topological sphere, the quotient map  $\mathbb{S} \rightarrow S_{P \perp Q}$  induces isomorphisms of homology, and hence imposes a preferred orientation on  $S_{P \perp Q}$ . We say the polynomials  $P$  and  $Q$  are *topologically mateable* if  $S_{P \perp Q}$  is a topological 2-sphere. In that case, the formal mating  $P \uplus Q$  induces a map

$$P \perp Q : S_{P \perp Q} \rightarrow S_{P \perp Q}$$

which is called the *topological mating* of  $P$  and  $Q$ . It can be shown that this map is a ramified self-cover of  $S_{P \perp Q}$ .

**1.7. Geometric mating.** Assume  $P$  and  $Q$  are topologically mateable. Denote by  $\mathcal{F}_{P \perp Q}$  the image of  $\overset{\circ}{\mathcal{K}}_P \cup \overset{\circ}{\mathcal{K}}_Q$  in  $S_{P \perp Q}$ . When this open set is nonempty, it carries a preferred complex structure induced by that on  $\overset{\circ}{\mathcal{K}}_P \cup \overset{\circ}{\mathcal{K}}_Q$ .

A geometric mating is a quadruple  $(P, Q, \phi, F)$  such that the following diagram commutes:

$$\begin{array}{ccc} S_{P \perp Q} & \xrightarrow{\phi} & \mathbb{P}^1 \\ P \perp Q \downarrow & & \downarrow F \\ S_{P \perp Q} & \xrightarrow{\phi} & \mathbb{P}^1. \end{array}$$

where

- $P : \mathbb{C} \rightarrow \mathbb{C}$  and  $Q : \mathbb{C} \rightarrow \mathbb{C}$  are monic polynomials of same degree  $d \geq 2$ ,
- $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a rational map of degree  $d \geq 2$  and
- $\phi : S_{P \perp Q} \rightarrow \mathbb{P}^1$  is an orientation-preserving homeomorphism which is conformal in  $\mathcal{F}_{P \perp Q}$ .

We shall say that  $F$  is a geometric mating of  $P$  and  $Q$  and shall use the notation  $F \cong P \perp Q$ .

## 2. FUNDAMENTAL QUESTIONS (PILGRIM-TAN LEI)

Let  $\mathbf{Poly}_d$ ,  $\mathbf{rat}_d$ ,  $\mathbf{top}_d$  denote the spaces of degree  $d$  monic polynomials, Möbius conjugacy classes of rational maps, and topological conjugacy classes of branched coverings of the sphere to itself, respectively; thus  $\mathbf{rat}_d \subset \mathbf{top}_d$ .

Mating may be viewed as a partially defined map

$$\perp : \mathbf{Poly}_d \times \mathbf{Poly}_d \dashrightarrow \mathbf{top}_d.$$

It is natural to formulate several questions.

- (1) **(Domains)** Determine when the pair  $(P, Q)$  is (a) Hausdorff, (b) topologically, (c) geometrically mateable. Identify obstructions to being mateable in each sense.
- (2) **(Images)** Given  $F$ , determine when  $F$  is the geometric mating of two polynomials. Identify obstructions to being a geometric, topological, or Hausdorff mating.
- (3) **(Fibers)** Given  $F$ , determine those  $(P, Q)$  such that  $(P, Q, \phi, F)$  is a geometric mating for some  $\phi$ .
- (4) **(Multiplicities)** Given  $(P, Q, F)$ , determine those  $\phi$  for which  $(P, Q, \phi, F)$  is a geometric mating.
- (5) **(Continuity)** Investigate the extent to which the convergence of  $(P_n, Q_n)$  to  $(P, Q)$  implies the convergence of  $P_n \perp Q_n$  to  $P \perp Q$ .

In the following sections, we discuss the interesting facts that mating, viewed as a map, is neither well-defined, surjective, injective, or continuous. We also pose numerous open problems.

**2.1. Domains.** On the one hand, flexible Lattès examples provide counterexamples to seemingly natural statements. The discussion in this paragraph is taken from [Mi, Section B-5]. The external rays of the Mandelbrot set of angles  $1/12$  and  $5/12$  land at parameter values  $f_{1/12}$  and  $f_{5/12}$ . Taking

$$P = f_{1/12}, \quad Q = f_{5/12} \quad \text{and} \quad F(z) = -\frac{1}{2} \left( z + \frac{1}{z} \right) + \sqrt{2},$$

it turns out that  $F \cong P \perp Q$ , and hence that  $G := F \circ F \cong P \circ P \perp Q \circ Q$ . However,  $G$  is a flexible Lattès example. It follows that there are uncountably many pairwise

holomorphically nonconjugate maps  $G_t$  such that  $G_t \cong P \circ P \amalg Q \circ Q$ , and hence that mating, viewed as a map, is not well-defined.

On the other hand, mating often behaves well. We organize our discussion along vaguely historical lines, treating cases in which the dynamical regularity is successively relaxed. We consider

- (1)  $P, Q$  both postcritically finite
- (2)  $P, Q$  both subhyperbolic
- (3)  $P, Q$  both geometrically finite
- (4) one or both of  $P, Q$  is geometrically infinite.

2.1.1. *Postcritically finite case.* The formal mating  $f := P \uplus Q$  is then a Thurston map. It may occur that it has obstructions that are *removable*; in this case, one considers a modified Thurston map  $f'$ . Other obstructions we call *nonremovable*.

By using Thurston's combinatorial characterization of rational functions, and ideas of M. Rees, Tan Lei [T2] and Shishikura [Shi] showed the following result.

**Theorem 1.** *Suppose  $d = 2$ , and suppose  $P(z) = z^2 + c_P$ ,  $Q(z) = z^2 + c_Q$  are postcritically finite. Then the following are equivalent.*

- (1)  $(P, Q)$  are geometrically mateable
- (2)  $(P, Q)$  are topologically mateable
- (3)  $c_P, c_Q$  do not lie in conjugate limbs of the Mandelbrot set.

That (2)  $\Rightarrow$  (3) is easy to see, since if (3) fails then there is a periodic ray-equivalence class in the formal mating that separates the sphere. However, in degree 3, Shishikura and Tan [ST] gave an explicit example of a pair  $(P, Q)$  of postcritically finite, hyperbolic polynomials which is topologically but not geometrically mateable (see also [C]).

2.1.2. *Subhyperbolic case.* Given a polynomial  $P_0$ , let  $\mathcal{QC}(P_0) \subset \mathbf{Poly}_d$  be the space of polynomials  $P$  for which  $P_0$  and  $P$  are quasiconformally conjugate on a neighborhood of their Julia sets. If  $P_0$  is e.g. hyperbolic,  $\mathcal{QC}(P_0)$  is the hyperbolic component containing  $P_0$ , and  $\mathcal{QC}(P_0)$  admits a smooth model as a family of Blaschke products. If now  $P_0$  and  $Q_0$  are e.g. two hyperbolic polynomials that yield a rational map  $F$  as their geometric mating, the hyperbolic component of  $F$  in  $\mathbf{rat}_d$  is essentially the product  $\mathcal{QC}(P_0) \times \mathcal{QC}(Q_0)$ . It follows that in this case mating can be extended in a reasonable way over the pairs of (sub)hyperbolic parameters corresponding to points in  $\mathcal{QC}(P_0) \times \mathcal{QC}(Q_0)$ . For example, taking  $P_0 = Q_0 = z^2$ , the spaces  $\mathcal{QC}(P_0), \mathcal{QC}(Q_0)$  are disks parameterized by the respective eigenvalues  $\lambda, \mu$  of the unique attracting fixed points, and the corresponding mating is, up to conjugacy,

$$F_{\lambda, \mu}(z) := \frac{z(z + \lambda)}{\mu z + 1}$$

which has a fixed point with multiplier  $\lambda$  at 0 and a fixed point with multiplier  $\mu$  at  $\infty$ .

2.1.3. *Geometrically finite case.* By letting  $(P, Q)$  tend to certain geometrically finite parameters on the boundary of  $\mathcal{QC}(P_0) \times \mathcal{QC}(Q_0)$ , results of Haïssinsky and Tan [HT] show that e.g. mating extends to quadratic polynomials with parabolic cycles; their results hold actually for a slightly wider class of maps.

2.1.4. *Geometrically infinite case.* Yampolsky and Zakeri [YZ] considered quadratic polynomials with Siegel disks. They showed that if

$$P_\lambda(z) := \lambda z + z^2 \quad \text{and} \quad Q_\mu(z) := \mu z + z^2$$

with  $\lambda = e^{2\pi i\alpha}$ ,  $\mu = e^{2\pi i\beta}$ ,  $\beta \neq -\alpha$  and  $\alpha, \beta$  irrational numbers which satisfy the Diophantine condition of bounded type, then  $P_\lambda$  and  $Q_\mu$  are geometrically mateable with  $F_{\lambda, \mu} \cong P_\lambda \perp\!\!\!\perp Q_\mu$ . Zhang [Z] expanded this result to the case where  $\alpha$  and  $\beta$  are of Petersen-Zakeri type ( $\log(a_n) \in \mathcal{O}(\sqrt{n})$  with  $a_n$  the entries of the continued fraction); see Section 5 for related questions.

Blé and Valdez [BV] showed that for  $\theta$  irrational of bounded type and  $c$  a real parameter depending in a certain way on  $\theta$ , the pair  $(z^2 + c, e^{2\pi i\theta} z + z^2)$  is geometrically mateable.

A *Yoccoz polynomial* is one that has a connected Julia set, all cycles repelling, and is not infinitely renormalizable. Luo [L] observed that the pair  $(z^2 - 1, z^2 + c)$  should be geometrically mateable whenever  $z^2 + c$  is Yoccoz and  $c$  does not lie in the 1/2-limb of the Mandelbrot set. Aspenberg and Yampolsky [AY] implemented Luo's program.

If one or more of  $\mathcal{K}_P$  and  $\mathcal{K}_Q$  fail to be locally connected, it is difficult to give a topological model for  $P \perp\!\!\!\perp Q$ . Dudko [D] gives an interpretation of matings of the form  $(z^2 - 1) \perp\!\!\!\perp (z^2 + c)$  for a large class of parameters  $c$  without assuming local connectivity of the corresponding Julia set.

In the next section, we consider topological questions lying at the foundation of mating.

2.2. **Images.** Not every map arises as a mating. The situation is best understood when  $F$  is hyperbolic and postcritically finite. In this case the following is known.

- (1)  $F$  is a mating of two (postcritically finite, hyperbolic) polynomials  $P$  and  $Q$  if and only if  $F$  has an *equator*. This is an oriented Jordan curve  $\gamma \subset \mathbb{P}^1 - \mathcal{P}_F$  such that  $F^{-1}(\gamma)$  is connected and, as an oriented curve, is homotopic in  $\mathbb{P}^1 - \mathcal{P}_F$  to  $\gamma$ .
- (2) The (bounded) Fatou components of  $P$  and  $Q$  survive under matings. If we color these components of  $P$  say white, and the ones of  $Q$  black, then the mating  $P \perp\!\!\!\perp Q$  maps each Fatou component to one of the same color. Since in the hyperbolic case each critical point is contained in a Fatou component, it follows that a necessary condition for  $F$  to be a mating is as follows. We must be able to color half of the critical points of  $F$  white, the other half black (counting multiplicities), such that the orbits of the white critical points is disjoint from the orbits of the black critical points. This condition is however not sufficient : the example considered by Tan Lei and Milnor in [MT] is, according to Wittner [W], not a mating.
- (3) Another necessary condition is as follows. There is a loop  $\gamma \subset \mathbb{P}^1 - \mathcal{P}_F$ , based at say  $b$ , such that for each  $n \in \mathbb{N}$ , the monodromy induced by  $\gamma$  on the set  $F^{-n}(b)$  is transitive. To see this, take  $\gamma$  to be an equator; see [K, Proposition 4.2]. This condition is again not sufficient.

It easily follows from (1) that a postcritically finite hyperbolic map with  $\#\mathcal{P}_F = 3$  is never a mating (unless it is conjugate to a polynomial, in which case it may be viewed as the mating of this polynomial with  $z^d$ ).

For non-hyperbolic maps, we have the following striking result from [Me3].

**Theorem 2** (Meyer). *Suppose  $F$  is a postcritically finite rational map without periodic critical points, so that  $\mathcal{J}_F = \mathbb{P}^1$ . Then every sufficiently high iterate  $F^{\circ N}$  is the geometric mating of two postcritically finite polynomials without periodic critical points.*

The image of the equator under the ray-equivalence quotient map from the formal mating to  $F^{\circ N}$  gives a semiconjugacy from  $z \mapsto z^{d^N}$  acting on the unit circle  $\mathbb{U}$  to  $F^{\circ N}$  acting on  $\mathbb{P}^1$ , i.e. an  $F^{\circ N}$ -invariant Peano curve; here  $d = \deg(F)$ . Moreover, the image is the limit of a Jordan curve passing through points of  $\mathcal{P}_F$  under a so-called pseudo-isotopy. The following related question appears in [K, Remark 3.9].

**Question 1.** *Is there a degree  $d$  critically finite rational map  $F$  such that (i) there is a semiconjugacy from  $z \mapsto z^d$  on  $\mathbb{U}$  to  $F : \mathcal{J}_F \rightarrow \mathcal{J}_F$ , but (ii) this semiconjugacy is not the limit of a pseudo-isotopy?*

Mashanova and Timorin [MT] identify arcs  $F_t$ ,  $t \in [0, 1]$  contained in the boundaries of hyperbolic components in the space of quadratic rational maps such that  $F_t$  is a continuous family of geometric matings.

In higher degrees, Tan Lei [T1] shows that within the set of rational maps arising from Newton's method applied to cubic polynomials, there is a large subset comprised of matings.

**2.3. Multiplicities.** Let  $(P, Q, \phi, F)$  be a geometric mating. If  $(P, Q, \psi, F)$  is another geometric mating, then  $\chi := \phi \circ \psi^{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a homeomorphism which is conformal on the Fatou set  $\mathcal{F}_F$  and commutes with  $F$ . Conversely, if  $\chi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is such a homeomorphism, then  $(P, Q, \chi \circ \phi, F)$  is a geometric mating. If  $F$  is a flexible Lattès example, the set of such  $\chi$  is countably infinite. If  $F$  is geometrically finite but not a flexible Lattès example, such a  $\chi$  is a Möbius transformation, hence there are at most  $\#\text{Aut}(F) \leq 3d$  such  $\chi$ .

If the polynomials  $P$  and  $Q$  are varied, the situation is a little more subtle. First, if  $\omega^{d-1} = 1$ , replacing  $(P, Q)$  by  $(P_\omega, Q_\omega)$  with

$$P_\omega(z) := \omega^{-1}P(\omega z) \quad \text{and} \quad Q_\omega(z) := \omega Q(\omega^{-1}z)$$

yields a canonical orientation-preserving homeomorphism (of order  $d - 1$ )

$$\chi_\omega : S_{P_\omega \perp Q_\omega} \rightarrow S_{P \perp Q}.$$

If they exist, the geometric matings  $(P, Q, \phi, F)$  and  $(P_\omega, Q_\omega, \phi \circ \chi_\omega, F)$  will be considered to be equivalent. Next, exchanging the roles of two topologically mateable polynomials  $P$  and  $Q$  yields a canonical orientation-preserving homeomorphism

$$\chi : S_{Q \perp P} \rightarrow S_{P \perp Q}.$$

If they exist, the geometric matings  $(P, Q, \phi, F)$  and  $(Q, P, \phi \circ \chi, F)$  will be considered to be equivalent. A rational map  $F$  is a *shared mating* if there are at least two inequivalent geometric matings  $(P_1, Q_1, \phi_1, F)$  and  $(P_2, Q_2, \phi_2, F)$ .

The existence of shared matings was first observed by Wittner [W]. However, it is becoming clear that shared matings are quite common (see [R4], [Me2], [Sha2]).

The *mating number* of a rational map  $F$  is the number of equivalence classes of geometric matings  $(P, Q, \phi, F)$ . If  $F$  is critically finite and hyperbolic, its mating number is, by the above observations, at most  $3d \cdot N^+ \cdot N^- \cdot (d-1)$ , where  $d = \deg(F)$  and  $N^\pm$  are the number of affine conjugacy classes of critically finite hyperbolic

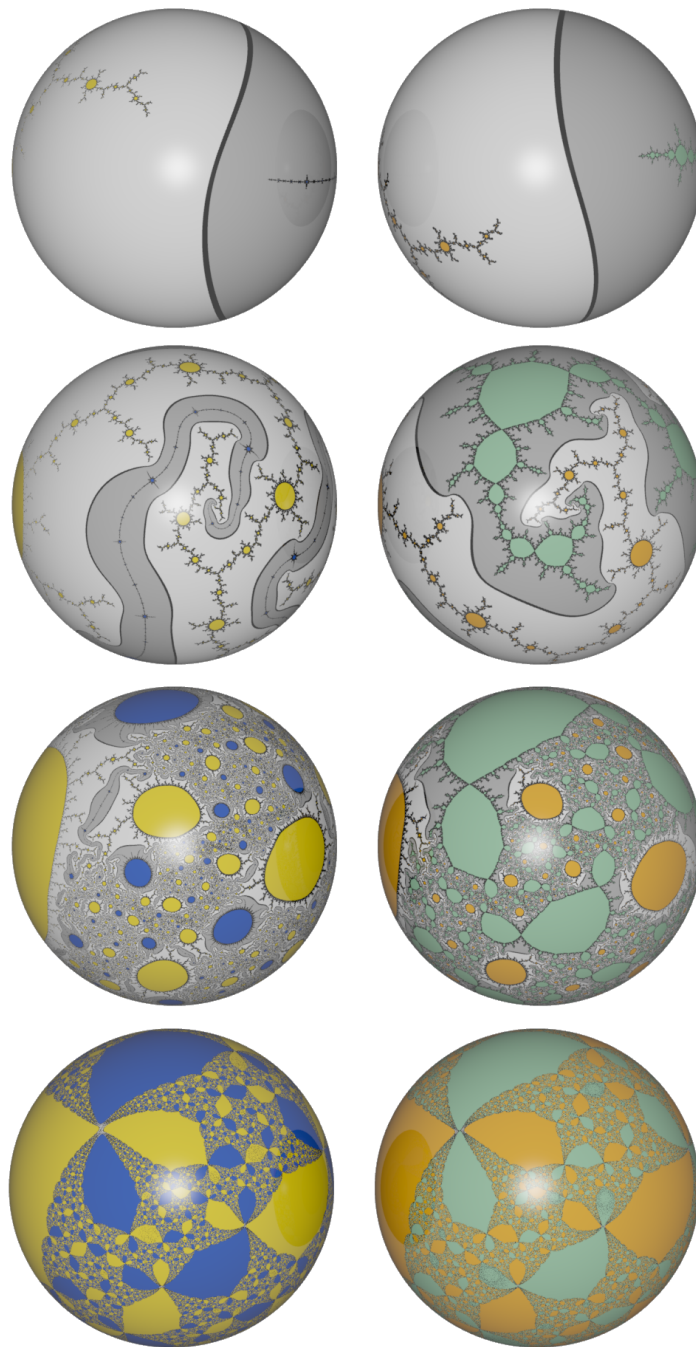


FIGURE 1. A shared mating of two quadratic polynomials having critical points of period 4 (pictures by I. Zidane). The rational map has a period 2 cluster. It can be seen as a geometric mating in 4 distinct ways. Here, we see two of them. Left: mating the left-most real quadratic polynomial of period 4 with the co-kokopelli. Right: mating the bi-basilica with the kokopelli.



polynomials with the appropriate dynamics on their critical orbits; the factor of  $(d - 1)$  accounts for the number of ways to glue together the dynamical planes of  $P$  and  $Q$ . The flexible Lattès examples have infinite mating number.

**Question 2.** *When is the mating number finite? When the mating number is finite, is there a bound in terms of the degree and the number of postcritical points?*

See Section 4 below for a more detailed discussion in the case of matings of quadratic polynomials.

**2.4. Continuity.** Mating is not continuous. If

$$P_\lambda(z) := \lambda z + z^2 \quad \text{and} \quad Q_\mu(z) := \mu z + z^2$$

with  $(\lambda, \mu) \in \mathbb{D}^2$  and  $\mu \neq \bar{\lambda}$ , then  $P_\lambda \perp\!\!\!\perp Q_\mu \cong F_{\lambda, \mu}$  where

$$F_{\lambda, \mu}(z) := \frac{z(z + \lambda)}{\mu z + 1}$$

fixes 0 with multiplier  $\lambda$  and  $\infty$  with multiplier  $\mu$ . Let  $\Re(s) > 1/2, t > 0$  and set

$$\lambda := \frac{1 - s + ti}{1 + ti} \quad \text{and} \quad \mu := \frac{1 - s - ti}{1 - ti}.$$

For fixed  $s$ , we have  $\lambda, \mu \rightarrow 1$  as  $t \rightarrow \infty$ . However, it can be shown that there exists a quadratic rational map  $F_s$  which has a fixed point of multiplier  $s/(s - 1/2)$  such that  $F_{\lambda, \mu} \rightarrow F_s$ . So the limit of the matings can depend on the manner of approach. This phenomenon is apparently widespread and can occur when mating families of polynomials parameterized by multipliers of attracting cycles of higher periods.

A. Epstein [Ep] has shown that other, much more subtle, types of discontinuities exist. However, in all known quadratic examples, the sequences  $(P_n, Q_n)$  leading to discontinuity have the property that both  $P_n$  and  $Q_n$  vary.

**Question 3.** *Can mating be discontinuous within a slice? That is, do there exist a polynomial  $P$  and polynomials  $Q, Q_n, n \in \mathbb{N}$  with  $Q_n$  converging to  $Q$  such that  $P \perp\!\!\!\perp Q_n$  converges to  $F \neq P \perp\!\!\!\perp Q$ ?*

At places where mating is not defined, limits of matings need not exist. Let  $P(z) = z^2 - 1$  and for  $Q_r(z) = rz + z^2$ . The pair  $(P, Q_{-1})$  is not topologically mateable; the period 2 external rays of angles  $1/3, 2/3$  glue together to form a ray equivalence class separating the sphere. So the point  $(P, Q_{-1})$  is not in the domain of mating. If  $-1 < r \leq 0$ , however, then

$$F_r(z) = (z + 1) \frac{z - (1 + r)}{rz + (1 + r)} \cong P \perp\!\!\!\perp Q_r.$$

A simple calculation shows that as  $r \searrow -1$ , the multiplier of  $F_r$  at a fixed-point tends to infinity, and so  $F_r \rightarrow \infty$  in  $\mathbf{rat}_2$ . An intuitive explanation is related to the nonexistence of the mating of  $P$  and  $Q_{-1}$ . For  $-1 < r \leq 0$ , there is a pair of arcs joining infinity to a common fixed-point of  $F_r$  separating the Julia set of  $F_r$ ; they are interchanged with rotation number  $1/2$ . As the multiplier  $r$  of  $F_r$  at infinity tends to  $-1$ , in order for a limit to exist, this pair of arcs must collapse to a point. See e.g. [T3], [P], and the references therein.

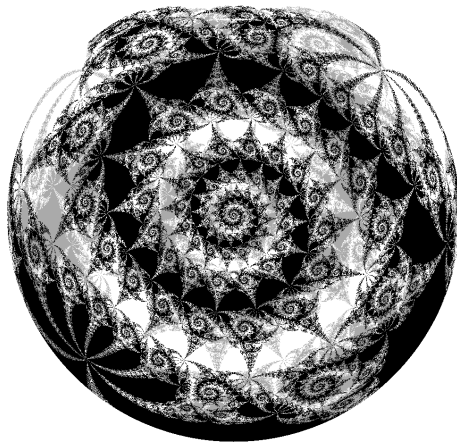


FIGURE 2. Mating of  $f_{1/255}$  with itself, found using the Medusa algorithm

**2.5. Computations.** Algebraic and numerical methods for finding solutions to the systems of polynomial equations associated to a geometric mating  $F$  will be effective only in the most simple cases; they will never, for example, locate the mating associated with two quadratic polynomials for which the critical points are periodic of period, say, eight.

In contrast, an implementation of Thurston's iterative algorithm is sometimes possible and can be used to find matings. Early versions were apparently used by Wittner and Shishikura. Hruska Boyd [Hr] and Bartholdi have carried this out as well (see Figure 2).

Another method, *slow mating*, gives a continuous interpolation of the discrete orbits in Thurston's algorithm (see Section 12 below).

### 3. TOPOLOGICAL MATINGS (EPSTEIN-MEYER)

**3.1. The structure of ray-equivalence classes.** The proof that postcritically finite quadratic polynomials are mateable if and only if they do not belong to conjugate limbs of the Mandelbrot set relies on Thurston's classification of postcritically finite branched covering of the sphere.

**Question 4.** *Is it possible to give a proof without appealing to Thurston's theorem, but rather by studying the ray-equivalence classes and showing they satisfy the hypothesis of Moore's theorem?*

Such an approach would require understanding the structure of ray-equivalence classes. If an equivalence class for  $\sim_{\text{ray}}$  contains a loop, the quotient space  $S_{P \perp Q}$  is not a 2-sphere. Thus we assume for now that no ray-equivalence class contains loops, whence each ray-equivalence class is a tree. There are examples where the first return to a periodic ray class acts by a combinatorial rotation which is not the identity. Sharland [Sha1] proved that if the mating of two hyperbolic polynomials is not obstructed, each periodic ray class contains at most one periodic branch point with non-zero combinatorial rotation number.

**Question 5.** *What do ray-equivalence classes look like? What do the periodic ones look like? How does the first return map to such a class act?*

If the ray-equivalence is not closed, we say that the mating is *Hausdorff-obstructed*.

**Question 6.** *Are there Hausdorff-obstructions to matings?*

Of course the above question can be asked in various settings. For example the answer (and/or the proof) might depend on whether the involved polynomials are postcritically finite or not. Furthermore the answer might depend on the degree. This seems unlikely, though a proof might be much easier in the quadratic case.

Let  $\simeq$  be the equivalence relation on  $\mathbb{R}/\mathbb{Z}$  induced by the Carathéodory loops  $\gamma_1, \gamma_2$ , that is, the smallest equivalence relation satisfying

$$s \simeq t \text{ if } \gamma_1(s) = \gamma_1(t) \text{ or } \gamma_2(s) = \gamma_2(t) \text{ or } \gamma_1(s) = \gamma_2(-t).$$

Note that  $\simeq$  is closed if and only if  $\sim_{\text{ray}}$  is closed.

**Question 7.** *If the equivalence classes of  $\simeq$  are uniformly bounded in size, must  $\simeq$  be closed?*

**3.2. Long ray connections.** Given an injective path contained in a ray-equivalence class, we may ask how often it crosses the equator. The *diameter* of a ray-equivalence class is the supremum over all such paths; *a priori* this quantity may be infinite. It is not difficult to show that if the ray-equivalence classes have uniformly bounded diameter then the ray-equivalence relation is closed. On the other hand, an infinite ray-equivalence class which is closed must separate, in which case the quotient space cannot be a 2-sphere (see [MP]).

**Question 8.** *Are there matings for which there are ray-equivalence classes with infinite diameter? Are there matings for which each ray-equivalence class has finite diameter, but such that the diameter of all equivalence classes is unbounded? If not, are there effective bounds on the diameter of the ray-equivalence classes?*

In [Me1, Theorem 6.1] it is shown that for all matings which arise in a certain way, the diameter of ray-equivalence classes is indeed uniformly bounded. The diameter of a rational ray-equivalence class is always finite: indeed, the rays landing at a given (pre)periodic point are rational numbers of the same denominator.

**Question 9.** *In matings of given degree, what is the maximum diameter for ray-equivalence classes of angles of given period? Is this quantity unbounded as the period tends to infinity?*

For a given angle  $\theta$ , consider the set  $\mathcal{X}_\theta$  of parameters  $c$  in the Mandelbrot set such that, for  $z \mapsto z^2 + c$ , the rays of angle  $\theta$  and some  $\vartheta \neq \theta$  land at the same point. For an odd-denominator rational  $\theta$ , the set  $\mathcal{X}_\theta$  is a nonempty finite union of

compact connected sets, the intersection of  $M$  with appropriate wakes associated to the various partners  $\vartheta$ . For even-denominator rational  $\theta$ , the set  $\mathcal{X}_\theta$  admits a similar, but somewhat more complicated description. Nothing seems to be known in the irrational case.

**Question 10.** *Is  $\mathcal{X}_\theta$  always nonempty? Can there be infinitely many components?*

**Question 11.** *Are there irrational angles which never participate in ray-equivalence classes of infinite diameter?*

**Question 12.** *Is there a combinatorially sensible place to expect to find long ray-chains? For example, given a ray-chain of some period, can we find longer ray-chains, of some higher period, which suitably shadows the connection? If we do this with enough care, can we extract an infinite limit?*

**3.3. Mating and tuning.** It was shown by Pilgrim that for hyperbolic quadratic (more generally, bicritical) rational maps with two attracting cycles, neither of which is fixed, the Fatou components are Jordan domains. One expects the intersection between any two Fatou component boundaries to be small - perhaps finite, perhaps involving specific internal angles - thereby leaving little opportunity for connections between the *small Julia sets* of tunings. Since mating suitably commutes with tuning when both operations are defined, this heuristic suggests that renormalizable maps are unlikely to furnish long(er) ray-chains.

**Question 13.** *For quadratic rational maps with two superattracting cycles, what are the possibilities for intersections of boundaries of immediate basins? Whatever the situation is, can the result be recovered for matings, using only ray combinatorics of polynomials? Can Pilgrim's result be recovered?*

**Question 14.** *For a hyperbolic quadratic polynomial, consider the Cantor set of rays landing on the boundary of the union of the immediate attracting basins. What can be said about the intersection of two such Cantor sets?*

#### 4. QUADRATIC RATIONAL MAPS (REES)

Let  $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a hyperbolic rational map of degree 2. Let  $\Omega_1$  and  $\Omega_2$  be the (possibly equal) connected components of the Fatou set  $\mathcal{F}_F$  which contain the critical points of  $F$ . According to [R2], the rational map is of

- type I if  $\Omega_1 = \Omega_2$ ; in that case  $\mathcal{F}_F = \Omega_1 = \Omega_2$  and  $\mathcal{J}_F$  is a Cantor set.
- type II if  $\Omega_1 \neq \Omega_2$  belong to the same periodic cycle of Fatou components.
- type III if  $\Omega_1$  belongs to a cycle of Fatou component and  $\Omega_2$  is strictly preperiodic to this cycle, or vice-versa.
- type IV if  $\Omega_1$  and  $\Omega_2$  belong to distinct cycles of Fatou components.

The notion of Wittner flips and captures arises in [W] and have been studied further by Rees [R3] and Exall [Ex]. The notion of period 1 and period 2 clusters arises in Sharland's thesis [Sha1].

**4.1. The overcount of matings.** Every geometric mating of critically periodic quadratic polynomials which is a rational map is the centre of a type IV hyperbolic component. The rational maps in such a hyperbolic component have two disjoint periodic orbits of attractive fixed points, of some periods  $m$  and  $n$ . It is possible to count the number of such hyperbolic components [KR]. It is also possible to

Ray Connection, Length 12

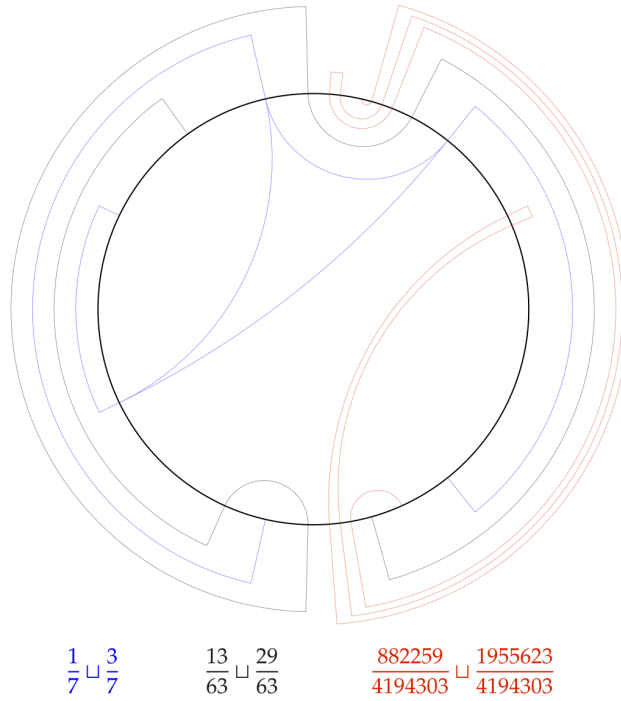


FIGURE 3. Example by Stuart Price. The largest known diameter of a ray-equivalence class for the mating of two postcritically finite quadratic polynomials has length 12: mating the rabbit polynomial with a real quadratic polynomial for which the critical point is periodic of period 22.

count the number of matings of critically periodic polynomials  $P(z) = z^2 + c_1$  and  $Q(z) = z^2 + c_2$  such that 0 is of period  $m$  under  $P$  and of period  $n$  under  $Q$ .

Surprisingly, the number of matings exceeds the number of hyperbolic components for all  $n \geq 5$  and sufficiently large  $m$ . In fact there is an excess even if one discounts the shared matings arising from the *Wittner flip*, those arising from *period one clusters*, when a mating with a starlike polynomial can be realised in another way, and a slight generalisation of this: those arising from a *period two cluster* when a mating with a critically periodic polynomial for which the periodic Fatou components accumulate on a period two repelling orbit can be realised in three other ways (see Figure 1). It was, in fact, the overcount which led to the discovery of the shared matings arising from period two clusters and removed the overcount in the case of  $n = 4$ . There must be many more shared matings to account for the overcount for all  $n \geq 5$ .

**Question 15.** *Can one give a simple description of more shared matings?*

Here is a brief description of the numerics. We write  $\eta'_{IV}(m, n)$  of type IV hyperbolic components with disjoint orbits of attractive periodic points of periods  $\geq 3$  and dividing  $n$  and  $m$ . For an integer  $q > 0$ , let  $\varphi(q)$  be the number of integers between 0 and  $q$  which are coprime to  $q$  (the usual Euler phi-function). Let  $\nu_q(n)$  be the number of minor leaves of period dividing  $n$  in a limb of period  $q$ , that is,

$$\nu_q(n) = \left\lfloor \frac{2^{n-1}}{2^q - 1} \right\rfloor \text{ if } q \nmid n \quad \text{and} \quad \left\lfloor \frac{2^{n-1}}{2^q - 1} \right\rfloor + 1 \text{ if } q|n.$$

Then  $\eta'_{IV}(m, n)$  is

$$\begin{aligned} & \left( \frac{7}{18}(2^{n-1} - 1) - \frac{1}{4} \sum_{3 \leq q \leq n} \frac{\varphi(q)\nu_q(n)}{2^q - 1} \right) \cdot 2^m + o(2^m) \text{ if } n \text{ is odd, and} \\ & \left( \frac{7}{9}(2^{n-2} - 1) - \frac{1}{4} \sum_{3 \leq q \leq n} \frac{\varphi(q)\nu_q(n)}{2^q - 1} \right) \cdot 2^m + o(2^m) \text{ if } n \text{ is even.} \end{aligned}$$

Meanwhile the number of critically periodic matings with critical points of periods  $\geq 3$ , and dividing  $m$  and  $n$  respectively is

$$\begin{aligned} & \left( \frac{4}{9}(2^{n-1} - 1) - \frac{1}{2} \sum_{3 \leq q \leq n} \frac{\varphi(q)\nu_q(n)}{2^q - 1} \right) \cdot 2^m + o(2^m) \text{ if } n \text{ is odd, and} \\ & \left( \frac{8}{9}(2^{n-2} - 1) - \frac{1}{2} \sum_{3 \leq q \leq n} \frac{\varphi(q)\nu_q(n)}{2^q - 1} \right) \cdot 2^m + o(2^m) \text{ if } n \text{ is even.} \end{aligned}$$

The shared matings coming from the Wittner flip and period two clusters reduce this by, respectively,

$$\frac{(n-2)\varphi(n)}{2(2^n - 1)} \cdot 2^m + o(2^m) \quad \text{and} \quad \frac{2n\varphi(n/2)}{2^n - 1} \cdot 2^m + o(2^m).$$

So, let  $\theta(n, m)$  is the number of matings, discounting known sharings. Then,  $\eta'_{IV}(n, m) - \theta(n, m)$  is

$$\left( \frac{(n-2)\varphi(n)}{2(2^n-1)} - \frac{1}{18}(2^{n-1}-1) + \frac{1}{4} \sum_{3 \leq q \leq n} \frac{\varphi(q)\nu_q(n)}{2^q-1} \right) 2^m + o(2^m)$$

if  $n$  is odd and

$$\left( \frac{(n-2)\varphi(n)}{2(2^n-1)} + \frac{2n\varphi(n/2)}{2^n-1} - \frac{1}{9}(2^{n-2}-1) + \frac{1}{4} \sum_{3 \leq q \leq n} \frac{\varphi(q)\nu_q(n)}{2^q-1} \right) \cdot 2^m + o(2^m)$$

if  $n$  is even. For  $n = 3, 4$  and  $5$  this is respectively

$$\frac{1}{21} \cdot 2^m + o(2^m), \quad \frac{6}{35} \cdot 2^m + o(2^m), \quad -\frac{156}{1085} \cdot 2^m + o(2^m)$$

But for  $n \geq 6$

$$\begin{aligned} \eta'_{IV}(n, m) - \theta(n, m) &\leq - \left( \frac{1}{18} - \frac{1}{4} \sum_{3 \leq q \leq n} \frac{q-1}{(2^q-1)^2} \right) \cdot 2^{m+n-1} \\ &\quad + \left( \frac{1}{9} + \frac{1}{8} \sum_{3 \leq q \leq n} \frac{q-1}{2^q-1} + \frac{(n-2)(3n-1)}{2(2^n-1)} \right) \cdot 2^m \\ &< \left( 1 - \frac{1}{27} \cdot 2^{n-1} \right) \cdot 2^m < 0. \end{aligned}$$

#### 4.2. Type II components and adjacent type IV components.

**Question 16.** *Are there type II hyperbolic components of quadratic rational maps whose centers are not periodic Wittner captures?*

**Question 17.** *Is there a postcritically finite quadratic rational map with Fatou components in a single periodic orbit of clusters, which is not a geometric mating?*

The second question is slightly weaker than the first, and can be considered without understanding the definition of periodic Wittner capture. If  $n \geq 3$ , the boundary of a type II hyperbolic component, with periodic Fatou component of period  $n$ , intersects the boundaries of at most three, and at least two, type IV hyperbolic components, with periodic Fatou components in two orbits of period  $n$ . If there are less than three, then the original hyperbolic component is unbounded. In all the known examples, all the type IV maps arising in this way are Thurston equivalent to matings. Are there any which are not? The representation as a mating is not usually unique, if it does exist, which, of course, raises other questions.

Any missing boundary points are certainly represented by inadmissible matings. If at least one of these boundary rational maps is equivalent to a mating, even an admissible mating, then the centre of the hyperbolic component is equivalent to a periodic Wittner capture.

For any type II hyperbolic component of quadratic rational maps with Fatou component of period  $n \geq 3$ , and any  $q \geq 2$ , there are  $6(2^{q-1}-1)$  type IV hyperbolic components with periodic Fatou components of periods  $n$  and  $nq_1$ , for  $q_1 > 1$  dividing  $q$ , such that the closures of the type II and type IV components intersect. The intersection contains a rational map with one critical point of period  $n$  and

an orbit of parabolic basins of period dividing  $nq$ . Much of this is proved in [R2], although there is no count there; and the count given here does need checking.

**Question 18.** *How many of the centers of these type IV hyperbolic components can be represented as matings?*

Such examples may occur even for  $n = 3$  and  $q = 2$ , but this has not been checked properly. There are two type II components of period 3, and hence there are twelve type IV hyperbolic components to consider for  $n = 3$  and  $q = 2$ . Of these, ten are represented by matings, eight of them in a unique way and two in two different ways. The last two of the twelve appear not to be represented by matings. This situation is probably reproduced for  $n = 3$  and any  $q > 2$ , but, again, this has not been checked properly.

## 5. QUADRATIC POLYNOMIALS WITH IRRATIONALLY INDIFFERENT FIXED POINTS (BUFF-KOCH)

There is a parallel between mating and constructions in the theory of hyperbolic 3-manifolds. One may view mating as a combination procedure. At a formal level, one can sometimes combine Kleinian groups  $G_1, G_2$  via so-called *Klein-Maskit* combinations into a larger group  $G := \langle G_1, G_2 \rangle$ . In contrast, under mating, the dynamics of  $P, Q$  typically does not faithfully embed in an even topological way into that of  $P \amalg Q$ . An exception is mating Blaschke products. For example, the problem of realizing  $(\lambda z + z^2) \amalg (\mu z + z^2)$  when  $|\lambda| = |\mu| = 1$  is formally analogous to the difficult problem of establishing Thurston's so-called Double Limit Theorem (see [Hu1] for more details).

More precisely, let  $\alpha$  and  $\beta$  be two irrational numbers with  $\beta \neq -\alpha$ . Set

$$P(z) := \lambda z + z^2 \quad \text{and} \quad Q(z) := \mu z + z^2 \quad \text{with} \quad \lambda := e^{2\pi i \alpha} \quad \text{and} \quad \mu := e^{2\pi i \beta}.$$

Both polynomials fix 0. The respective multipliers are  $\lambda$  and  $\mu$ . Let  $F$  be the rational map defined by

$$F(z) := \frac{z(z + \lambda)}{\mu z + 1}$$

which fixes 0 with multiplier  $\lambda$  and  $\infty$  with multiplier  $\mu$ .

**Question 19.** *To which extent can  $F$  be seen as a mating of  $P$  and  $Q$ ?*

The following question is due to Milnor [Mi].

**Question 20.** *If  $\mathcal{J}_P$  and  $\mathcal{J}_Q$  are locally connected, then are  $P$  and  $Q$  topologically mateable, and do we have  $F \cong P \amalg Q$ ?*

As already mentioned, Yampolsky and Zakeri [YZ] proved that  $F \cong P \amalg Q$  when  $\alpha$  and  $\beta$  are of bounded type. Zhang [Z] expanded this result to the case where  $\alpha$  and  $\beta$  are of Petersen-Zakeri type.

When at least one of the Julia sets is not locally connected, we can still expect to see the dynamics of  $P$  and  $Q$  reflected in that of  $F$ . We propose the following questions.



**Question 21.** *Are there continuous surjective maps  $\phi : \mathcal{J}_P \rightarrow \mathcal{J}_F$ ,  $\psi : \mathcal{J}_Q \rightarrow \mathcal{J}_F$  such that the following three diagrams commute?*

$$\begin{array}{ccc} \mathcal{J}_P & \xrightarrow{P} & \mathcal{J}_P \\ \phi \downarrow & & \downarrow \phi \\ \mathcal{J}_F & \xrightarrow{F} & \mathcal{J}_F \end{array} \qquad \begin{array}{ccc} \mathcal{J}_Q & \xrightarrow{Q} & \mathcal{J}_Q \\ \psi \downarrow & & \downarrow \psi \\ \mathcal{J}_F & \xrightarrow{F} & \mathcal{J}_F \end{array}$$

$$\begin{array}{ccc} \mathbb{Q}/\mathbb{Z} & \xrightarrow{\theta \mapsto -\theta} & \mathbb{Q}/\mathbb{Z} \\ \gamma_P \downarrow & & \downarrow \gamma_Q \\ \mathcal{J}_P & & \mathcal{J}_P \\ & \searrow \phi & \swarrow \psi \\ & \mathcal{J}_F & \end{array}$$

It is known that if  $P$  has a Siegel disk around 0 (this occurs if and only if  $\alpha$  is a Brjuno number), then  $F$  has a Siegel disk around 0. One may wonder whether the converse holds.

**Question 22.** *Is there a local conjugacy between the dynamics of  $P$  near 0 and the dynamics of  $F$  near 0? If it exists, what is its regularity? For example, if  $F$  has a Siegel disk around 0, does  $P$  have a Siegel disk around 0?*

It is conjectured that if  $P$  has a Siegel disk around 0, then its boundary is a Jordan curve.

**Question 23.** *Assume  $P$  has a Siegel disk  $\Delta$  around 0 so that  $F$  has a Siegel disk  $D$  around 0. Are  $\overline{\Delta}$  and  $\overline{D}$  homeomorphic? Does the boundary of  $\Delta$  contain the critical point of  $P$  if and only if the boundary of  $D$  contains a critical point of  $F$ ?*

It follows from the Fatou-Shishikura Inequality that the two critical orbits of  $F$  are disjoint. In addition, if  $F$  has a Siegel disk around 0 the boundary of the Siegel disk is contained in the closure of at least one critical orbit. Otherwise 0 is contained in the closure of at least one critical orbit.

**Question 24.** *Are the closures of the two critical orbits of  $F$  disjoint? In particular, if  $F$  has two Siegel disks  $D_0$  and  $D_\infty$  around 0 and  $\infty$ , do we have  $\overline{D}_0 \cap \overline{D}_\infty = \emptyset$ ?*

Lastly, we wish to understand the dynamics of a point randomly chosen with respect to the Lebesgue measure. For each critical point  $c$  of  $F$  we set

$$B_c := \{z \in \mathbb{P}^1 \mid \text{dist}(F^{\circ n}(z), \mathcal{O}_F(c)) \xrightarrow{n \rightarrow \infty} 0\}.$$

**Question 25.** *Assume  $\mathcal{J}_P$  and  $\mathcal{J}_Q$  have positive Lebesgue measure. Does  $F$  fail to be ergodic with respect to the Lebesgue measure? More precisely, if  $c_1$  and  $c_2$  are the critical points of  $F$ , are  $B_{c_1}$  and  $B_{c_2}$  disjoint and do they have positive Lebesgue measure?*

## 6. TWISTED MATINGS OF QUADRATIC POLYNOMIALS (BUFF-KOCH)

Let  $P$  be a monic polynomial of degree  $d \geq 2$ . Then for any integer  $k \geq 1$ , and any  $\alpha = d^k - 1$  root of unity the polynomial

$$P_\alpha^k := z \mapsto \alpha \cdot P^{\circ k}(z/\alpha)$$

is monic with  $\mathcal{J}_{P_\alpha^k} = \alpha \cdot \mathcal{J}_P$ .

**Question 26.** *Given two quadratic polynomials  $P$  and  $Q$ , and an integer  $k \geq 1$ , for which  $(\alpha, \beta)$  are  $P_\alpha^k$  and  $Q_\beta^k$  mateable?*

If  $P(z) = z^2 + c_1$  and  $Q(z) = z^2 + c_2$  are postcritically finite, and  $k = 1$ , the answer is well-known:  $P$  and  $Q$  are mateable if and only if  $c_1$  and  $c_2$  do not belong to conjugate limbs of the Mandelbrot set, ([T2], [R1], [R3], [Shi]). It follows from this result that if  $P(z) = z^2 + c_1$  and  $Q(z) = z^2 + c_2$  are two postcritically finite polynomials for which  $c_1$  and  $c_2$  are in conjugate limbs, then for all  $k \geq 2$ , and  $\alpha = \beta$ , then  $P_\alpha^k$  and  $Q_\beta^k$  are not mateable.

In the case of  $P(z) = Q(z) = z^2 - 1$ , as long as  $k \geq 2$  and  $\alpha \neq \beta$ , the polynomials  $P_\alpha^k$  and  $Q_\beta^k$  are mateable, see [BEK].

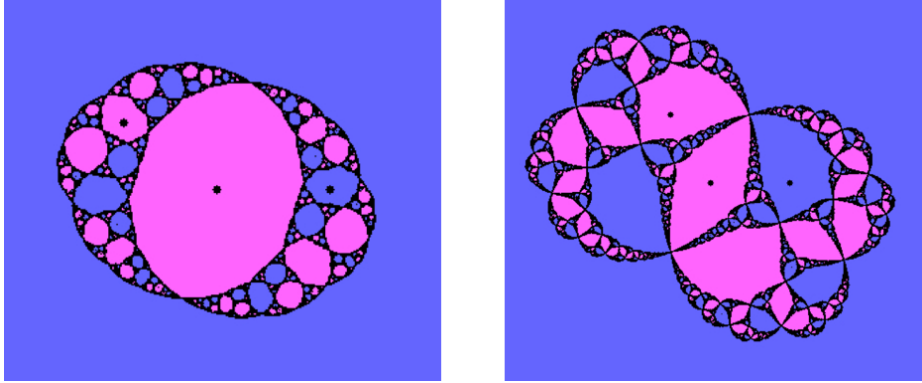


FIGURE 4. Twisted matings of the basilica,  $P = Q : z \mapsto z^2 - 1$ .  
 Left: a geometric mating of  $P_1^2$  and  $Q_\alpha^2$  with  $\alpha = \exp(-2\pi i/3)$ .  
 Right: a geometric mating of  $P_1^4$  and  $Q_\beta^4$  with  $\beta = \exp(-2\pi i/5)$ .

## 7. SLOW MATINGS (BUFF-KOCH-MEYER)

**7.1. Equipotential gluing.** Let  $P$  and  $Q$  be two monic polynomials with connected Julia sets. Let  $h_P : \mathbb{C} \rightarrow [0, +\infty)$  and  $h_Q : \mathbb{C} \rightarrow [0, +\infty)$  be the associated Green functions. Given  $\eta > 0$ , let  $U_P(\eta)$  and  $U_Q(\eta)$  be defined by

$$U_P(\eta) := \{z \in \mathbb{C} \mid h_P(z) < \eta\} \quad \text{and} \quad U_Q(\eta) := \{w \in \mathbb{C} \mid h_Q(w) < \eta\}.$$

Let  $\Sigma_\eta$  be the Riemann sphere obtained by identifying

$$\text{böt}_P(z) \in U_P(\eta) \quad \text{and} \quad \text{böt}_Q(w) \in U_Q(\eta) \quad \text{when} \quad zw = \exp(\eta).$$

The restrictions  $P : U_P(\eta) \rightarrow U_P(2\eta)$  and  $Q : U_Q(\eta) \rightarrow U_Q(2\eta)$  induce a holomorphic map  $f_\eta : \Sigma_\eta \rightarrow \Sigma_{2\eta}$ . We wish to know whether the family  $f_\eta$  has a limit as

$\eta \rightarrow 0^+$ . For this, we need to normalize the family  $f_\eta$ . Choose  $x \in \mathcal{K}_P$  and  $y \in \mathcal{K}_Q$ . For  $\eta > 0$  let  $\phi_\eta : \Sigma_\eta \rightarrow \mathbb{P}^1$  be the conformal isomorphism satisfying

$$\phi_\eta(x) = 0, \quad \phi_\eta(y) = \infty \quad \text{and} \quad \phi_\eta(\exp(\eta/2)) = 1.$$

The map

$$F_\eta := \phi_{2\eta} \circ f_\eta \circ \phi_\eta^{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

is a rational map.

**Question 27.** *For which polynomials  $P$  and  $Q$ , and which points  $x$  and  $y$  does the family of rational maps  $F_\eta$  converge uniformly on  $\mathbb{P}^1$  to a rational map  $F$ ?*

**Question 28.** *Assume  $\mathcal{K}_P$  and  $\mathcal{K}_Q$  are locally connected, and  $F_\eta \rightarrow F$  as  $\eta \rightarrow 0^+$ . Are  $P$  and  $Q$  mateable, and is it true that  $F$  is a geometric mating of  $P$  and  $Q$ ?*

**7.2. Holomorphic motions.** The strong form of Moore’s theorem says that the semi-conjugacy from the formal mating to the topological mating can be obtained as the end of a pseudo-isotopy, i.e., each ray-equivalence class can be continuously deformed to a point. Is it possible to do this deformation “in a nice way”? If so, what can be said about this deformation? Particularly nice would be to do the “ray-shrinking deformation” in a *holomorphic motion*.

Consider the sphere  $\mathbb{S}$  divided by the equator in two hemispheres. Each hemisphere is (conformally equivalent to) a unit disk. Draw the laminations from the two polynomials  $P$  and  $Q$  whose mating we consider into each hemisphere (more precisely map the laminations of the unit disk into each hemisphere). Consider now a gap (i.e. a component of the complement) of one lamination. Assume for simplicity that it is an ideal triangle. We can think of this gap as consisting conformally of three half-strips that are glued together at the center of the gap. Into each half-strip we put a Beltrami-field of very long, thin, ellipses in the direction of the strip. When solving the Beltrami-equation these ellipses will be deformed to be round, causing everything in the strip to be “pulled together”.

Formally note that the map  $x + iy \mapsto \frac{1}{K}x + iy$  is a solution of the Beltrami equation with  $\mu = \frac{1-K}{1+K}$ . Copy this Beltrami fields into each of the three half-strips of the gap. More precisely we map the half-strip  $[0, \infty) \times [0, 1]$  conformally to “one third” of the gap and use the conformal map to transfer the Beltrami-field.

Similarly if the gap is an ideal  $n$ -gon we divide it into  $n$  conformal half-strips and copy the Beltrami-field as above into each half-strip.

Now we can vary the construction above by setting  $\mu$  on  $[0, \infty) \times [0, 1]$  to be constant to another number  $z_0 \in \mathbb{D}$ . The solution of the resulting Beltrami-equation yields a *holomorphic motion*.

**Question 29.** *Consider first the solutions obtained from setting  $\mu = (1-K)/(1+K)$  as in the first paragraph. Does the solution for large real  $K$  tend to the pseudo-isotopy from Moore’s theorem? More precisely: show that the solution  $h_K$  (of the Beltrami equation) converges for large  $K$  to a continuous map  $h_\infty$ . Show that preimages of  $h_\infty$  are precisely the equivalence classes of the ray-equivalence relation.*

If the above indeed can be carried out one would get a very natural, as well as powerful, description of the mating.

These holomorphic motions would be natural candidates for extremal examples for *Brennan’s conjecture* a deep conjecture from the theory of conformal maps.

## 8. MISCELLANY (MEYER)

**8.1. Semi-conjugacies.** A mating induces a semi-conjugacy between the multiplication by  $d$  acting on  $\mathbb{R}/\mathbb{Z}$  and the rational map  $F$  acting on  $\mathcal{J}_F$ , i.e., there is a continuous, surjective map  $\gamma: \mathbb{R}/\mathbb{Z} \rightarrow \mathcal{J}_F$  such that  $\gamma(d \cdot \theta) = F \circ \gamma(\theta)$  for all  $z \in \mathbb{R}/\mathbb{Z}$ .

There is a parallel in the theory of hyperbolic 3-manifolds. Suppose  $S$  is a closed surface and  $\rho: \pi_1(S) \rightarrow PSL_2(\mathbb{C})$  is a discrete faithful representation with image a Kleinian group  $G$ . By [Mj1, Thm. 5.14] (see also [Mj2]), there is an associated semiconjugacy  $\mathbb{U} \rightarrow \mathbb{P}^1$  from the action of  $\pi_1(S)$  on the boundary of its Cayley graph to the action of  $G$  on its limit set. Called the *Cannon-Thurston map*, it is the analog of the Carathéodory loop; see also the main result of [CT] for the case of 3-manifolds fibering over the circle. This result establishes the fact that the limit sets arising are always locally connected, contrasting with our present setting of polynomials.

In the case when  $F$  is postcritically finite and  $\mathcal{J}_F = \mathbb{P}^1$ , there often is also a semi-conjugacy between the multiplication by  $-d$  acting on  $\mathbb{R}/\mathbb{Z}$  and the rational map  $F$  acting on  $\mathbb{P}^1$ . Indeed every sufficiently high iterate  $F^n$  is a mating of two postcritically finite polynomials without periodic critical points (see [Me1], [Me2]). This phenomenon seems to be as common as the standard semi-conjugacies (which are associated with matings). One may speculate that there is some *other* type of mating from which this phenomenon arises. This should be related to *orientation reversing equators* of the kind one encounters for the map  $\mathbb{U} \ni z \mapsto z^{-2} \in \mathbb{U}$ .

**Question 30.** *Give a proper definition of the “matings” that occur as above.*

**8.2. Hausdorff dimension.**

**Question 31.** *Assume  $F$  is the geometric mating of postcritically finite polynomials  $P$  and  $Q$ . Is there a relation between the Hausdorff dimension of the Julia sets  $\mathcal{J}_P$ ,  $\mathcal{J}_Q$  and  $\mathcal{J}_F$ ? For example is it true that  $\max(\text{Hdim } \mathcal{J}_P, \text{Hdim } \mathcal{J}_Q) \leq \text{Hdim } \mathcal{J}_F$ ?*

Of course the question could also be asked in other settings, i.e., when  $P, Q$  are not postcritically finite. Then however the answer is no, as the following example shows. The mating of two complex conjugate perturbations of  $z^2$  (whose Julia sets have Hausdorff dimension greater than 1) is a Blaschke product (whose Julia set has Hausdorff dimension 1).

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