Parabolic bifurcations

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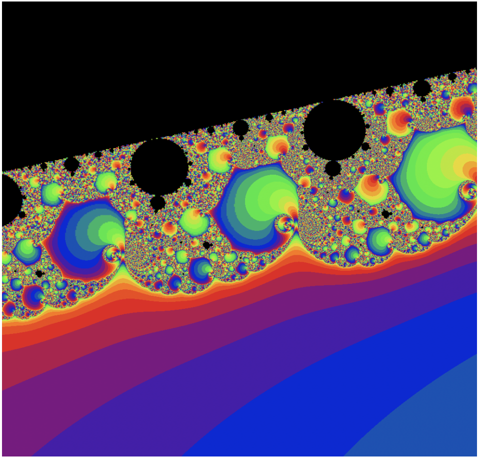
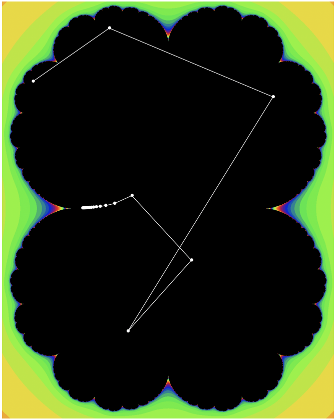
Around the Mandelbrot set
a conference celebrating the 60th birthday of
Mitsuhiro Shishikura

Parabolic bifurcations in complex dimension 1

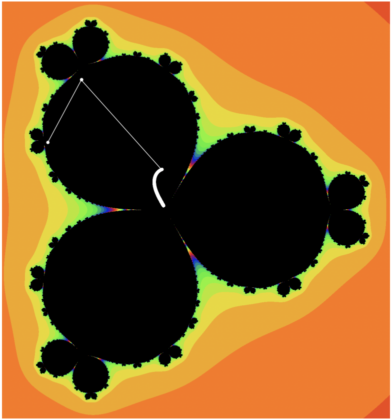


Parabolic basins

- $f(z) = z - z^{k+1} + \mathcal{O}(z^{k+1})$, $k \geq 1$.
- $\mathcal{B} = \{z \in \mathbb{C} \ : \ f^{\circ n}(z) \underset{\neq}{\longrightarrow} 0\}$.



$k = 1$



$k = 3$

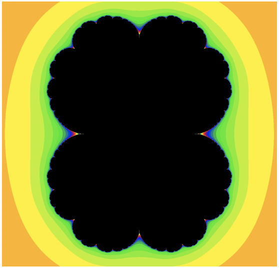
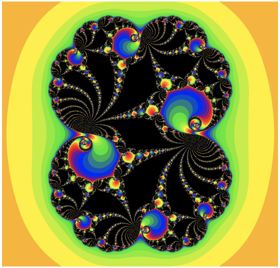
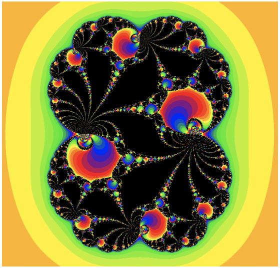
Geometric limits

- $f(z) = z + z^2 + \mathcal{O}(z^3)$ has a parabolic basin $\mathcal{B}$.
- $\mathcal{L}_\sigma : \mathcal{B} \to \mathbb{C}$ is the Lavaurs map with phase $\sigma \in \mathbb{C}$.
- $f_\varepsilon(z) = f(z) + \varepsilon^2$.

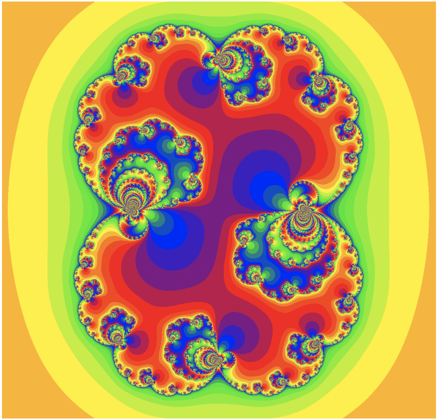
Theorem (Lavaurs)

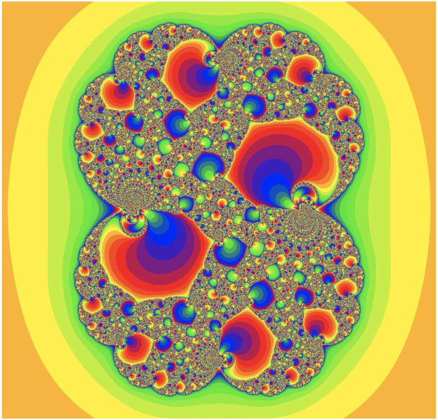
Assume $\mathbb{N} \ni T_n \to +\infty$ *and* $\mathbb{C}^* \ni \varepsilon_n \to 0$ *satisfy* $T_n - \dfrac{\pi}{\varepsilon_n} \to \sigma$.

Then, $f_{\varepsilon_n}^{\circ T_n} \to \mathcal{L}_\sigma$ *locally uniformly in* $\mathcal{B}$.

Dynamical enrichment





Parabolic renormalization

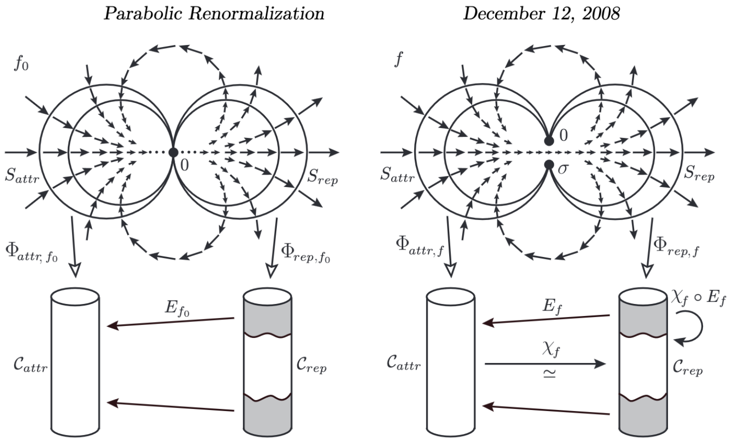
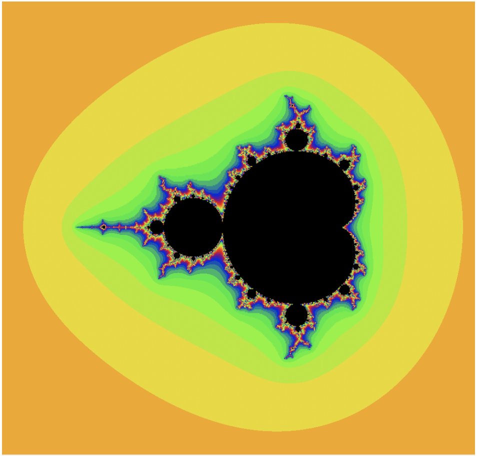


Figure 2: Perturbation of parabolic fixed point: before (left) and after (right)

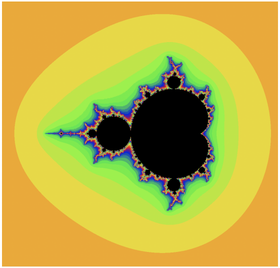
Hausdorff dimension and Lebesgue measure

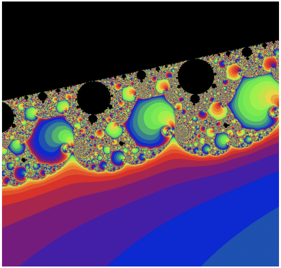


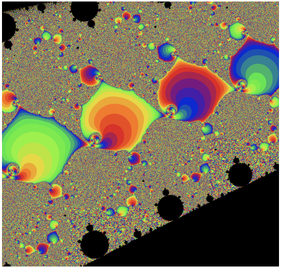
Theorem (Shishikura)

$$H\text{-}dim(\partial M) = 2.$$

Moreover, for any open set U which intersects ∂M, $H\text{-}dim(\partial M \cap U) = 2$.







As for the area (the 2-dimensional Lebesgue measure), it is conjectured that ∂M and J_c (for any c) have area zero. There are partial results: the set of parameters in ∂M for which P_c are not infinitely renormalizable has area zero [Sh1]. If P_c has no irrationally indifferent periodic point and is not infinitely renormalizable, then the Julia set J_c has area zero [Ly2] and [Sh1].

The renormalization for parabolic fixed points and their perturbation

Hiroyuki Inou and Mitsuhiro Shishikura

Dedicated to the memory of Adrien Douady

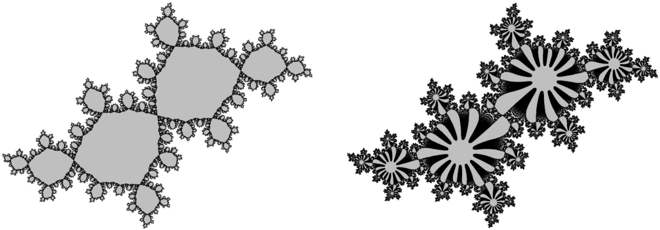
Abstract

For holomorphic maps of one variable with a parabolic fixed point, the parabolic renormalization $\mathcal{R}_0$ is defined in terms of Fatou coordinates and horn maps. A class $\mathcal{F}_1$ of such maps is proposed so that it is invariant under $\mathcal{R}_0$, which acts as a uniform contraction with respect to a certain metric. The near-parabolic renormalization $\mathcal{R}$ is also defined for the perturbation of these maps, and it amounts to taking a first return map on a certain fundamental region. It is also shown that $\mathcal{R}$ is hyperbolic on the space of maps whose multiplier is sufficiently close to 1 . These results will help us to analyze the behavior of orbits of near the fixed points, especially irrationally indifferent ones. Buff and Chéritat [BC] used our result as one of main tools in their construction of a quadratic polynomial with Julia set of positive Lebesgue measure.

Positive area

Theorem (B-Chéritat)

There exist quadratic polynomials which have a Julia set of positive area.



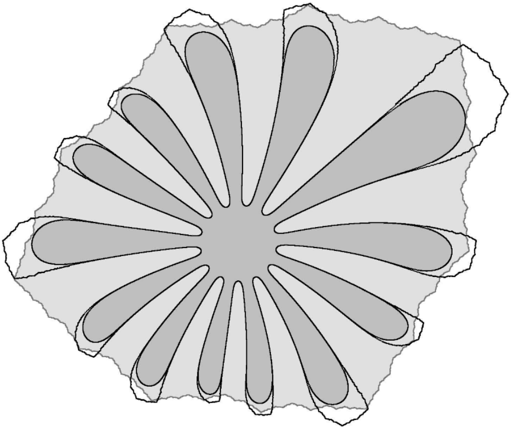
- The strategy is due to Douady.
- Examples are of the form $P_\alpha(z) = \mathrm{e}^{2\pi \mathrm{i}\alpha} z + z^2$ with $\alpha \in \mathcal{S}_N$, where $\mathcal{S}_N$ is the set of bounded type irrational numbers whose continued fraction has entries at least N.

Peturbed Siegel disks

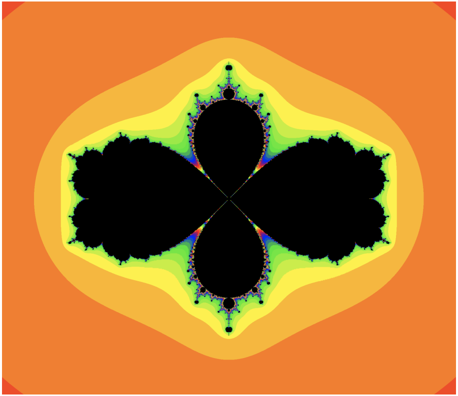
Proposition (B-Chéritat)

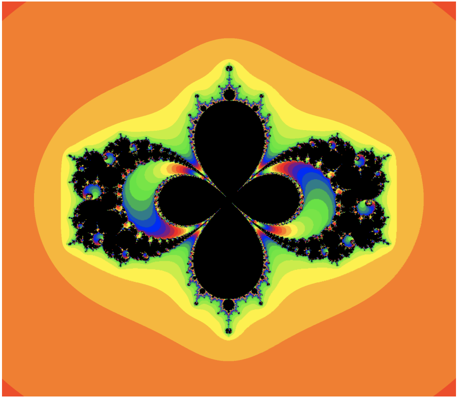
There exists N such that as $\alpha' \in \mathcal{S}_N \to \alpha \in \mathcal{S}_N$, we have

$$\partial\big(\mathcal{PC}(P_{\alpha'}), \overline{\Delta}_\alpha\big) \to 0.$$

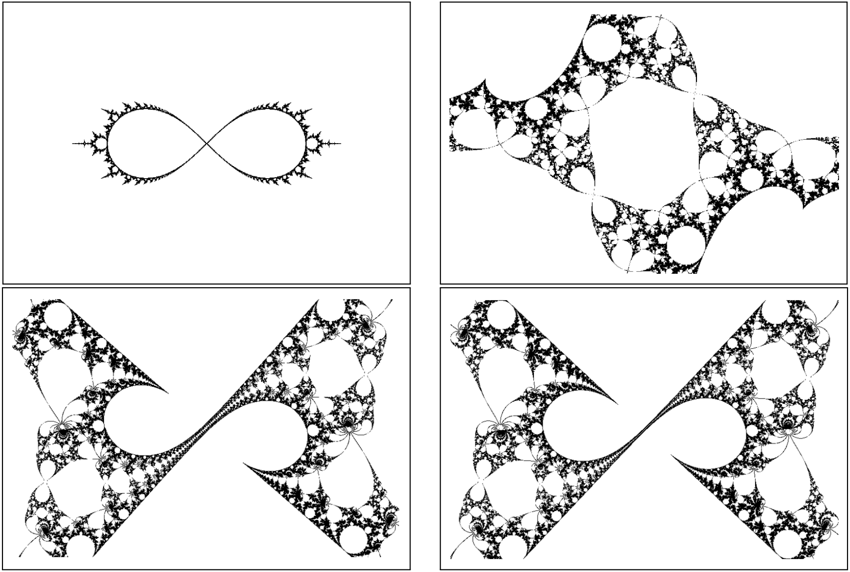


Enrichments in parameter spaces





Parameter spaces : $e^{2\pi i/q} z/(1 + az + z^2)$, $a \in \mathbb{C}$



Résidu itératif

- $g(z) = \dfrac{e^{2\pi i p/q} z}{1 - az + z^2}$ with $a \in \mathbb{C}$.
- If $g(z) \underset{\text{formally}}{\sim} e^{2\pi i p/q} z \cdot \left(1 + z^{\nu q} + \alpha z^{2\nu q}\right)$ then

$$\text{résit}(g) := \frac{\nu q + 1}{2} - \alpha.$$

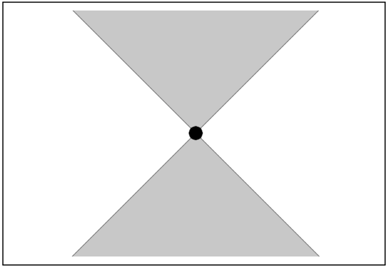
- $R_{p/q}(a) := \begin{cases} \text{résit}(g) & \text{if } \nu = 1 \\ \infty & \text{if } \nu = 2. \end{cases}$

Examples.

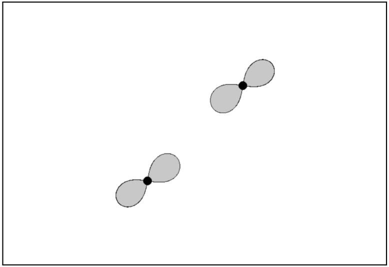
$$R_{0/1}(a) = \frac{1}{a^2}, \quad R_{1/2}(a) = \frac{a^2+2}{4}$$

$$R_{1/4}(a) = \frac{i a^6 + (15 - 2i)a^4 - (6 + 32i)a^2 - 6}{2(a^2 - i)^2}.$$

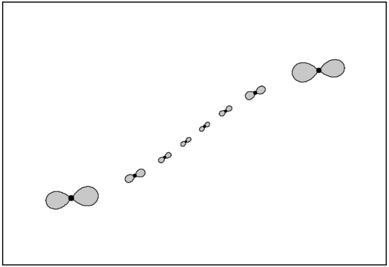
Résidu itératif



$p/q = 0/1$



$p/q = 1/4$



$p/q = 1/10$

Résidu itératif

Theorem (B-Ecalle-Epstein)

If $q \geq 2$ then $R_{p/q} : \mathbb{C} \to \widehat{\mathbb{C}}$ is a rational function of degree $2q - 2$. Every pole of $R_{p/q}$ is double: there is a pole at infinity and the poles in $\mathbb{C}$ correspond to maps such that $\nu = 2$.

Theorem (B-Ecalle-Epstein)

There is a meromorphic transcendental function $\mathcal{R} : \mathbb{C}^* \to \widehat{\mathbb{C}}$, such that

$$\left(\frac{1}{q}\right)^2 R_{1/q} \to \mathcal{R}.$$

The function $\mathcal{R}$ has an essential singularity at 0, a double pole at infinity and infinitely many poles accumulating 0.

- $\mathcal{R}$ is a meromorphic function of $b = 1/a^2$.

Theorem (Work in progress with Petersen)

The poles of $\mathcal{R}$ are double poles. In the b-plane, the poles of $\mathcal{R}$ are contained in the vertical strip $\{0 < \mathrm{Re}(b) < 1/2\}$. They form a sequence (b_n) with the asymptotic behavior

$$b_n = -n\frac{\mathrm{i}}{2\pi} + \frac{1}{4} + \frac{\mathrm{i}}{4\pi} + \mathrm{o}(1).$$

Parabolic bifurcation with $\nu \geq 2$

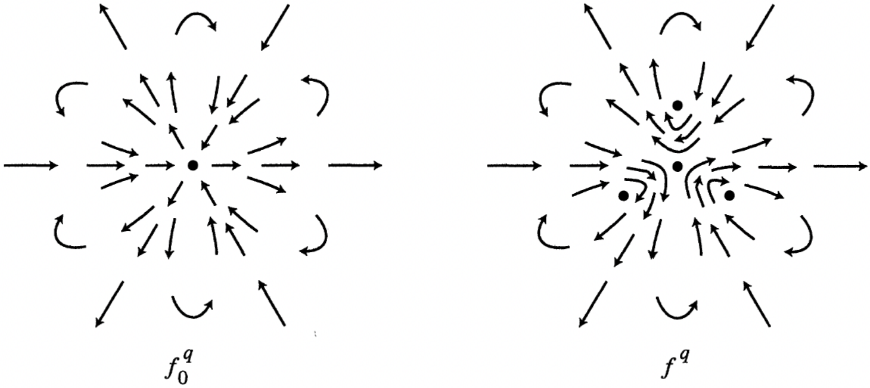
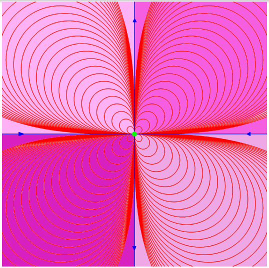
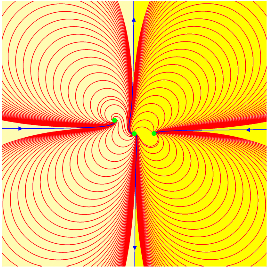


FIGURE 8.

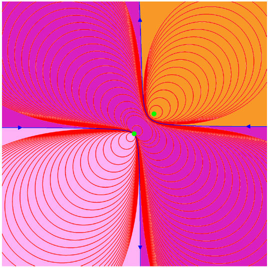
Real-time trajectories of polynomial vector fields



$-z^3 \partial/\partial z$

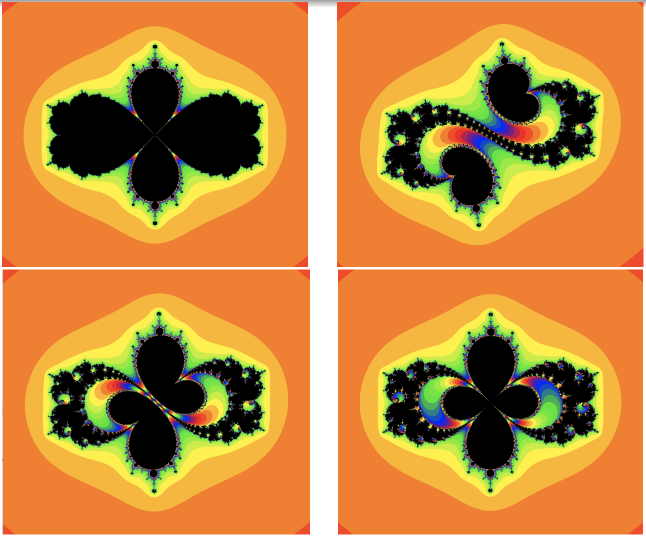


$-z(z-0.3)(z+0.3-0.2\mathrm{i})\partial/\partial z$



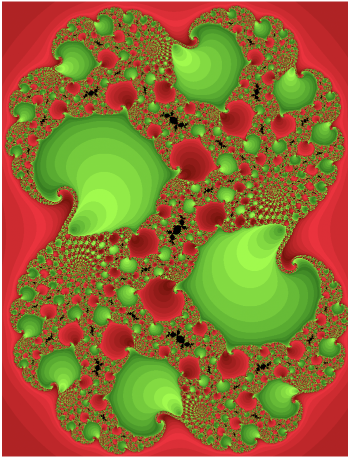
$-z^2(z-0.3-0.3\mathrm{i})\partial/\partial z$

Parameter spaces : $e^{2\pi i/q}z + az^2 + z^3$, $a \in \mathbb{C}$



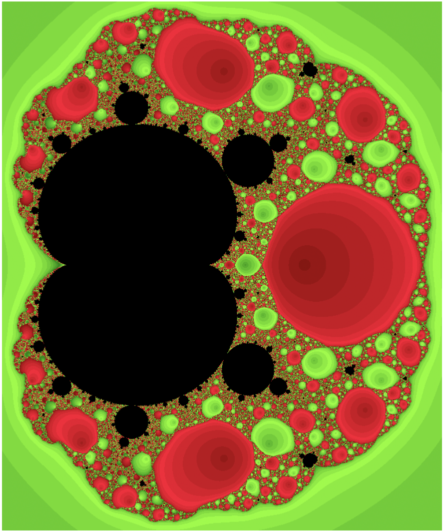
- Alex Kapiamba and Runze Zhang are studying parabolic enrichments for cubic polynomials.

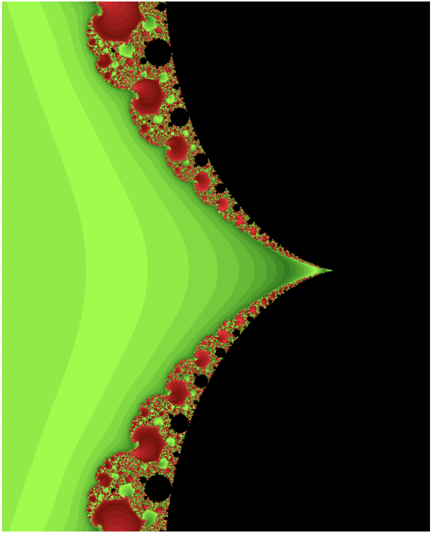
Parabolic bifurcations at infinity



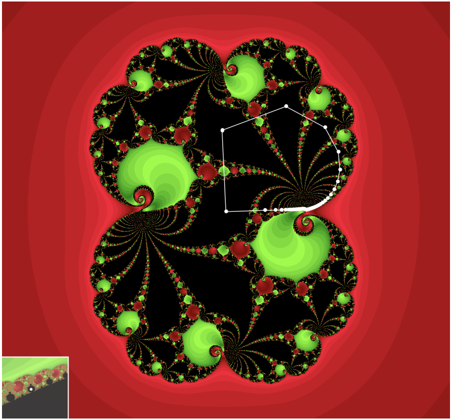
Quadratic rational maps with a critical point of period 2

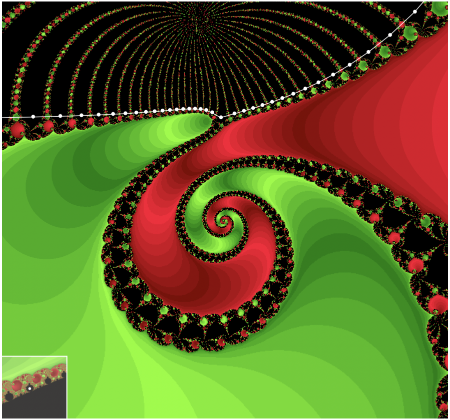
- $f_a(z) = \dfrac{a}{z+z^2}$ with $a \in \mathbb{C} \setminus \{0\}$.





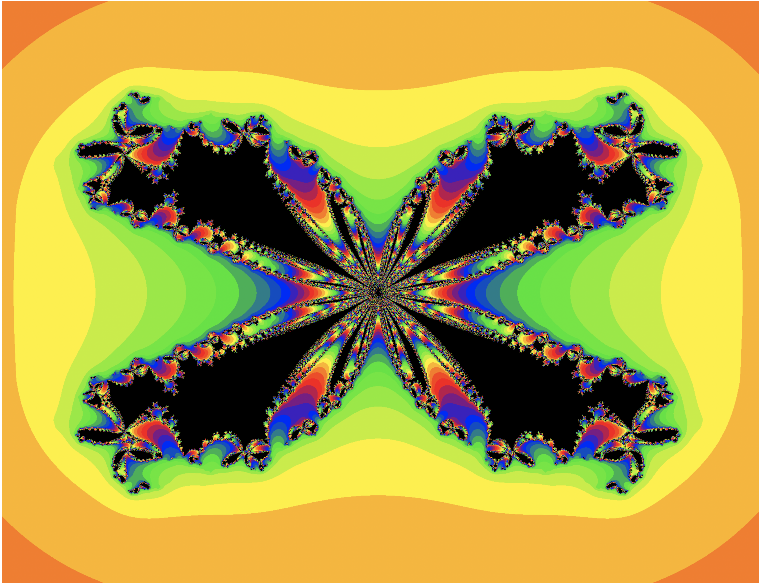
Quadratic rational maps with a critical point of period 2





- Caroline Davis and Alex Kapiamba are studying parabolic bifurcations for rescaling limits at infinity in $\mathrm{Per}_n(0)$.

Parabolic bifurcations in complex dimension 2



Semi-parabolic Bifurcations in Complex Dimension Two

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Abstract: Parabolic bifurcations in one complex dimension demonstrate a wide variety of interesting dynamical phenomena. In this paper we consider the bifurcations of a holomorphic diffeomorphism in two complex dimensions with a semi-parabolic, semi-attracting fixed point.

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Semi-parabolic bifurcation in complex dimension 2

- $F_{a,\varepsilon}(x,y) = \big((1+a)x - ay + x^2 + \varepsilon^2, x + \varepsilon^2\big)$.

Theorem (Bedford-Smillie-Ueda)

The map $\varepsilon \mapsto K^+(F_{a,\varepsilon})$ is discontinuous.

Semi-parabolic Bifurcations in Complex Dimension Two 5

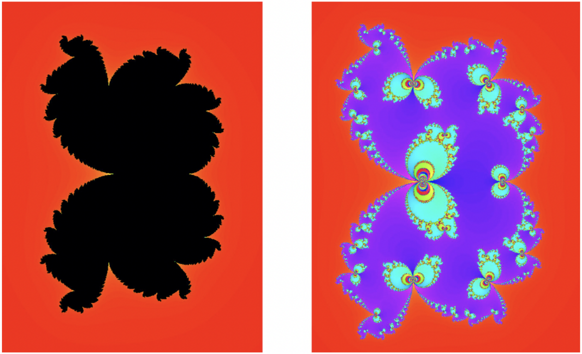


Fig. 1. The discontinuity of the map $\epsilon \mapsto K^+(F_{a,\epsilon})$ illustrated by showing complex linear slices in $\mathbf{C}^2$ for two nearby parameter values. $F_{a,\epsilon}$ is given by Eq. (1.3) with $a = .3$; $\epsilon = 0$ (*left*), $a = .3$, $\epsilon = .05$ (*right*)

Wandering domains in complex dimension 2

Theorem (Astorg-B-Dujardin-Peters-Raissy)

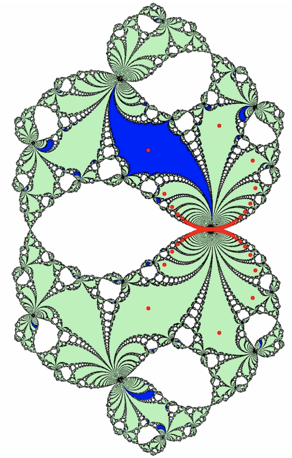
If $a < 1$ is sufficiently close to 1, the skew product $P : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$P(w, z) = \left(w - w^2, z + z^2 + az^3 + \frac{\pi^2}{4} w\right)$$

has a wandering Fatou component.

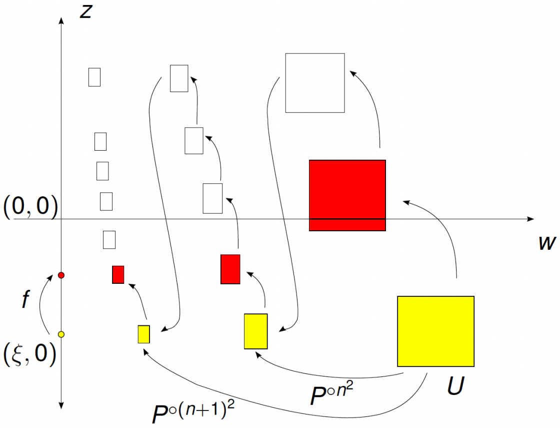
- The strategy is due to Lyubich.
- The map is a skew product.
- $P(0, z) = (0, z + z^2 + az^3)$ has a parabolic fixed point.

Wandering domains in complex dimension 2



- If $a < 1$ is sufficiently close to 1, the (phase $\sigma = 0$) Lavaurs map $\mathcal{L} : \mathcal{B} \to \mathbb{C}$ has an attracting fixed point.

Wandering domains in complex dimension 2



Spiralling domains for germs tangent to the identity

Theorem (work in progress with Raissy)

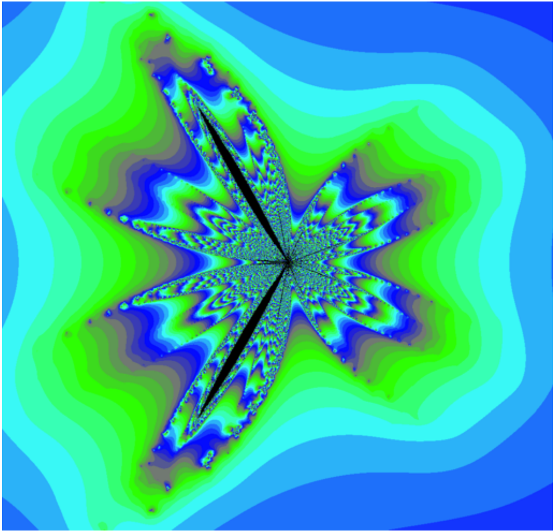
The polynomial endomorphisms

$$F_a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + a\begin{pmatrix} x(x-y) \\ y(x-y) \end{pmatrix}, \quad a \in \mathbb{R} \setminus \{0\}$$

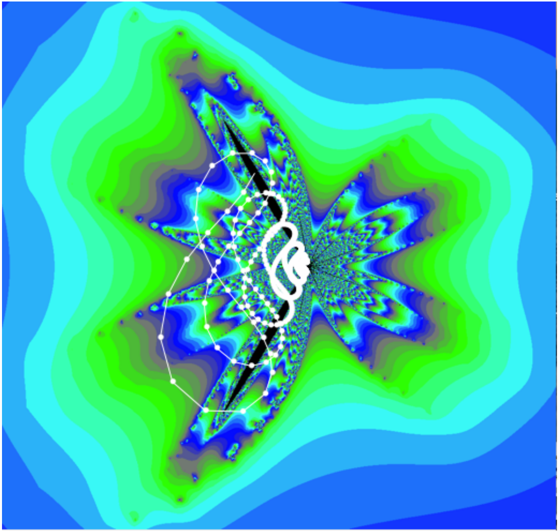
have infinitely many spiralling domains contained in distinct Fatou components.

- homogeneous vector fields in $\mathbb{C}^2$,
- affine (or similarity) surfaces.

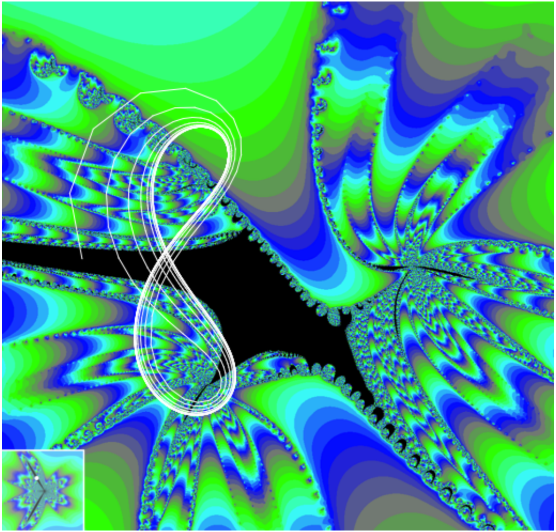
The dynamics of F_a for $a = 0.1$



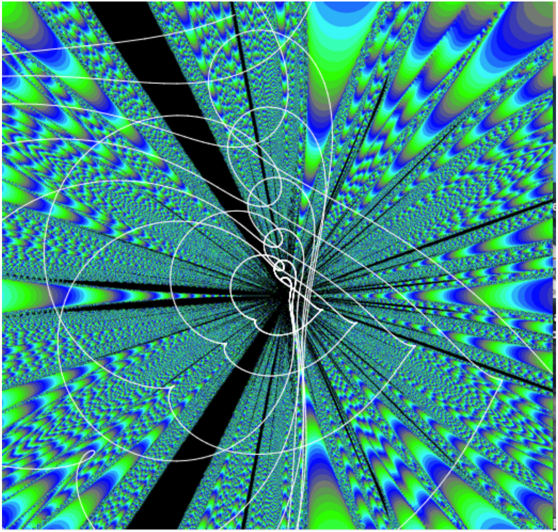
The dynamics of F_a for $a = 0.1$



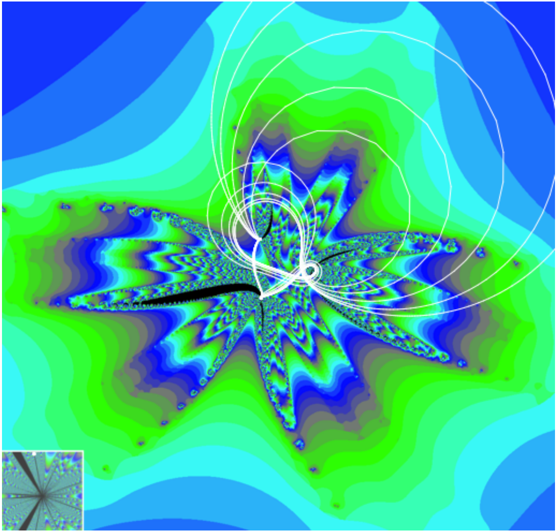
The dynamics of F_a for $a = 0.1$



The dynamics of F_a for $a = 0.1$



The dynamics of F_a for $a = 0.1$



Germs tangent to the identity

Assumptions:

- $\vec{v}$ is a homogeneous vector field of degree k on $\mathbb{C}^2$:

$$\vec{v} := U\partial_x + V\partial_y$$

  with U and V homogeneous polynomials of degree $k+1$;

- $$\Phi := xV - yU$$

  vanishes on $k+2$ *characteristic directions*, counting multiplicities;

- $$F(\boldsymbol{x}) = \boldsymbol{x} + \vec{v}(\boldsymbol{x}) + \mathrm{O}\left(\|\boldsymbol{x}\|^{k+2}\right).$$

Observation :

- Near $\mathbf{0}$, orbits of F shadow real-time trajectories of $\vec{v}$.

- A trajectory for $\vec{\boldsymbol{v}}$ is a solution of the differential equation

$$\dot{\gamma} = \vec{\boldsymbol{v}} \circ \gamma.$$

- Complex-time trajectories are Riemann surfaces which cover $\mathbb{CP}^1$ minus the characteristic directions.

Proposition (Abate-Tovena)

We may equip $\mathbb{CP}^1$ with the structure of an affine surface $\mathbf{S}_{\vec{\boldsymbol{v}}}$ so that the projection to $\mathbf{S}_{\vec{\boldsymbol{v}}}$ of real-time trajectories of $\vec{\boldsymbol{v}}$ are geodesics.

Affine surfaces and geodesics

Definition (Affine surface)

An *affine surface* **S** is a Riemann surface whose change of charts are affine maps $z \mapsto \lambda z + \mu$ with $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$.

Example : **C** is the complex plane with its canonical affine structure.

Definition (Affine map)

A map between affine surfaces is an *affine map* if its expression in affine charts is of the form $z \mapsto \lambda z + \mu$.

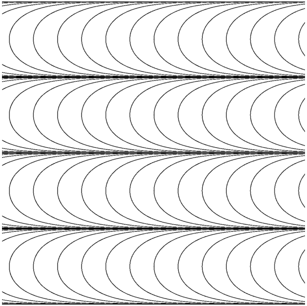
Definition (Geodesic)

A curve $\delta : I \to$ **S** defined on an interval $I \subseteq \mathbb{R}$ is a *geodesic* if δ is the restriction of an affine map $\varphi : U \to$ **S** defined on an open subset $U \subseteq$ **C**.

An example

- The dilation plane $\widetilde{\mathbf{C}}$ with underlying Riemann surface $\mathbb{C}$, whose affine charts are the restrictions of

$$\exp(z) : \widetilde{\mathbf{C}} \to \mathbf{C} \setminus \{0\}.$$



A family of parallel geodesics in $\widetilde{\mathbf{C}}$.

Non linearity

- The non linearity of a holomorphic map $\varphi : \mathbf{S} \to \mathbf{T}$ with non vanishing derivative is the 1-form $\mathcal{N}_\varphi$ defined on **S** by

$$\mathcal{N}_\varphi := \mathrm{d}(\log \varphi') = \frac{\mathrm{d}\varphi'}{\varphi'}.$$

- $\mathcal{N}_\varphi = 0$ if and only if φ is an affine map.
- If $\varphi : \mathbf{S} \to \mathbf{T}$ and $\psi : \mathbf{T} \to \mathbf{U}$ are holomorphic maps, then

$$\mathcal{N}_{\psi\circ\varphi} = \mathcal{N}_\varphi + \varphi^*(\mathcal{N}_\psi).$$

Affine surface of a homogeneous vector field

- $\vec{v} = U\partial_x + V\partial_y$ is homogeneous of degree k.
- $\Phi = xV - yU$.
- $z : \mathbb{CP}^1 \ni [x : y] \mapsto \dfrac{x}{y} \in \widehat{\mathbb{C}}$.
- $f(z) = \dfrac{U(x,y)}{V(x,y)}$.
- $p(z) = \dfrac{\Phi(x,y)}{y^{k+2}}$.

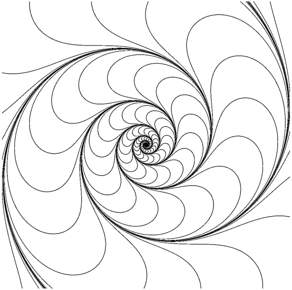
Proposition

The non linearity of $z : \mathbf{S}_{\vec{v}} \to \mathbf{C}$ is

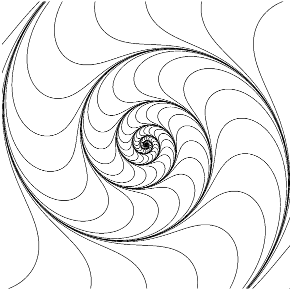
$$\nu := \left(\frac{p'(z)}{p(z)} - \frac{k}{z - f(z)}\right) \mathrm{d}z.$$

Affine surface of a homogeneous vector field

- Singularities of ν are characteristic directions.
- Assume there is a simple pole and let ρ be the residue.



$\mathrm{Re}(\rho) > 1$



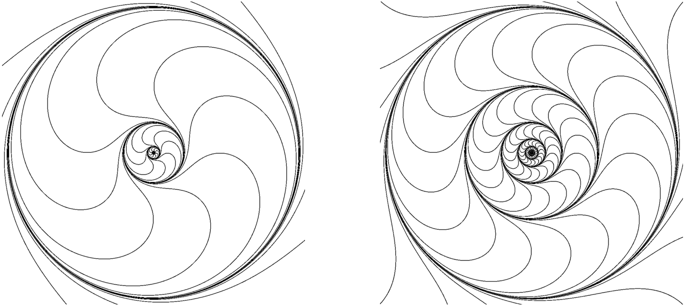
$\mathrm{Re}(\rho) < 1$

Proposition (Écalle, Hakim)

If ν has a simple pole and $\mathrm{Re}(\rho) > 1$, there is a parabolic domain on which orbits converge to **0** tangentially to the characteristic direction.

Affine surface of a homogeneous vector field

- Singularities of ν are characteristic directions.
- Assume there is a simple pole and let ρ be the residue.



$\rho = 1 - 2\mathrm{i}$ $\rho = 1 - 4\mathrm{i}$

Proposition (Rivi,Rong)

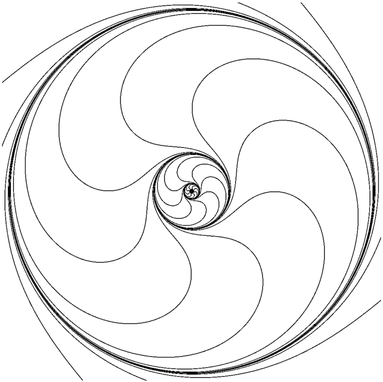
If ν has a simple pole and $\mathrm{Re}(\rho) = 1$, there is a parabolic domain on which orbits converge to **0** spiralling around the characteristic direction.

Closed geodesics

- A geodesic $\delta : I \to \mathbf{S}$ is *closed* if there exists $\lambda \in (0, +\infty)$ and $t_0 < t_1$ in I such that

$$\delta(t_1) = \delta(t_0) \quad \text{and} \quad \dot\delta(t_1) = \lambda\dot\delta(t_0).$$

- Such a geodesic is *attracting* if $\lambda \in (0, 1)$.



Spiralling domains associated to attracting geodesics

- If an affine surface contains an attracting closed geodesic, it contains an *attracting dilation cylinder* foliated by attracting closed geodesic.

Proposition (work in progress with Raissy)

Assume $F(\boldsymbol{x}) = \boldsymbol{x} + \vec{\boldsymbol{v}}(\boldsymbol{x})$ with $\vec{\boldsymbol{v}}$ homogeneous. If $\mathbf{S}_{\vec{\boldsymbol{v}}}$ contains an attracting dilation cylinder $\mathcal{C}$, then F has a spiralling domain in which orbits converge to $\mathbf{0}$, spiralling towards an attracting closed geodesic of $\mathcal{C}$.

Proposition

Assume $a \in \mathbb{R} \setminus \{0\}$ and

$$\vec{\boldsymbol{v}} := \big(y^2 + ax(x-y)\big)\partial_x + \big(x^2 + ay(x-y)\big)\partial_y.$$

Then, $\mathbf{S}_{\vec{\boldsymbol{v}}}$ contains infinitely many non homotopic attracting dilation cylinders.

Neutral geodesics

Theorem (work in progress with Raissy)

If $a \in \left(\frac{k+1}{2}, +\infty\right)$, *the polynomial endomorphisms*

$$F_a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^{k+1} \\ x^{k+1} \end{pmatrix} + ax^k y^k \begin{pmatrix} x \\ y \end{pmatrix}$$

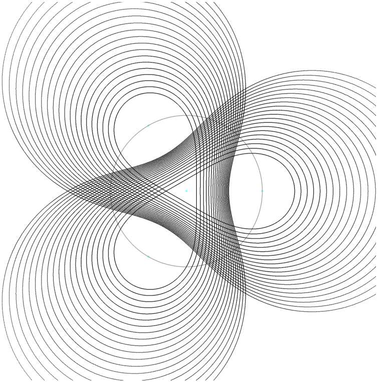
has infinitely many spiralling domains contained in distinct Fatou components.

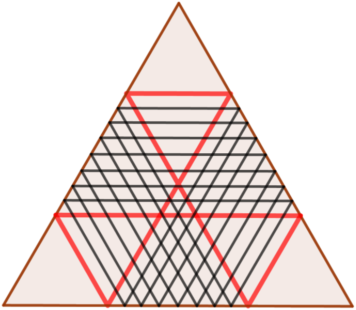
- Polygonal billiards.
- $k = 1$: equilateral triangle
- $k = 3$: pentagonal billiard.

Polygonal billiards

Proposition (Valdez)

The real-time dynamics of $y^{k+1}\partial_x + x^{k+2}\partial_y$ is controlled by the billiard dynamics in a regular polygon with $k+2$ sides.





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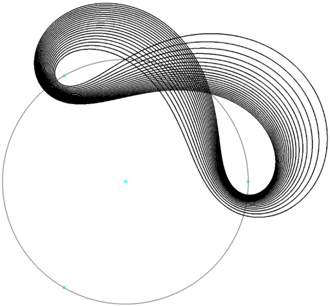
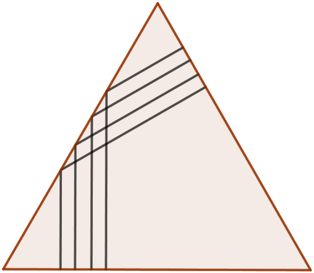
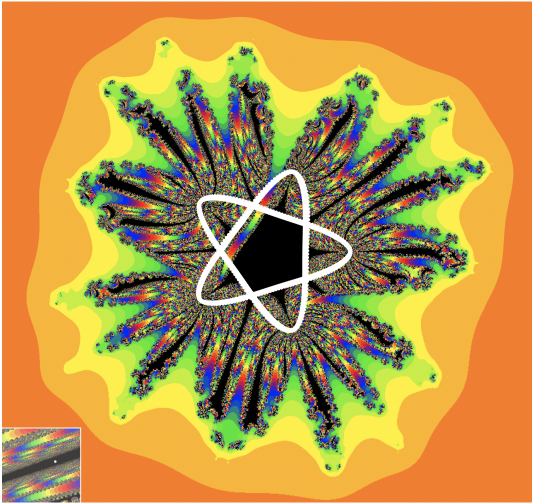
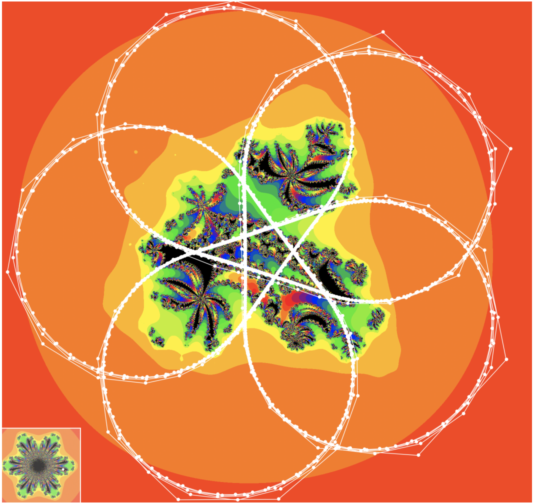




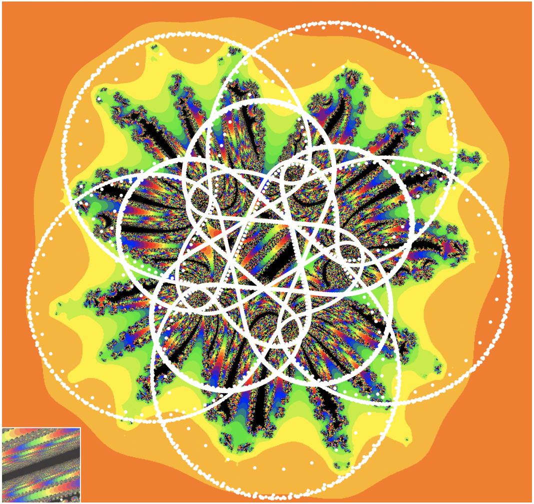
Pentagonal billiards



Pentagonal billiards

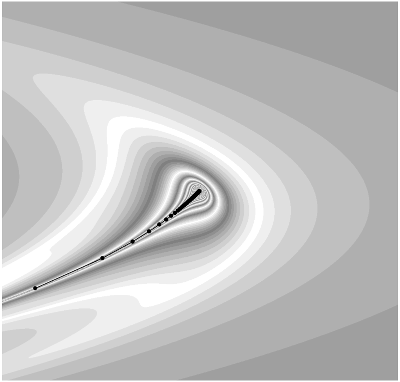


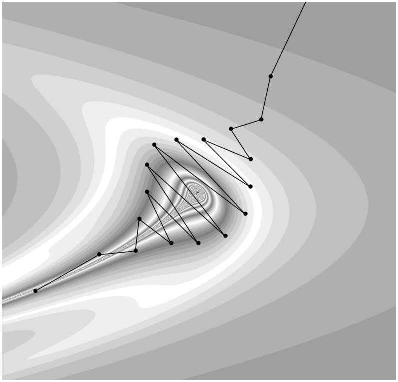
Pentagonal billiards



Hénon maps

- $H_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x + y^2 \end{pmatrix}.$



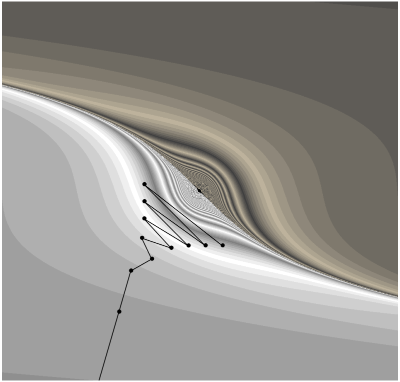


Question

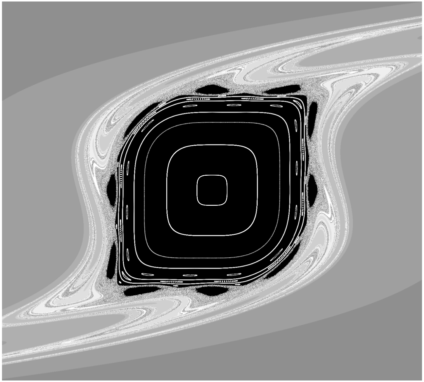
Can we describe the dynamics of H_2 near the origin in $\mathbb{C}^2$?

Hénon maps

- $H_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+y^3 \end{pmatrix}$.



$\{(x,y)\in\mathbb{R}^2\}$



$\{(x,y)\in(1+\mathrm{i})\mathbb{R}\times(-1+\mathrm{i})\mathbb{R}\}$

Hénon maps

- H_3 has small cycles.
- H_3 has Herman rings.

Question

Does H_2 have small cycles?

Question

Does H_2 have Herman rings?

Numerical experiments

- Repelling parabolic curve:

$$\Phi(z) = \lim_{n\to+\infty} H_2^{\circ n}\left(\frac{1}{n-z}, \frac{1}{n-z}\right).$$

- $C = \frac{1}{\sqrt[3]{2}}\mathrm{B}\left(\frac{1}{3}, \frac{1}{3}\right)$, $\alpha = \frac{1}{2} + \frac{1}{C}$ and $\beta = \frac{1}{2} - \frac{1}{C}$.

Conjecture

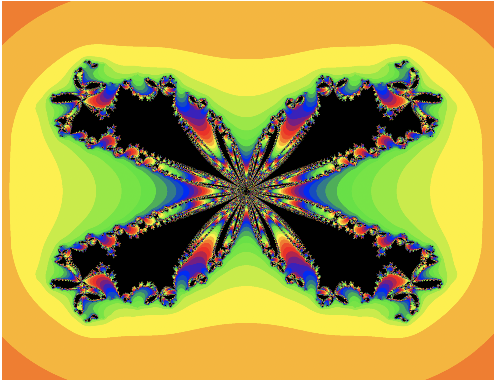
There exists a constant $\gamma \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} H_2^{\circ n}\left(\frac{C}{n-x}, -\frac{C}{n-y}\right) = \Phi(\alpha x + \beta y + \gamma).$$

Question

What can we deduce from such a result?

Thank you for your attention



Happy birthday Mitsu