The semiclassical Garding inequality

We give a proof of the semiclassical Garding inequality (Theorem 4.1) using as the only black box the Calderon-Vaillancourt Theorem.

1 Anti-Wick quantization

For $(q,p) \in \mathbb{R}^d \times \mathbb{R}^d$ define $U(q,p)$ by

$$U(q,p)\varphi(x) = e^{ix\cdot p}\varphi(x-q),$$

say for φ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Fix then a function η such that

$$\eta \in \mathcal{S}(\mathbb{R}^d), \qquad ||\eta||_{L^2(\mathbb{R}^d)} = 1, \tag{1.1}$$

and define the family of Schwartz functions $(\eta_{q,p})_{(q,p)\in\mathbb{R}^{2d}}$ by

$$\eta_{q,p} = U(q,p)\eta.$$

Denote the L^2 inner product by

$$(\psi,\varphi)_{L^2} = \int_{\mathbb{R}^d} \overline{\psi}\varphi.$$

Lemma 1.1. *For all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the function $T\varphi$ defined by*

$$T\varphi(q,p) := (\eta_{q,p},\varphi)_{L^2}$$

belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Furthermore, the linear map

$$T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{2d})$$

is continuous.

Proof. For all $\alpha, \beta \in \mathbb{N}^d$, we have

$$\partial_p^\beta \partial_q^\alpha T\varphi(q,p) = (-1)^{|\alpha|}(-i)^{|\beta|} \int e^{-ix\cdot p}(\partial^\alpha \overline{\eta})(x-q)x^\beta \varphi(x)dx.$$

If we multiply this expression by q^δ, with $\delta \in \mathbb{N}^d$, and write

$$q^\delta = (q-x+x)^\delta = \sum_{\delta'+\delta''=\delta} \frac{\delta!}{\delta'!\delta''!}(q-x)^{\delta'}x^{\delta''},$$

we see that $q^\delta \partial_p^\beta \partial_q^\alpha T\varphi$ is a linear combination of

$$\int e^{-ix\cdot p}(q-x)^{\delta'}(\partial^\alpha \overline{\eta})(x-q)x^{\beta+\delta''}\varphi(x)dx.$$

Finally, by integrations by part we obtain that $p^\mu q^\delta \partial_p^\beta \partial_q^\alpha T\varphi$ is a linear combination of

$$\int e^{-ix\cdot p}\partial_x^\mu \left((q-x)^{\delta'}(\partial^\alpha\overline{\eta})(x-q)x^{\beta+\delta''}\varphi(x)\right)dx$$

and the result follows easily. $\square$

The formal adjoint T^* which maps *a priori* $\mathcal{S}'(\mathbb{R}^{2d})$ to $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$(\Psi, T\varphi)_{L^2(\mathbb{R}^{2d})} = (T^*\Psi, \varphi)_{L^2(\mathbb{R}^d)},$$

where the inner product in the right hand has to be taken in the distributions sense. However, if $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$, this is actually a standard integral for Fubini's Theorem easily shows that

$$\begin{aligned}(\Psi, T\varphi)_{L^2(\mathbb{R}^{2d})} &= \iint \overline{\Psi(q,p)}\left(\int \overline{\eta_{q,p}(x)}\varphi(x)dx\right)dqdp \\ &= \int \overline{\left(\iint \Psi(q,p)\eta_{q,p}(x)dqdp\right)}\varphi(x)dx,\end{aligned}$$

hence that

$$(T^*\Psi)(x) = \iint \Psi(q,p)\eta_{q,p}(x)dqdp.$$

Lemma 1.2. *The map T^* is continuous from $\mathcal{S}(\mathbb{R}^{2d})$ to $\mathcal{S}(\mathbb{R}^d)$.*

The proof of this lemma is elementary and uses the same techniques as Lemma 1.1. It is left to the reader as an exercise. The important result is the following inversion formula.

Proposition 1.3. *For all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,*

$$T^*T\varphi = (2\pi)^d\varphi.$$

Proof. Fix $x \in \mathbb{R}^d$ and $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi \equiv 1$ near 0. The result then follows by dominated convergence and Fubini's Theorem since

$$\begin{aligned}T^*T\varphi(x) &= \lim_{\epsilon\to 0}\iiint e^{-ip\cdot y}\overline{\eta}(y-q)\varphi(y)e^{ip\cdot x}\eta(x-q)\chi(\epsilon p)dydqdp \\ &= \lim_{\epsilon\to 0}\iint \epsilon^{-d}\widehat{\chi}\left(\frac{y-x}{\epsilon}\right)\overline{\eta}(y-q)\eta(x-q)\varphi(y)dydq \\ &= \lim_{\epsilon\to 0}\iint \widehat{\chi}(z)\overline{\eta}(x+\epsilon z-q)\eta(x-q)\varphi(x+\epsilon z)dzdq \\ &= \int \widehat{\chi}\times ||\eta||_{L^2}^2 \times \varphi(x) = (2\pi)^d\varphi(x),\end{aligned}$$

using the second assumption in (1.1). $\square$

Corollary 1.4. *For all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\begin{aligned}(\psi,\varphi)_{L^2(\mathbb{R}^d)} &= (2\pi)^{-d}(T\psi, T\varphi)_{L^2(\mathbb{R}^{2d})} \\ &= (2\pi)^{-d}\iint \overline{(\eta_{q,p},\psi)}_{L^2}(\eta_{q,p},\varphi)_{L^2}dqdp.\end{aligned}$$

In particular

$$||\varphi||_{L^2(\mathbb{R}^d)}^2 = (2\pi)^{-d}\iint |(\eta_{q,p},\varphi)_{L^2}|^2 dqdp.$$

Proof. The result is obtained either by polarization from $||\varphi||^2_{L^2(\mathbb{R}^d)} = (2\pi)^{-d}||T\varphi||^2_{L^2(\mathbb{R}^{2d})}$ or by inserting the decomposition $\varphi = \iint (\eta_{q,p},\varphi)_{L^2}\eta_{q,p}dqdp$ into $(\psi,\varphi)_{L^2(\mathbb{R}^d)}$ and using Fubini's Theorem. $\square$

Now for all $a \in L^\infty(\mathbb{R}^{2d})$,

$$B_a(\psi,\varphi) := (2\pi)^{-d}\iint a(q,p)\overline{(\eta_{q,p},\psi)}_{L^2}(\eta_{q,p},\varphi)dqdp$$

is a well defined sesquilinear form on $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ and Corollary 1.4 together with the Cauchy-Schwarz inequality show that

$$|B_a(\psi,\varphi)| \le ||a||_{L^\infty(\mathbb{R}^{2d})}||\varphi||_{L^2(\mathbb{R}^d)}||\psi||_{L^2(\mathbb{R}^d)}. \tag{1.2}$$

Therefore there exists a unique bounded operator A on $L^2(\mathbb{R}^d)$ such that

$$B_a(\psi,\varphi) = (\psi, A\varphi)_{L^2(\mathbb{R}^d)},$$

for all $\varphi,\psi \in \mathcal{S}(\mathbb{R}^d)$.

Definition 1.5. *The operator A is the anti-Wick quantization of a and is denoted by*

$$A = Op^{\mathrm{aW}}(a).$$

As a straightforward consequence of (1.2), we see that

$$||Op^{\mathrm{aW}}(a)||_{L^2\to L^2} \le ||a||_{L^\infty}. \tag{1.3}$$

Note that Corollary 1.4 also shows that

$$Op^{\mathrm{aW}}(1) = I.$$

Next, by writting

$$(\psi, Op^{\mathrm{aW}}(a)\varphi)_{L^2} = B_a(\psi,\varphi) = \overline{B_{\overline{a}}(\varphi,\psi)} = \overline{(\varphi, Op^{\mathrm{aW}}(\overline{a})\psi)}_{L^2}$$

we get

$$Op^{\mathrm{aW}}(a)^* = Op^{\mathrm{aW}}(\overline{a}). \tag{1.4}$$

The following last property is then straightforward

$$a \ge 0 \ \ a.e. \qquad \Rightarrow \qquad Op^{\mathrm{aW}}(a) \ge 0, \tag{1.5}$$

and is the most important one. Note that all these properties are independent of η.

2 Wigner's function

Definition 2.1 (Pseudo-differential operator)**.** *Given $a \in \mathcal{S}(\mathbb{R}^{2d})$, the pseudo-differential operator of symbol a, $Op(a)$, is the operator defined by*

$$Op(a)\varphi(x) = (2\pi)^{-d}\int e^{ix\cdot\xi}a(x,\xi)\widehat{\varphi}(\xi)d\xi,$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Pseudo-differential operators are actually defined for wider classes of symbols than $\mathcal{S}(\mathbb{R}^{2d})$ but the latter will be sufficient here.

Fix now $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$. For any $a \in \mathcal{S}(\mathbb{R}^{2d})$, a straightforward application of Fubini's Theorem shows that

$$(\psi, Op(a)\varphi)_{L^2(\mathbb{R}^d)} = \iint_{\mathbb{R}^{2d}} aW_{\psi,\varphi},$$

where

$$W_{\psi,\varphi}(x,\xi) = (2\pi)^{-d} e^{ix\cdot\xi}\overline{\psi(x)}\widehat{\varphi}(\xi). \tag{2.1}$$

Definition 2.2. *$W_{\psi,\varphi}$ is the Wigner function associated to φ and ψ.*

In the sequel we fix a single η satisfying (1.1) and set

$$W(x,\xi) = W_{\eta,\eta}(x,\xi). \tag{2.2}$$

We further assume that

$$\eta \quad \text{is even and real valued,} \tag{2.3}$$

which implies the easily verified property that

$$W \quad \text{is even.} \tag{2.4}$$

The important relationship between anti-Wick quantization and the Wigner function is the following one.

Proposition 2.3. *For all $a \in \mathcal{S}(\mathbb{R}^{2d})$, we have*

$$Op^{\mathrm{aW}}(a) = Op(a * \overline{W}).$$

Proof. By (2.1), (2.3) and (2.4), we have

$$a * \overline{W}(x,\xi) = (2\pi)^{-d} \iint a(q+x, p+\xi) e^{-iq\cdot p}\eta(q)\widehat{\eta}(p)dqdp$$

so that

$$Op(a*\overline{W})\varphi(x) = (2\pi)^{-2d} \iiint e^{ix\cdot\xi - iq\cdot p} a(q+x, p+\xi)\eta(q)\widehat{\eta}(p)\widehat{\varphi}(\xi)dqdpd\xi.$$

On the other hand, using the fact that

$$(\eta_{q,p}, \varphi)_{L^2} = (2\pi)^{-d} \int e^{iq\cdot(\xi-p)}\widehat{\eta}(\xi-p)\widehat{\varphi}(\xi)d\xi,$$

where we also used (2.3), we have

$$\begin{aligned}
Op^{\mathrm{aW}}(a)\varphi(x) &= (2\pi)^{-d} \iint (\eta_{q,p}, \varphi)_{L^2} a(q,p)\eta_{q,p}(x)dx \\
&= (2\pi)^{-2d} \iiint e^{iq\cdot(\xi-p)}\widehat{\eta}(\xi-p)\widehat{\varphi}(\xi)a(q,p)e^{ip\cdot x}\eta(x-q)dqdpd\xi \\
&= (2\pi)^{-2d} \iiint e^{-i(q+x)\cdot p}\widehat{\eta}(-p)\widehat{\varphi}(\xi)a(q+x,p+\xi)e^{i(p+\xi)\cdot x}\eta(-q)dqdpd\xi \\
&= (2\pi)^{-2d} \iiint e^{-iq\cdot p + ix\cdot\xi}\widehat{\eta}(p)\widehat{\varphi}(\xi)a(q+x,p+\xi)\eta(q)dqdpd\xi
\end{aligned}$$

by changing q into $q+x$ and p into $p+\xi$ to get the third line and using (2.3) for the last one. This completes the proof. $\square$

3 Semiclassical scaling

For $h \in (0,1]$, the semiclassical pseudo-differential quantization is given by

$$Op_h(a)\varphi(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} a(x, h\xi)\widehat{\varphi}(\xi) d\xi.$$

In other words, if we set

$$a_h(x,\xi) := a(x, h\xi)$$

we have, according to Definition 2.1,

$$Op_h(a) = Op(a_h).$$

Next, if η satisfies (1.1) and (2.3) then so does

$$\eta_h(x) := h^{-d/4}\eta\left(\frac{x}{h^{1/2}}\right),$$

and the corresponding Wigner function is

$$W_{\eta_h,\eta_h}(x,\xi) \quad = \quad (2\pi)^{-d} e^{ix\cdot\xi}\eta\left(\frac{x}{h^{1/2}}\right)\widehat{\eta}(h^{1/2}\xi).$$

Lemma 3.1. *For all $a \in \mathcal{S}(\mathbb{R}^{2d})$ and all $h \in (0,1]$*

$$(a_h * \overline{W}_{\eta_h,\eta_h})(x,\xi) = (a * \overline{W}_h)(x, h\xi),$$

with

$$W_h(x,\xi) = h^{-d} W\left(\frac{x}{h^{1/2}}, \frac{x}{h^{1/2}}\right).$$

where W is defined by (2.2).

We omit the very simple proof of this lemma which follows from an elementary change of variable.

If we finally define $Op_h^{\mathrm{aW}}(a)$ to be the anti-Wick quantization of a associated to η_h namely

$$Op_h^{\mathrm{aW}}(a)\varphi = (2\pi)^{-d} \iint a(q,p)\left((\eta_h)_{q,p}, \varphi\right)_{L^2} (\eta_h)_{q,p}(x) dq dp,$$

then Proposition 2.3 and Lemma 3.1 show that

$$Op_h^{\mathrm{aW}}(a) = Op_h\left(a * \overline{W}_h\right). \tag{3.1}$$

4 The semiclassical Garding inequality

Theorem 4.1. *There exists $C, N \geq 0$ such that, for all $a \in \mathcal{S}(\mathbb{R}^{2d})$ satisfying*

$$a \geq 0$$

we have

$$\mathrm{Re}\left(\varphi, Op_h(a)\varphi\right)_{L^2} \geq -Ch \max_{|\alpha+\beta|\leq N} ||\partial_x^\alpha \partial_\xi^\beta a||_{L^\infty} ||\varphi||_{L^2}^2,$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and all $h \in (0,1]$.

The non elementary tool for the proof is the Calderon-Vaillancourt Theorem which we recall for completeness.

Theorem 4.2. *There exists $C, N_{\rm CV} \geq 0$ such that, for all $a \in \mathcal{S}(\mathbb{R}^{2d})$, we have*

$$||Op_h(a)\varphi||_{L^2} \leq C \max_{|\alpha+\beta|\leq N_{\rm CV}} ||\partial_x^\alpha \partial_\xi^\beta a||_{L^\infty} ||\varphi||_{L^2},$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and all $h \in (0,1]$.

We now prove Theorem 4.1.

Proposition 4.3. *There exists $C \geq 0$ such that*

$$||a * \overline{W}_h - a||_{L^\infty} \leq Ch \max_{|\alpha+\beta|=2} ||\partial_x^\alpha \partial_\xi^\beta a||_{L^\infty},$$

for all $a \in \mathcal{S}(\mathbb{R}^{2d})$ and all $h \in (0,1]$.

Proof. Set for simplicity $W^{\alpha\beta}(x,\xi) := x^\alpha \xi^\beta W(x,\xi)$ and consider its scaled version

$$W_h^{\alpha\beta}(x,\xi) := h^{-d} W^{\alpha\beta}\left(\frac{x}{h^{1/2}}, \frac{\xi}{h^{1/2}}\right).$$

Note that

$$x^\alpha \xi^\beta W_h = h^{\frac{|\alpha|+|\beta|}{2}} W_h^{\alpha\beta}. \tag{4.2}$$

Recall next that $\iint W_h = 1$ and observe that, since W_h is even,

$$\iint x_j W_h = \iint \xi_j W_h = 0 \qquad j = 1, \ldots, d.$$

Thus, by using the Taylor formula

$$a(q+x, p+\xi) = a(q,p) + \sum_{j=1}^d x_j \partial_{x_j} a(q,p) + \xi_j \partial_{\xi_j} a(q,p) + \sum_{|\alpha+\beta|=2} x^\alpha \xi^\beta r_{\alpha\beta}(x,\xi,q,p),$$

where $||r_{\alpha\beta}||_{L^\infty(\mathbb{R}^{4d})} \leq C ||\partial_x^\alpha \partial_\xi^\beta a||_{L^\infty(\mathbb{R}^{2d})}$, we see that

$$(a * \overline{W}_h - a)(q,p) = h \sum_{|\alpha+\beta|=2} \iint \overline{W}_h^{\alpha\beta}(x,\xi) r_{\alpha\beta}(x,\xi,q,p) dx d\xi,$$

using (4.2). The result follows since $W_h^{\alpha\beta}$ is a bounded family in $L^1(\mathbb{R}^{2d})$ (as h varies in $(0,1]$) because $W^{\alpha\beta} \in \mathcal{S}(\mathbb{R}^{2d})$. $\square$

Proof of Theorem 4.1. It suffices to write that

$$\begin{aligned} Op_h(a) &= Op_h\left(a - a * \overline{W}_h\right) + Op_h(a * \overline{W}_h) \\ &= Op_h\left(a - a * \overline{W}_h\right) + Op_h^{\rm aW}(a) \end{aligned}$$

by (3.1). The second operator is non negative (and self-adjoint) whereas the first one satisfies the estimate

$$\begin{aligned} ||Op_h\left(a - a * \overline{W}_h\right)\varphi||_{L^2} &\leq C \max_{|\alpha+\beta|\leq N_{\rm CV}} ||\partial_x^\alpha \partial_\xi^\beta (a - a * \overline{W}_h)||_{L^\infty} ||\varphi||_{L^2} \\ &\leq Ch \max_{|\alpha+\beta|\leq N_{\rm CV}+2} ||\partial_x^\alpha \partial_\xi^\beta a||_{L^\infty} ||\varphi||_{L^2} \end{aligned}$$

by the Calderon-Vaillancourt Theorem and Proposition 4.3. This completes the proof. $\square$