The Calderon-Vaillancourt Theorem

What follows is a completely self contained proof of the Calderon-Vaillancourt Theorem on the L^2 boundedness of pseudo-differential operators.

1 The result

Definition 1.1. *The symbol class S^0_{00} is the space of smooth functions b on $\mathbb{R}^d \times \mathbb{R}^d$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta b(x,\xi)| \leq C_{\alpha\beta}, \qquad x,\xi \in \mathbb{R}^d,$$

for all $\alpha, \beta \in \mathbb{N}^d$.

For the next definition, we recall that $\widehat{\varphi}(\xi) = \int e^{-iy\cdot\xi}\varphi(y)dy$.

Definition 1.2 (Pseudo-differential operator)**.** *Given $b \in S^0_{00}$, the pseudo-differential operator of symbol b, $Op(b)$, is the operator defined by*

$$Op(b)\varphi(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} b(x,\xi)\widehat{\varphi}(\xi)d\xi,$$

for all φ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

Theorem 1.3 (Calderon-Vaillancourt)**.** *There exists $C, N_{\mathrm{CV}} > 0$ such that for all $b \in S^0_{00}$ and all $\varphi \in \mathcal{S}(\mathbb{R}^d)$*

$$||Op(b)\varphi||_{L^2(\mathbb{R}^d)} \leq C \max_{|\alpha+\beta|\leq N_{\mathrm{CV}}} ||\partial_x^\alpha \partial_\xi^\beta b||_{L^\infty} ||\varphi||_{L^2(\mathbb{R}^d)}. \tag{1.1}$$

The next sections are devoted to the proof of this theorem.

2 The Schur estimate

Let $K \in \mathcal{S}(\mathbb{R}^{2d})$ and consider the associated operator

$$Au(x) = \int_{\mathbb{R}^d} K(x,y)u(y)dy,$$

defined for any $u \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$. Introduce the norms

$$||A||_{l-\mathrm{Schur}} := \sup_x \int |K(x,y)|dy, \qquad ||A||_{r-\mathrm{Schur}} := \sup_y \int |K(x,y)|dx.$$

Proposition 2.1. *For all $p \in [1,\infty]$ and all $u \in L^p(\mathbb{R}^d)$,*

$$||Au||_{L^p} \leq ||A||_{l-\text{Schur}}^{1-1/p} ||A||_{r-\text{Schur}}^{1/p} ||u||_{L^p},$$

with the convention that $C_1^{1-1/p} C_2^{1/p} = C_1$ if $p = \infty$.

Proof. Assume that $p < \infty$. Observe that for each $x \in \mathbb{R}^d$, Hölder's inequality yields

$$\begin{aligned} \int |K(x,y)u(y)|dy &= \int |K(x,y)|^{1-1/p} |K(x,y)|^{1/p} |u(y)|dy \\ &\leq \left(\int |K(x,y)|dy \right)^{1-1/p} \left(\int |K(x,y)||u(y)|^p dy \right)^{1/p}. \end{aligned}$$

Hence we have

$$\left(\int |K(x,y)u(y)|dy \right)^p \leq ||A||_{l-\text{Schur}}^{p-1} \int |K(x,y)||u(y)|^p dy.$$

By integrating this inequality with respect to x and using the Fubini Theorem, we obtain the result. If $p = \infty$, the estimate is obvious. □

3 The Cotlar-Knapp-Stein criterion

Consider a countable family $(A_j)_{j\in\mathbb{N}}$ of bounded operators on $L^2(\mathbb{R}^d)$. We will actually assume that each A_j is compact, which will not be a restriction for the final application. The only reason for this (non necessary) additional condition is that the spectral theorem for self-adjoint operators is maybe more elementary, or at least more popular, for compact operators than the general theorem of Von Neumann.

For simplicity, $||\cdot||$ denotes the operator norm on $L^2(\mathbb{R}^d)$.

Proposition 3.1. *Assume that*

$$\sup_j \sum_k ||A_j^* A_k||^{1/2} \leq M, \qquad \sup_k \sum_j ||A_k A_j^*||^{1/2} \leq M.$$

Then, if we set

$$S_N = \sum_{j \leq N} A_j,$$

we have

$$||S_N|| \leq M, \qquad N \in \mathbb{N}.$$

Proof. Consider the self-adjoint (and compact) operator

$$H_N := S_N^* S_N.$$

The Spectral Theorem then yields

$$||S_N||^2 = \sup_{||\varphi||_{L^2}=1} (S_N^* S_N \varphi, \varphi) = ||H_N|| = \max \sigma(H_N)$$

as well as

$$||H_N^m|| = ||H_N||^m, \qquad m \in \mathbb{N},$$

so that

$$||S_N|| = ||H_N^m||^{1/2m}. \tag{3.1}$$

One then writes

$$H_N^m = \sum_{j_1}\sum_{k_1}\cdots\sum_{j_m}\sum_{k_m} A_{j_1}^* A_{k_1}\cdots A_{j_m}^* A_{k_m},$$

where all indices are taken between 0 and $N-1$, and observes that

$$||A_{j_1}^* A_{k_1}\cdots A_{j_m}^* A_{k_m}^*|| \leq \begin{cases} ||A_{j_1}^* A_{k_1}||\cdots||A_{j_m}^* A_{k_m}|| \\ \text{and} \\ ||A_{j_1}^*||||A_{k_m}||||A_{k_1}A_{j_2}^*||\cdots||A_{k_{m-1}}A_{j_m}^*|| \end{cases}.$$

Therefore, since $\min(a,b) \leq (ab)^{1/2}$ for all $a, b \geq 0$ and $||A_k|| \leq M$ for all k, we have

$$\begin{aligned} ||H_N^m|| &\leq M\sum_{j_1}\sum_{k_1}||A_{j_1}^* A_{k_1}||^{1/2}\sum_{j_2}||A_{k_1}A_{j_2}^*||^{1/2}\cdots\sum_{j_m}||A_{k_{m-1}}A_{j_m}^*||^{1/2}\sum_{k_m}||A_{j_m}^* A_{k_m}||^{1/2} \\ &\leq M\sum_{j_1<N} M^{2m-1}. \end{aligned}$$

Using (3.1), we thus have $||S_N|| \leq N^{1/2m}M$ and get the result by letting $m \to \infty$. □

4 Phase space translations

For $(q,p) \in \mathbb{R}^d \times \mathbb{R}^d$ define $U(q,p)$ by

$$U(q,p)\varphi(x) = e^{ix\cdot p}\varphi(x-q),$$

say for φ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. These operators are obviously unitary on $L^2(\mathbb{R}^d)$. They also satisfy the relations

$$U(q,p)^* = e^{-iq\cdot p}U(-q,-p), \tag{4.1}$$

and

$$U(q_1,p_1)U(q_2,p_2) = e^{-ip_1\cdot q_2}U(q_1+q_2,p_1+p_2), \tag{4.2}$$

which are both easily seen by elementary calculations.

We also define $\tau_{q,p}$ by

$$\tau_{q,p}a(x,\xi) = a(x-q,\xi-p).$$

Lemma 4.1. *For all* $(q,p) \in \mathbb{R}^d \times \mathbb{R}^d$ *and* $a \in \mathcal{S}(\mathbb{R}^{2d})$,

$$U(q,p)Op(a)U(q,p)^* = Op\left(\tau_{q,p}a\right).$$

Proof. It follows from (4.1) and the Fubini Theorem since

$$\begin{aligned} U(q,p)Op(a)U(q,p)^*\varphi(x) &= e^{ip\cdot x}(2\pi)^{-d}\iint e^{i(x-q)\cdot\xi}a(x-q,\xi)e^{-iy\cdot\xi}e^{-iq\cdot p}e^{-ip\cdot y}\varphi(y+q)dyd\xi \\ &= (2\pi)^{-d}\iint e^{i(x-z)\cdot\zeta}a(x-q,\zeta-p)\varphi(z)dzd\zeta \;=\; Op\left(\tau_{q,p}a\right)\varphi(x) \end{aligned}$$

by the change of variables $\xi = \zeta - p$ and $y + q = z$. □

Lemma 4.2. *There exists* $\chi \in C_0^\infty(\mathbb{R}^{2d})$ *such that*

$$\sum_{(q,p)\in\mathbb{Z}^{2d}} \tau_{q,p}\chi \equiv 1.$$

Proof. The result has nothing to do with the dimension and easily follows from the existence of $\theta \in C_0^\infty(\mathbb{R})$ such that $1 = \sum_{j\in\mathbb{Z}} \theta(x-j)$. We can construct the latter by choosing $\theta_0 \in C_0^\infty(\mathbb{R})$ such that $\theta \geq 0$ and $\theta \equiv 1$ on $[0,1]$ so that the following smooth and 1 periodic function

$$\Theta(x) := \sum_{j\in\mathbb{N}} \theta_0(x-j)$$

is bounded from below by 1 since $x-j$ belongs to $[0,1]$ for some j. One then obtains θ by considering $\theta = \theta_0/\Theta$ and then χ with $\chi = \theta\otimes\cdots\otimes\theta$. $\square$

5 Elementary symbolic calculus

In this section we give the minimal symbolic calculus properties required for the proof of the Calderon-Vaillancourt Theorem.

We start by observing that, if $a \in \mathcal{S}(\mathbb{R}^{2d})$, the operator $Op(a)$ (see Definition 1.2) has a kernel

$$K_{Op(a)}(x,y) = (2\pi)^{-d}\widehat{a}(x, y-x), , \tag{5.1}$$

where $\widehat{a}$ denotes the Fourier transform of a with respect to ξ. This follows from the Fubini theorem by expanding $\widehat{\varphi}(\xi)$ into $\int e^{-iy\cdot\xi}\varphi(y)dy$ in the definition of $Op(a)\varphi$.

Clearly, this kernel belongs to $\mathcal{S}(\mathbb{R}^{2d})$ and thus so does the kernel of $Op(a)^*$ which is given by

$$K_{Op(a)^*}(x,y) = (2\pi)^{-d}\overline{\widehat{a}}(y, x-y). \tag{5.2}$$

It will be sufficient for the present purpose to show that $Op(a)^*$ is of the form $Op(a^*)$ for some Schwartz function a^* depending continuously on a in the Schwartz space. To describe this continuity, we introduce the (semi)norms of the Schwartz space,

$$||a||_{N,\mathcal{S}} := \max_{|\alpha+\beta|\leq N} ||\langle x\rangle^N \langle\xi\rangle^N \partial_x^\alpha \partial_\xi^\beta a||_{L^\infty}, \tag{5.3}$$

where $N \in \mathbb{N}$.

Proposition 5.1 (Stablity of $Op(\mathcal{S}(\mathbb{R}^{2d}))$ by adjunction)**.** *There exists an antilinear map* $a \mapsto a^*$ *on* $\mathcal{S}(\mathbb{R}^{2d})$ *such that*

$$(Op(a)^*\psi, \varphi)_{L^2} = (\psi, Op(a^*)\varphi)_{L^2}, \qquad \psi,\varphi \in \mathcal{S}(\mathbb{R}^d),$$

and continuous in the sense that for all $N_1 \in \mathbb{N}$ *there exists* $C > 0$ *and* $N_2 \in \mathbb{N}$ *such that*

$$||a^*||_{N_1,\mathcal{S}} \leq C||a||_{N_2,\mathcal{S}}. \tag{5.4}$$

Proof. By (5.2), we look for a Schwartz function a^* such that

$$\widehat{a^*}(x, y-x) = \overline{\widehat{a}}(y, x-y),$$

ie $\widehat{a^*}(x,z) = \overline{\widehat{a}}(x+z,-z) = \overline{\widehat{a}}(x+z,z)$. Taking the inverse Fourier transform, we get

$$a^*(x,\xi) = (2\pi)^{-d} \int e^{i\xi\cdot z} \overline{\widehat{a}}(x+z,z)dz.$$

This function depends continuously on a in the Schwartz space since, by expanding $x^\delta = (x+z-z)^\delta$ by the binomial formula, one easily checks that $x^\delta \xi^\mu \partial_x^\alpha \partial_\xi^\beta a^*$ is a linear combination of

$$\int e^{i\xi\cdot z} z^{\delta'} (x+z)^{\delta''} \partial_z^\mu \left(z^\beta \partial_x^\alpha \overline{\widehat{a}}(x+z,z) \right) dz,$$

with $\delta' + \delta'' = \delta$. This implies that $||x^\delta \xi^\mu \partial_x^\alpha \partial_\xi^\beta a^*||_{L^\infty}$ is bounded by some seminorm of $\widehat{a}$ in $\mathcal{S}(\mathbb{R}^{2d})$ hence by some seminorm of a. □

6 Proof of the theorem

Using Lemma 4.2, we write first

$$b = \sum_{(q,p)\in\mathbb{Z}^{2d}} b\tau_{q,p}\chi = \sum_{(q,p)} \tau_{q,p} \left(\chi \tau_{-q,-p} b \right). \tag{6.1}$$

For simplicity, we set

$$b_{q,p} = \chi \tau_{-q,-p} b.$$

We also introduce

$$\begin{aligned} B_{q,p} &:= Op\left(b\tau_{q,p}\chi\right) && (6.2)\\ &= U(q,p) Op(b_{q,p}) U(q,p)^*. && (6.3) \end{aligned}$$

the second line being a consequence of Lemma 4.1.

To be in position to use the Cotlar-Knapp-Stein criterion, we mainly need the following result.

Proposition 6.1. *There exist $C > 0$ and $N_{\mathrm{CV}} > 0$ such that, for all $(q_1,p_1),(q_2,p_2) \in \mathbb{Z}^{2d}$ and all $b \in S^0_{00}$, we have*

$$||B_{q_1,p_1} B^*_{q_2,p_2}||_{L^2\to L^2} \leq C \langle q_1 - q_2 \rangle^{-2d-2} \langle p_1 - p_2 \rangle^{-2d-2} \max_{|\alpha+\beta|\leq N_{\mathrm{CV}}} ||\partial_x^\alpha \partial_\xi b||^2_{L^\infty}, \tag{6.4}$$

and

$$||B^*_{q_1,p_1} B_{q_2,p_2}||_{L^2\to L^2} \leq C \langle q_1 - q_2 \rangle^{-2d-2} \langle p_1 - p_2 \rangle^{-2d-2} \max_{|\alpha+\beta|\leq N_{\mathrm{CV}}} ||\partial_x^\alpha \partial_\xi b||^2_{L^\infty}. \tag{6.5}$$

The proof goes in two steps. The first remark is that the family $(b_{q,p})_{(q,p)\in\mathbb{Z}^{2d}}$ is bounded in $\mathcal{S}(\mathbb{R}^{2d})$. To state this property more precisely, we use the notation (5.3).

Lemma 6.2. *For all $N \in \mathbb{N}$, there exists $C > 0$ such that, for all $(q,p) \in \mathbb{Z}^{2d}$ and all $b \in S^0_{00}$,*

$$||b_{q,p}||_{N,\mathcal{S}} \leq C \max_{|\alpha+\beta|\leq N} ||\partial_x^\alpha \partial_\xi^\beta b||_{L^\infty}. \tag{6.6}$$

Proof. It is an easy consequence of the Leibniz rule, using that $\langle x \rangle^N \langle \xi \rangle^N$ is bounded on the support of χ. □

Lemma 6.3. *For all $D > 0$, there exist $C > 0$ and $N > 0$ such that,*

$$||Op(a_1)U(q,p)Op(a_2)U(q,p)^*|| \leq C\langle q\rangle^{-D}\langle p\rangle^{-D}||a_1||_{N,\mathcal{S}}||a_2||_{N,\mathcal{S}},$$

for all $a_1, a_2 \in \mathcal{S}(\mathbb{R}^d)$ and all $q, p \in \mathbb{Z}^d$.

Proof. By Lemma 4.1, we have $U(q,p)Op(a_2)U(q,p)^* = Op(\tau_{q,p}(a_2))$ so the kernel $K_{q,p}$ of the operator $Op(a_1)U(q,p)Op(a_2)U(q,p)^*$ is given by

$$\begin{aligned} K_{q,p}(x,y) &= (2\pi)^{-2d}\int \widehat{a}_1(x,z-x)\widehat{\tau_{q,p}a}_2(z,y-z)dz \\ &= (2\pi)^{-2d}\int \widehat{a}_1(x,z-x)e^{ip\cdot(z-y)}\widehat{a}_2(z-q,y-z)dz, \end{aligned}$$

and it is not hard to check that it is a Schwartz function. By Proposition 2.1, it is then sufficient to show that

$$|K_{q,p}(x,y)| \leq C_D\langle q\rangle^{-D}\langle p\rangle^{-D}\langle x-y\rangle^{-d-1}||a_1||_{N,\mathcal{S}}||a_2||_{N,\mathcal{S}}.$$

To get the latter, we compute $(x-y)^\alpha q^\beta p^\gamma K_{q,p}$ by expanding

$$q^\beta = (q-z+z-x+x)^\beta, \qquad (x-y)^\alpha = (x-z+z-y)^\alpha$$

using the binomial law, and integrations by part to handle the term p^γ. We obtain a linear combination of integrals of the form

$$\int e^{ip\cdot z}\partial_z^\gamma\left(x^{\beta'}(z-x)^{\beta''+\alpha'}\widehat{a}_1(x,z-x)(y-z)^{\alpha''}(z-q)^{\beta'''}\widehat{a}_2(z-q,y-z)\right)dz,$$

with $\beta' + \beta'' + \beta''' = \beta$ and $\alpha' + \alpha''$. These integrals are bounded by seminorms of a_1 and a_2 (uniformly with respect to $(x-y), q, p$) and the result follows then easily. □

Proof of Proposition 6.1. We have

$$\begin{aligned} ||B_{q_1,p_1}B^*_{q_2,p_2}||_{L^2\to L^2} &= ||U(q_1,p_1)Op(b_{q_1,p_1})U(q_1,p_1)^*U(q_2,p_2)Op(b_{q_2,p_2})^*U(q_2,p_2)||_{L^2\to L^2} \\ &= ||Op(b_{q_1,p_1})U(q_1,p_1)^*U(q_2,p_2)Op(b_{q_2,p_2})^*U(q_2,p_2)^*U(q_1,p_1)||_{L^2\to L^2} \\ &= ||Op(b_{q_1,p_1})U(q_2-q_1,p_2-p_1)Op(b_{q_2,p_2})^*U(q_2-q_1,p_2-p_1)^*||_{L^2\to L^2} \end{aligned}$$

using the unitarity of $U(q_1,p_1)$ to get the second line and the identities (4.1) and (4.2) to get the third one. Then, by writing $Op(b_{q_2,p_2})^* = Op\left(b^*_{q_2,p_2}\right)$ according to Proposition 5.1, Lemma 6.3 gives the estimate

$$||B_{q_1,p_1}B^*_{q_2,p_2}||_{L^2\to L^2} \leq C\langle q_1-q_2\rangle^{-2d-2}\langle p_1-p_2\rangle^{-2d-2}||b_{q_1,p_1}||_{N_1,\mathcal{S}}||b^*_{q_2,p_2}||_{N_1,\mathcal{S}}$$

for some N_1 depending only on d. Using (5.4) and (6.6), the seminorms in the right hand side can be replaced by $\max_{|\alpha+\beta|\leq N_2}||\partial_x^\alpha\partial_\xi^\beta b||^2_{L^\infty}$ and we get (6.4). The proof of (6.5) is similar. □

We recall that we assumed that the operators A_j in Proposition 3.1 were compact[1]. This condition is fullfilled by the operators $B_{q,p}$.

Proposition 6.4. *For all $(q,p) \in \mathbb{Z}^{2d}$, $B_{q,p}$ is compact on $L^2(\mathbb{R}^d)$.*

[1] we recall that is assumption is only for simplicity and can be removed

Proof. Recall that $B_{q,p}$ is given by (6.3). Since $b_{q,p}$ belongs to $C_0^\infty(\mathbb{R}^d)$, the kernel of $Op(b_{q,p})$ belongs to the Schwartz class (see (5.1)), hence to $L^2(\mathbb{R}^{2d})$, and thus is Hilbert-Schmidt. □

Set now $b_N = \sum_{|q|+|p|} b\tau_{q,p}\chi$ so that

$$Op(b_N) = \sum_{|q|+|p|\leq N} B_{q,p}.$$

Lemma 6.5. *For all* $\psi, \varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$(\psi, Op(b_N)\varphi)_{L^2} \to (\psi, Op(b)\varphi)_{L^2}, \qquad N \to \infty.$$

Proof. It suffices to observe that

$$\begin{aligned}(\psi, Op(b_N)\varphi)_{L^2} &= (2\pi)^{-d} \iint e^{ix\cdot\xi}\overline{\psi(x)} b_N(x,\xi)\widehat{\varphi}(\xi) d\xi dx \\ &\to (2\pi)^{-d} \iint e^{ix\cdot\xi}\overline{\psi(x)} b(x,\xi)\widehat{\varphi}(\xi) d\xi dx,\end{aligned}$$

where the first line follows from Fubini's Theorem and the second one by dominated convergence since $b_N \to b$ pointwise with $||b_N||_\infty$ bounded. □

Proof of Theorem 1.3. By Propositions 3.1, 6.1 and 6.4, there exists $C > 0$ and N_{CV} such that, for all N and all b

$$||Op(b_N)||_{L^2\to L^2} \leq C \max_{|\alpha+\beta|\leq N_{\mathrm{CV}}} ||\partial_x^\alpha \partial_\xi^\beta b||_{L^\infty}.$$

Thus, for all $\psi, \varphi \in \mathcal{S}(\mathbb{R}^d)$, the Cauchy-Schwarz inequality yields

$$|(\psi, Op(b_N)\varphi)_{L^2}| \leq C \max_{|\alpha+\beta|\leq N_{\mathrm{CV}}} ||\partial_x^\alpha \partial_\xi^\beta b||_{L^\infty} ||\varphi||_{L^2(\mathbb{R}^d)} ||\psi||_{L^2(\mathbb{R}^d)}. \tag{6.7}$$

By Lemma 6.5, we can let N go to ∞ and thus replace b_N by b in the left hand side of (6.7). Taking then the supremum over those ψ such that $||\psi||_{L^2} = 1$, we get the result. □