Dynamical Localization for Delone-Anderson operators

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Anderson localization

The dynamics of a particle moving in a material, represented by $\psi \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, $\int |\psi|^2 = 1$, is governed by :

$$\partial_t \psi(t,x) = -iH\psi(t,x), \quad \textit{Schrödinger equation,}$$

its temporal evolution given by

$$\psi(t,x) = e^{-itH}\psi(0,x).$$

If the medium is a perfect crystal, the spectrum of H is a reunion of bands of a.c. spectrum *(extended states)*.

1958 P.W. Anderson *"Absence of diffusion in certain random lattices"* (Phys. Rev.) (Nobel 1977)
Anderson Localization : absence of diffusion of waves in a solid with impurities (ex. alloys, amorphous solids)

Dynamical Localization

The Anderson Model :

$$H_\omega = H_0 + \lambda V_\omega \quad \text{on } l^2(\mathbb{Z}^d) \text{ or } L^2(\mathbb{R}^d),$$

where $H_0 = -\Delta$ and $V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j u(x-j)$, ω_j i.i.d. random variables.

Anderson Localization : p.p. spectrum with exponentially decaying eigenfunctions.

Dynamical Localization : moments of wave packets stay spatially localized in time.

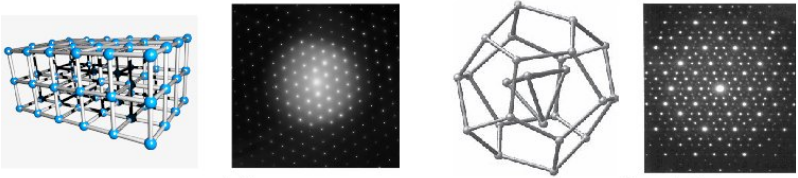
We say $E \in \Sigma_{DL} \subset \mathbb{R}$ (region of ***Dynamical Localization***, **DL**) if H_ω exhibits strong **DL** in a neighborhood I of E, that is, if for all $\mathcal{X} \in \mathcal{C}^\infty_{c,+}(I)$ we have

$$\sup_{u\in\mathbb{Z}^2} \mathbb{E}\left(\sup_{t\in\mathbb{R}} \| \langle X-u\rangle^{p/2} e^{-itH_\omega} \mathcal{X}(H_\omega)\chi_u \|_2^2 \right) < \infty \qquad \text{for every } p \geq 0$$

Quasicrystals

1984 ('82) D. Shechtman, I. Blech, D. Gratias, J.W. Cahn, *"Metallic phase with long-range orientational order and no translation symmetry"*, Phys. Rev. Letters.

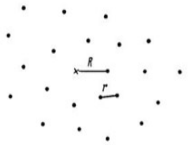
Diffraction patterns



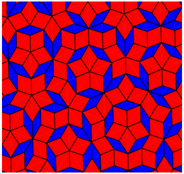
crystal quasicrystal

A ***Delone* set** D of parameters (r,R) is a pure point set in $\mathbb{R}^d$, uniformly discrete (r) and relatively dense (R).

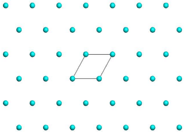
Delone sets



Delone set



Penrose tiling



lattice

Delone operators $\sim$ *Delone dynamical systems* : $\Omega = \overline{\{D + t : t \in \mathbb{R}^d\}}$,
$\omega \in \Omega$ a *Delone* set,

$$H(\omega) = -\Delta + \sum_{\gamma \in \omega} f(\cdot - \gamma)$$

- The spectrum of $H(\omega)$ is generically purely singular continuous.
- Discontinuity in the Integrated Density of States due to compactly supported eigenfunctions

'90, '00's : A. Hof, R. Moody, J.C. Lagarias, B. Solomyak, D. Lenz- P.Stollmann, P. Muller-C. Richard.

Delone-Anderson operators

Consider the operator $H_\omega = H_0 + \lambda V_\omega$, on $L^2(\mathbb{R}^d)$, where $\lambda > 0$,

i) $H_0 = -\Delta$

ii) H_0 is the Landau Hamiltonian H_B, of constant magnetic field $B > 0$,

$$H_B := (-i\nabla - \mathbf{A})^2, \quad \mathbf{A} = \frac{B}{2}(x_2, -x_1)$$

- $V_\omega(x) = \sum_{\gamma \in D} \omega_\gamma u(x - \gamma)$,
  - D is a (r,R)-***Delone*** set
  - $\{\omega_\gamma\}$ iid r.v. with continuous prob. density μ, $\mathrm{supp}\mu = [-m, M]$, $0 \leq m, M$.
  - In case (ii), $u \in \mathcal{C}_c^2(\mathbb{R}^d)$

Let σ_ω be its spectrum , $\omega \in \Omega$.

($\nexists E$) (No ergodicity) A priori, *there does not exist* a family of unitary operators $\{U_\gamma\}$ associated to an ergodic group of translations τ_γ acting on Ω s.t.

$$H_{\tau_\gamma(\omega)} = U_\gamma H_\omega U_\gamma^*$$

Finite-volume properties

Finite volume operators : $H_{\omega,x,L} = H_{0,x,L} + \lambda V_{\omega,x,L}$ on $L^2(\Lambda_{x,L})$, of spectrum $\sigma_{\omega,x,L}$. For the ergodic setting take $x = 0$.

(UWE) H_ω satisfies a *uniform Wegner estimate* in an open interval I, with Hölder exponent s, if for every $E \in I$, there exists a constant Q_W, bounded on compact subsets of I and $0 < s \leq 1$ s.t.

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}\{\text{dist}\,(\sigma_{\omega,x,L}, E) \leq \eta\} \leq Q_W \eta^s L^d$$

Other *structural* properties :
(SLI) *Simon-Lieb type inequality*, (NE) *Number of eigenvalues*, (GEE) *Generalized Eigenfunction Expansion*, (EDI) *Eigenfunction decay inequality*.

The Multiscale Analysis (MSA)

Goal : Prove localization by studying the decay of the finite-volume resolvent from the center of the box to its boundary as measured by

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\|_{x,L}$$

(ILSE) *Initial Length Scale Estimate* : For $\theta > d/s$, there exists $\mathcal{L}_\theta(E)$ s.t. for $\mathcal{L} > \mathcal{L}_\theta(E)$ we have

$$\inf_{x\in\mathbb{Z}^d} \mathbb{P}\left\{ \|\Gamma_{x,L} R_{\omega,x,L}(E)\chi_{x,L/3}\|_{x,L} \leq \frac{1}{L^\theta} \right\} > 1 - \frac{1}{841^d}$$

$$\Sigma_{MSA} = \{E \in \mathbb{R} : H_\omega \text{ satisfies (UWE) and (ILSE) in } E\}$$

Theorem (Germinet-Klein'98, R.-M. '11)

For H_ω a Delone-Anderson operator, $\Sigma_{MSA} \subset \Sigma_{DL}$.

Quantitative unique continuation principles

- H_0 satisfies a ***unique continuation property*** (UCP) if for any $E \in \mathbb{R}$ and any $\varphi \in \mathcal{D}(H_0)$, if φ satisfies $(H_0 - E)\varphi = 0$ and it vanishes on some open set, then $\varphi \equiv 0$.
- ***Quantitative UCP*** (QUCP) : Let $\Lambda_L = \Lambda_{x,L}$, $P_{0,L}(I) = \chi_I(H_{0,L})$ and **D a lattice**. There exists a constant $C_{UCP}(u,I,d) > 0$ such that

$$P_{0,L}(I) \sum_{\gamma \in D \cap \Lambda_L} u(x-\gamma) P_{0,L}(I) \geq C_{UCP}(u,I,d) P_{0,L}(I)$$

that is, if $\varphi \in \operatorname{Ran} P_{0,L}(I)$ and **D is a lattice**, then

$$\sum_{\gamma \in D \cap \Lambda_L} \|\varphi\|^2_{B(\gamma,\delta)} \geq C_{UCP}(I,d) \|\varphi\|^2_{\Lambda_L}$$

Proof : Floquet decomposition of H_0.
Applications : *Wegner estimates* (Combes-Hislop-Klopp'03, Combes-Hislop-Klopp'07). *Anderson and Dynamical localization* (Bourgain-Kenig'05, Germinet-Klein'11). *Perturbation of the ground state energy* (Boutet de Monvel-Lenz-Stollmann'09).

QUCP for kinetic energy operators

i. Spatial averaging for $H_0 = -\Delta$
As in Bourgain-Kenig '05, Germinet-Klein-Hislop '07. Compare the potential V_L to the averaged $\bar{V}_L$, given by

$$\bar{V}_L(\cdot) := \frac{1}{R^d} \int_{\Lambda_R(0)} V_L(\cdot - a)da \geq \frac{C_u}{R^d} \chi_{\Lambda_L}(\cdot)$$

$\bar{V}$ is a good approximation of V in the bottom of the spectrum of $-\Delta$.

ii. Enlargement of obstacles for $H_0 = (-i\nabla - \mathbf{A})^2$.
Adapt Combes-Hislop-Klopp-Raikov '04 : for each $n \in \mathbb{N}$, $0 < \varepsilon < K$, $\delta > 1$ and $\eta > 0$ there exists a constant C_0 such that

$$P_{0,n}\chi_{\Lambda_\varepsilon}P_{0,n} \geq C_0\left(P_{0,n}\chi_{\Lambda_K}P_{0,n} - \eta P_{0,n}\chi_{\Lambda_{\delta K}}P_{0,n}\right).$$

obtain finite volume version + exponentially small error as in Germinet-Klein-Schenker '07.

Theorem (Germinet–Müller–R.-M.'12, R.-M. '11)

i. *For $d \geq 1$, let $H_0 = -\Delta$ and supp $\mu = [0,1]$. There exists $E^*(R) > 0$ (uniform in λ) s.t. for any subinterval $[E-\eta, E+\eta] \subset [0, E^*(R)]$ there exists a constant $Q_W = Q_W(\lambda, R, r, u, d)$ and a finite scale $\mathcal{L}_* = \mathcal{L}_*(R)$ s.t. for $L > \mathcal{L}_*$ we have*

$$\sup_x \mathbb{P}\{\mathrm{dist}(\sigma_{\omega,x,L}, E) \leq \eta\} \leq Q_W \|\mu\|_\infty \eta L^d.$$

ii. *For $d = 2$, let H_0 be the Landau Hamiltonian and supp $\mu = [-M, M]$, $M > 0$. For any energy $E \in \mathbb{R}$ and $\eta < 1/2$ there exist a constant $Q_W = Q_W(B, \lambda, R, r, I_0, u)$ and a finite scale $\mathcal{L}_*$ s.t. for and $L > \mathcal{L}_*$, the same result holds.*

Proof : Follow Combes-Hislop-Klopp '07. Take $\tilde{I}$ so that $I \subset \tilde{I} \subset \mathbb{R}$, and decompose :

$$\mathrm{tr}\; P_{\omega,L}(I) = \mathrm{tr}\; P_{\omega,L}(I) P_{0,L}(\tilde{I}^c) + \mathrm{tr}\; P_{\omega,L}(I) P_{0,L}(\tilde{I})$$

Use Combes-Thomas esimates in the first term of the r.h.s., as for the second, use QUCP. □

QUCP for $H_0 = -\Delta + V_0$

Let $H_{0,L} = -\Delta_L + V_{0,L}$ with V_0 bounded, and $E_0 = \inf \sigma(H_0)$

Theorem (R.-M– Veselic' '12)

If φ is an eigenfunction of the operator $H_{0,L}$ in an interval I, and D is a Delone set, we have

$$\sum_{\gamma \in D \cap \Lambda_L} \|\varphi\|^2_{B(\gamma,\delta)} \geq C_{UCP}(I,d)\, \|\varphi\|^2_{\Lambda_L}$$

i) Uniform Wegner estimate for all energies : for each $E_* \in \mathbb{R}$ there exists a constant C_W such that, for all $E \leq E_*$ and $\eta \leq 1/2$

$$\sup_x \mathbb{P}\{dist(\sigma_{\omega,x,L}, E) \leq \eta\} \leq C_W\, \|\mu\|_\infty \eta\, |\log \eta|^d L^d$$

ii) Perturbation of the bottom of the spectrum : denote by $\lambda^L(t) = \inf \sigma(H_{t,L})$ the bottom of the spectrum of $H_{t,L} := -\Delta_L + V_{0,L} + tV_L$ on $\Lambda_L(x)$ with Dirichlet boundary conditions. Then

$$\forall\, t \in (0,1] : \quad \lambda^L(t) \geq \lambda^L(0) + C_{UCP}(u,I,d) \cdot t$$

Proof : Local estimate (Germinet-Klein'11) : Let φ satisfy an eigenfunction eq. on $G\subset\mathbb{R}^d$, $R:=\mathrm{dist}(x,\Theta)$, with $B(x,\delta)$, Θ and $B(x,12R)$ in G, then

$$\|\varphi\|^2_{B(x,\delta)}\geq C\left(R,\frac{\|\varphi\|_G}{\|\varphi\|_\Theta}\right)\|\varphi\|^2_\Theta \tag{1}$$

Decompose $\Lambda_L=\bigcup_k\Lambda_1(k)$. We say $\Lambda_1(k)$ is a *dominant* box (site) if

$$\|\varphi\|^2_{\Lambda_1(x)}\geq C_T\,\|\varphi\|^2_{\Lambda_T(k)}\,.$$

Then $\|\varphi\|^2_{\Lambda_L}<2\sum_{\text{dominant sites}}\|\varphi\|^2_{\Lambda_1(k)}$, so it is enought to obtain QUCP for dominant unit boxes. We split a dominant box into $(10)^d$ boxes of side $(1/10)$, there exists at least one *maximal* box $\Lambda_{1/10}$, such that

$$\|\varphi\|^2_{\Lambda_{1/10}}\geq\frac{1}{(10)^d}\,\|\varphi\|^2_{\Lambda_1(k)}$$

Now, consider a belt A at a distance $1/10$ from $\Lambda_{1/10}$ such that for any $B(x,\delta)\subset A$, we can apply (1) with $G=\Lambda_T(k)$, $\Theta=\Lambda_{1/10}$, $R\in[1/10,\sqrt{d}]$:

$$\|\varphi\|^2_{B(x,\delta)}\geq C\left(R,\,\|\varphi\|_{\Lambda_T(k)}/\|\varphi\|_{\Lambda_{1/10}}\right)\|\varphi\|^2_{\Lambda_{1/10}}$$

By the definition of maximal box, we get

$$\|\varphi\|^2_{B(x,\delta)}\geq C\left(d,C_T\right)\|\varphi\|^2_{\Lambda_1(k)} \tag{2}$$

In particular, by the definition of a dominant box, $\|\varphi\|^2_{B(x,\delta)}\geq C'\,\|\varphi\|^2_{\Lambda_T(k)}$.
It remains the case $B(x,\delta)\in A^c\Lambda_1(k)\setminus A$. For any cube of side $1/10$ in A that is at a distance at least $1/10$ from A^c, (2) holds. Pick one, call it $\Lambda'_{1/10}$. Then for $B(x,\delta)\subset A^c$, we can apply (1) with $G=\Lambda_T(k)$, $\Theta=\Lambda'_{1/10}$ and $R\in[1/10,\sqrt{d}]$. Then

$$\|\varphi\|^2_{B(x,\delta)}\geq C\left(R,\,\|\varphi\|_{\Lambda_T(k)}/\|\varphi\|_{\Lambda'_{1/10}}\right)\|\varphi\|^2_{\Lambda'_{1/10}}$$

Since in $\Lambda'_{1/10}$ (2) holds, we have

$$\|\varphi\|^2_{B(x,\delta)}\geq C'(d,C_T)\,\|\varphi\|^2_{\Lambda_1(k)}\,. \tag{3}$$

. □

ILSE for Delone-Bernoulli model

Let $H_\omega = -\Delta + V_0 + V_\omega$, with V_0 bounded, $V_\omega = \sum_{\gamma \in D} \omega_\gamma u(x-\gamma)$, where $\omega_\gamma \in \{0,1\}$ **Bernoulli r.v.**, $\inf \sigma(H_\omega) = \inf \sigma(H_0) = E_0$ a.s.

Decompose $\Lambda_L = \bigcup_j \Lambda_K(j)$, consider the probability of finding at least one $\omega_{\gamma_j} = 1$ in each cube $\Lambda_K(j)$:

$$\text{if } K = (\log L)^{1/d} \text{ and } V_L = \sum_{\gamma_j \in \Lambda_K(j)} u(x-\gamma_j) \Longrightarrow \mathbb{P}(V_{\omega,L} \geq V_L) \geq \mathbf{1} - \mathbf{L}^{-\mathbf{p}}$$

$\lambda_L(t) = \inf \sigma(H_{t,L})$, then $\inf \sigma(H_{\omega,L}) \geq \lambda_L(1)$ with probability $1 - L^{-p}$.

- By QUCP (ii) :

$$\mathbb{P}\left(\inf \sigma(H_{\omega,L}) \geq E_0 + C_{UCP}(L)\right) \geq 1 - L^{-p}$$

Applying Combes-Thomas estimate in an interval $[E_0, E_0 + \frac{C_{UCP}(L)}{2}]$ gives a decay of the local resolvent of order

$$-C_{UCP}(L) \cdot L = -(\log L)^{-(\log L)^{\frac{4}{3d}}} \cdot L$$

- **ILSE** if $d \geq 2$. Germinet-Klein'11 gives Localization.

Integrated Density of States (IDS) for H_ω

(UE) Assumption on D : $\mathcal{X}_D$ is *uniquely ergodic*. For example, D is linearly repetitive.

Definition (Eigenvalue counting function)

Let $\{\Lambda_L\}_{L\in\mathbb{N}}$ be a sequence of concentric cubes in $\mathbb{R}^d$. We define for any energy $E \in \mathbb{R}$,

$$\nu_L(E) = \frac{1}{|\Lambda_L|}\sharp\{\text{e.v. of } H_{\omega,L} \leq E\}$$

Theorem (Existence of IDS, Germinet–Müller–R.-M.'12)

For every $E \in \mathbb{R}$, $\nu(E) := \lim_{L\to\infty} \nu_L(E)$ *exists for a.e.* $\omega \in \Omega$.

Proof : Application of Ergodic Theorem in *Ergodic properties of randomly coloured point sets* by Müller-Richard '11.

Proof : Coloured point sets

Consider the base space $\Gamma = \mathbb{R}^d$ and the group $T = \mathbb{R}^d$ acting on Γ as a translation. Given a Delone set $D \subset \Gamma$, take its closed T-orbit

$$\mathcal{X}_D = \overline{\{x + D : x \in T\}} \ni P$$

Consider the *colour space* $\mathbb{A} = [-m, M]$ and define the probability space $(\Omega_P, \mathcal{A}_P, \mathbb{P}_P)$ where

$$\Omega_P = \bigotimes_{\gamma \in P} \mathbb{A}$$

Then $P^\omega = \{(\gamma, \omega_\gamma) : \gamma \in P\}$ is the coloured point set P^ω with colour realization $\omega \in \Omega_P$.

The closed T-orbit of D^ω in the space $\Gamma \times \mathbb{A}$ is given by

$$\hat{\mathcal{X}}_D = \{x + D^\omega : x \in T\}$$

For any element $P^\omega \in \hat{\mathcal{X}}_D$ the action is given by the translations

$$x + P^\omega = (x + P)^{\tau_x \omega}$$

where $\tau_x : \Omega_P \to \Omega_{x+P}$ is defined as $\tau_x \omega(x + p) = \omega(p)$ for all $p \in P$. □

Lifshitz tails for $H_0 = -\Delta$

Dirichlet-Neumann bracketing : For every fixed L we have

$$\int_{X_P} \mathbb{E}_{\Omega_P}(\nu^D_{\Lambda_L,P^\omega}(E))d\mu(P) \leq \nu(E) \leq \int_{X_P} \mathbb{E}_{\Omega_P}(\nu^N_{\Lambda_L,P^\omega}(E))d\mu(P)$$

Lemma (Germinet–Müller–R.-M.'12)

There exist constants $C_{a,b,c,\ldots}$ such that for $L = [\beta E^{-1/2}]$, $\beta > 0$, we have for L big enough :

$$\mathbb{E}_{\Omega_P}(\nu^D_{\Lambda_L,P^\omega}(E)) \geq C_{\beta,r,d} E^{d/2} e^{-C_{\beta,\alpha,r,d} E^{-d/2} \ln E}$$

$$\mathbb{E}_{\Omega_P}(\nu^N_{\Lambda_L,P^\omega}(E)) \leq C e^{-C_{\beta,R,r,d} E^{-d/2}}$$

where the bounds are uniform for all $P \in X_D$.

We obtain Lifshitz tails for energies $E \approx 0$:

$$\lim_{E \searrow 0} \frac{\ln|\ln(\nu(E))|}{\ln(E)} = -\frac{d}{2}$$

Localization for Delone-Anderson operators

Recall $H_\omega = H_0 + \lambda V_\omega$ on $L^2(\mathbb{R}^d)$ with $\lambda > 0$, $V_\omega(x) = \sum_{\gamma \in D} \omega_\gamma u(x-\gamma)$:

- D is a (r,R)-*Delone* set,
- $\{\omega_\gamma\}$ iid random variables with probability density μ.

Theorem (Germinet–Müller–R.-M.'12, R.-M.'11)

i. *For $d \geq 1$ and $H_0 = -\Delta$, supp $\mu = [0,1]$:*
Dynamical localization at the bottom of the spectrum $[0, E^*(R,\lambda)]$,
where $E^*(R,\lambda) = C_{d,\lambda} R^{-(2d+2)} (\ln R)^{-2/d} > 0$.

ii. *For $d \geq 2$ and $H_0 = -\Delta + V_0$, and supp $\mu = \{0,1\}$.*
Dynamical localization at the bottom of the spectrum
$[E_0, E_0 + E^*(R,\lambda)]$.

ii. *For $d = 2$ and H_B the Landau Hamiltonian, $B > 0$, supp $\mu = [-M, M]$,*
$M > 0$:
Dynamical localization in the band edges of the n-th Landau band.

References

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