# Decay of linear fields using spin lowering and spin raising processes

Jérémie Joudioux

Max-Planck-Institut für Gravitationsphysik
Albert-Einstein-Institute

february, 9th. 2012

4ème Rencontre du GDR CNRS « Dynamique quantique »
Toulouse

# Introduction

- Many results where (components of) Dirac fields are reduced to a scalar wave equation : Fackerell-Ipser equation ('72) for Maxwell equations, Teukolsky equation ('73) for linearized gravity (spin 2), for linearized gravity on type D (Aksteiner-Andersson '10).
- Recent result (Blue '07) for Maxwell equations ; decay of the field obtained by the control of one component of the field.

**Purpose**

Study Dirac fields of arbitrary spin by reducing the study to a scalar wave equation, by methods extendible to curved space-time.

- Reference result by Christodoulou-Klaineman (90) for Maxwell equations (spin 1) and linearized gravity (spin 2) on flat background.
- Methods by Penrose to reduce the spin fields to a wave equation ;
  - Construction of a potential (spin raising ; 1965)
  - Using the symmetries of space-time (spin lowering ; 1975)

# Decay of solutions of the linear wave equation

- Background : flat space-time.
- Obtained by energy estimates (Klainerman 83-85) or by conformal compactification (Penrose 65).
- Problem : $\Box\chi = 0$ + Initial data $(\chi_0, \partial_t \chi)$ in certain weighted Sobolev spaces.
- Obtain decay estimates in two directions :
  - Interior decay : along time directions $(t > 3r)$ ;
  - Exterior decay : along null directions $(\frac{r}{3} < t < 3r)$.

# Decay of solutions of the linear wave equation 2

## Theorem

*Let* $s_0 \geq 2$*. Let* $u$ *be a solution of the wave equation with initial date in* $H_{0,k}(\mathbb{R}^3) \times H_{0,k+1}(\mathbb{R}^3)$*. Then*

1. *for* $t > 3r$

$$|u(t,x)| \leq \frac{||u(0)||_{0,s_0}}{(1+t)^{\frac{3}{2}}},$$

2. *for* $\frac{r}{3} < t < 3r$ *:*

$$|u(t,x)| \leq \frac{||u(t)||_{0,s_0}}{(t-r)^{\frac{1}{2}} r^1},$$

# Decay of solutions of the linear wave equation 3

## Theorem

*Let* $s_0 \geq 2$, $(j,k,l) \in \mathbb{N}^3$. *Let* $u$ *be a solution of the wave equation with data in* $H_{0,s_0+j+k+l}$. *Then*

1. *for* $t > 3r$
$$|\nabla^i u(t,x)| \leq \frac{||u(0)||_{0,s_0+j}}{(1+t)^{\frac{3}{2}+j}},$$
2. *for* $\frac{r}{3} < t < 3r$ :
$$|(\partial_u)^j (\partial_v)^k \nabla^l_{\mathbb{S}^2_r} u(t,x)| \leq \frac{||u(0)||_{0,s_0+j+k+l}}{(t-r)^{\frac{1}{2}+j} r^{1+k+l}},$$

$u = t - r$ *and* $v = t + r$.

# Dirac equations of arbitrary spin

- Background : flat spacetime
- Spinor $\phi_A$ of spin $\frac{1}{2}$ : element of $\mathbb{C}^2$ ;
- Symmetric spinor $\phi_{A...F} = \phi_{(A...F)}$ of spin $s$ ($2s$ indices) ; element of Sym $\left(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2\right)$
- Dirac equation (ZRM) : first order equation for symmetric spinors :

$$\begin{cases} \nabla^{AA'}\phi_{A...F} = 0 \\ \phi_{A...F} = \phi_{(A...F)} \end{cases}$$

- Spin 1 : Maxwell equations : Spin 2 ; linearized gravity (Bianchi identity for the Weyl tensor)
- Does not make sense without geometric assumption on curved background for spin $> 1$

# Plan

1. Using potentials : spin raising
2. Using the symmetry of space-time : spin lowering
3. Decay of linear fields

# Using potentials : spin raising

- Idea : Represent Maxwell fields with potentials :

$$\text{Ex} : \mathrm{d}F = 0, \mathrm{d}^\star F = 0, \text{ then } F = \mathrm{d}A \text{ with } \mathrm{d}^\star A = 0$$

- Problem : introduce a second order potential : Hertz Potential
  Let $P$ be a two form satisfying the wave equation :

$$\Box P = \mathrm{d}G + \mathrm{d}^\star W, \text{G, W, gauge functions}$$
$$\text{Then } F = \mathrm{d}\mathrm{d}^\star P \text{ satisfies } (\mathrm{d} + \mathrm{d}^\star)\, F = 0$$

- These solutions cannot be charged : F cannot be a Coulomb field.
- The procedure can be done on curved space-time (Cohen-Kegeles 76)

# Potential for linear fields – Spin raising

- Purpose : do the same for spin $s$ fields :
- Spin raising : Penrose ('65), Eastwood-Penrose-Ward ('81), Eastwood ('85) : let $\phi_{A\dots F}$ a solution of the Dirac equation ; there exists a potential $\xi^{A'\dots F'}$ :

$$\phi_{A\dots F} = \nabla_{AA'}\dots\nabla_{FF'}\xi^{A'\dots F'}$$
$$\text{with } \Box\xi^{A'\dots F'} = 0.$$

- On flat space-time, $\xi^{A'},\dots,\zeta^{A'}$ 2s-constant spinors :

$$\phi_{A\dots F} = \sum \xi^{A'}\dots\zeta^{F'}\nabla_{AA'}\dots\nabla_{FF'}\chi \text{ where } \Box\chi = 0$$

# Spin lowering

- Idea from Penrose ('60) : reducing the spin of the equation.
- $\phi_{A\ldots F}$ a spin-$s$ ZRM field and $\xi^{A\ldots C}$ a spin $r$ field such that :

$$\nabla^{AA'}\phi_{A\ldots F} = 0 \text{ and } \nabla_{A'}^{(A}\xi^{B\ldots E)} = 0$$

then :

- if $r < s$, $\nabla^{AA'}\left(\phi_{A\ldots F}\xi^{C\ldots F}\right) = 0$
- if $r = s$, $\Box\left(\phi_{A\ldots F}\xi^{A\ldots F}\right) = 0$

# Twistors

- Consider a $r$-spinor satisfying the twistor equation :

$$\nabla_{A'}^{(A}\xi^{B\ldots E)} = 0$$

- for spin, $\frac{1}{2}$ :

$$\nabla_{A'}^{(A}\xi^{B)} = 0$$

  the set of solutions is described by, on flat space-time :

$$\mathbb{T} = \left\{(\omega^A, \pi_{A'})|\xi^A = \omega^A + x^{AA'}\pi_{A'}, \left(\omega^A, \pi_{A'}\right) \text{constant spinors}\right\}.$$

  where $x^{AA'} = x^a = t\partial_t + r\partial_r$.

- This equation has geometrical constraints : conformally flat (Petrov type O) or Petrov type N.

# Newman-Penrose tetrad.

- Newman-Penrose tetrad :

$$l^a = \frac{1}{\sqrt{2}}\left(\partial_t + \partial_r\right) \qquad n^a = \frac{1}{\sqrt{2}}\left(\partial_t - \partial_r\right)$$
$$m^a = \frac{1}{r\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin(\theta)}\partial_\psi\right) \quad \overline{m}^a = \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin(\theta)}\partial_\psi\right).$$

- for the metric $\eta = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\omega_{\mathbb{S}^2}$, $l^a, n^a, m^a$ are null vectors and :

$$\eta_{ab} l^a n^b = 1\,, \eta_{ab} m^a \overline{m}^b = -1\,, (l^a, n^a) \perp (m^a, \overline{m}^a)$$

- $(o^A, \iota^A)$ : normalized dyad arising from $(l^a, n^a, m^a, \overline{m}^a)$ :

$$l^a = o^A \overline{o}^{A'} \quad m^a = o^A \overline{\iota}^{A'}$$
$$n^a = \iota^A \overline{\iota}^{A'} \quad \overline{m}^a = \iota^A \overline{o}^{A'}$$
$$o_A \iota^A = 1$$

# Twistor and contraction with a field

- $\phi_{A\ldots F}$ a symmetric field of valence $s$ :

$$\phi_{A\ldots F} = \sum_{i=0}^{2s} \phi_i \underbrace{o_{(A} \ldots o_C}_{i \text{ times}} \underbrace{\iota_D \ldots \iota_{F)}}_{2s-i \text{ times}}, \ \phi_i \in \mathbb{C}.$$

- $\pi_{A'} = \alpha \bar{o}_{A'} + \beta \bar{\iota}_{A'}$
- Contraction of $\phi_{A\ldots F}$ with the twistor $x^{AA'}\pi_{A'} = \frac{1}{\sqrt{2}} \left( \beta(t+r) o^A - \alpha(t-r)\iota^A \right)$ :

$$x^{AA'}\pi_{A'} \ldots x^{FF'}\pi_{F'}\phi_{A\ldots F} = \sum_{i=0}^{2s} c_i \alpha^{2s-i} \beta^i (t+r)^{2s-i} (t-r)^i \phi_i$$

# Extension to curved space-time

- Existence of 1-twistors implies geometrical constraints : no extension to curved background.
- On type D background (Kerr background), there exists a 2-twistor (Killing spinor) :

$$\kappa_{AB} \propto o_A \iota_B$$

- gives control on one component (spin weight zero component).
- Use symmetry operators to control other components.

# Peeling

- Obtained by Sachs ('61) and Newman-Penrose ('62).
- Idea : all components in null directions of a ZRM field does not decay at the same rate.
- $$\phi_{A\ldots F} = \sum_{i=0}^{2s} \phi_i \underbrace{o_{(A} \ldots o_C}_{i \text{ times}} \underbrace{\iota_D \ldots \iota_{F)}}_{2s-i \text{ times}}, \ \phi_i \in \mathbb{C}.$$

  then, along any null directions,

  $$\phi_i \sim \frac{1}{r^{1+2s-i}}$$

  where $r$ is an affine parameter along null rays.

## Decay result for linear fields

- Consider the Cauchy problem for spin $s > 0$ :

$$\begin{cases} \nabla^{AA'}\phi_{A...F} = 0 \\ \phi_{A...F}|_{t=0} \in H_{\delta,k}(\mathbb{R}^3) \end{cases}$$

  where $k \geq 2$

$$||\phi||^2_{\delta,k} = \sum_{l=0}^{k} \int_{\{t=0\}} (1+r^2)^{\delta+l} |\nabla^l \phi|^2 d^3x$$

- The decay is obtained by using the decay of the scalar wave equation and, separately, spin raising and spin lowering.

# Potential and initial data

- Starting from Cauchy data for the Cauchy problem :

$$\begin{cases} \nabla^{AA'}\phi_{A\ldots F} = 0 \\ \phi_{A\ldots F}|_{t=0} \in H_{\delta,s} \end{cases}$$

- Regularity of the potential on the initial time slice $t = 0$ :
  - Hertz potential, assuming that it exists :

$$\textbf{A1} \ ||\phi_{A\ldots F}||_{\delta,k} \lesssim ||\chi^{A'\ldots F'}||_{\delta,k+2s}.$$

  - Contraction with $2s$ twistors :

$$\textbf{A2} \ ||\xi^{A}\ldots\zeta^{F}\phi_{A\ldots F}||_{\delta,k} \lesssim ||\phi_{A\ldots F}||_{\delta+2s,k}.$$

- Criteria on the potentials to have decay, for $k \geq 2$ :

$$||\chi^{A'\ldots F'}||_{0,k} < \infty \text{ or } ||\xi^{A}\ldots\zeta^{F'}\phi_{A'\ldots F'}||_{0,k} < \infty$$

# Decay for linear fields

## Proposition : time decay and

Assume that the initial data satisfy the constraints equations and **A1** or **A2** :

- Interior decay $(t > 3r)$ :

$$|\phi_i| \leq \frac{C}{(1+t)^{\frac{3}{2}+2s}}$$

- Exterior decay $(3r > t > \frac{r}{3})$ :

$$|\phi_i| \leq \frac{C}{(1+|t-r|)^{1+i} r^{\frac{1}{2}+2s-i}}$$

# Decay for linear fields 2

**Proposition : time decay and peeling**

Let $(j,k,l)\in\mathbb{N}^3$ :

- Interior decay ($t>3r$) :

$$|\nabla^j\phi_i|\leq\frac{C}{(1+t)^{\frac{3}{2}+2s+j}}$$

- Exterior decay ($3r>t>\frac{r}{3}$) :

$$|(\partial_u)^j(\partial_v)^k\nabla^l_{\mathbb{S}^2_r}\phi_i|\leq\frac{C}{(1+|t-r|)^{1+i+j}r^{\frac{1}{2}+2s-i+k+l}}$$

## Outlook : Flat space-time

- Finish the energy estimates between the potentials and the field.
- Optimality of the decay result : regularity, decay of initial data, decay along timelike curves.
- Add charges on flat space-time : can one split the solution as :

  static part + radiating part represented with a potential ?
- Describe in this context all the charges and Coulomb solutions of the fields.

## Outlook : Kerr background

- Represent uncharged field with a potential.
- Obtain decay for the 0-spin weight component of the integer spin field using the contraction with the Killing spinor, for spin 1 and spin 2.
- Using symmetry operators (combining both methods), represent uncharged fields with a potential.