# Almost global existence for Klein-Gordon equations with small Cauchy data on a Toeplitz structure

Rafik Imekraz

Université Cergy-Pontoise
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# Previous works 1/2

- Bourgain and Bambusi study the following equation

$$(\partial_t^2 - \partial_x^2 + m^2)v = \partial_v f(v), \quad x \in S^1$$

- almost global existence for small solutions :

  for **generic** $m > 0$, $\forall r > 1 \quad \forall s \gg 1$

  if $||v(0,\cdot)||_{H^s} + ||\partial_t v(0,\cdot)||_{H^{s-1}} = \varepsilon \ll 1$ then

$$\forall t \in [-C\varepsilon^{-r}, C\varepsilon^{-r}] \quad ||v(t,\cdot)||_{H^s} + ||\partial_t v(t,\cdot)||_{H^{s-1}} \leq 2\varepsilon$$

- Sobolev norms of solutions have weak growth for long time

- Bambusi-Grébert's generalization for wave equation with potential
$$(\partial_t^2 - \partial_x^2 + V(x))v = \partial_v f(v), \quad x \in S^1$$
here $V$ is a **generic** potential (instead of $m^2$)
- Delort-Szeftel study the following non-hamiltonian equation
$$(\partial_t^2 - \Delta + V(x) + m^2)v = f(x, v, \partial_t v), \quad x \in S^d$$
solution bounded by $2\varepsilon$ in a time existence of order $\varepsilon^{-p}$ where $p = p(f)$

- "Tour de force" by Delort-Szeftel : $S^d \Rightarrow$ Zoll manifold $X$

- Bambusi-Delort-Grébert-Szeftel prove the almost global existence for Zoll manifolds
$$(\partial_t^2 - \Delta + V(x) + m^2)v = \partial_v f(v), \quad x \in X$$

- Grébert-R.I.-Paturel show the almost global existence for the harmonic oscillator

$$i\partial_t\psi = (-\Delta + ||x||^2 + M)\psi + \partial_{\overline{\psi}} f(\psi), \quad x \in \mathbb{R}^d$$

- R.I. for the quartic oscillator

$$i\partial_t\psi = (-\partial_x^2 + x^4 + M)\psi + \partial_{\overline{\psi}} f(\psi), \quad x \in \mathbb{R}$$

- in both cases, $M : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a “generic" compact operator (instead of the parameter $m^2$ in the Klein-Gordon equation)

Understand the spectral analysis of the linear part :

- good knowledge of spectrum of $\sqrt{-\Delta + V}$, roughly we want that two eigenvalues are uniformly separated (see later)
- good multlinear estimates on eigenfunctions

- The Szegö projector on $L^2(S^1)$ :

$$\pi\left(\sum_{n\in\mathbb{Z}} a_n e^{in\bullet}\right) = \sum_{n\geq 0} a_n e^{in\bullet}$$

appears in the Szegö equation $i\partial_t\psi = \pi(|\psi|^2\psi)$ (see Gérard-Grellier's papers)

- A version of $\pi$ on the real line $\mathbb{R}$ appears in the nonlinearities of PDE studied in Pocovnicu's thesis (2011)

- Motivation : understand dynamics of solutions of analogue equations in the Heisenberg group

- Almost global existence for Klein-Gordon equations which deal with Szegö projectors in nonlinearities, for instance

$$(\partial_t^2 - \Delta_{S^{2d-1}} + m^2)v = \pi(|v|^2 v), \qquad x \in S^{2d-1}$$

where $\pi : L^2(S^{2d-1}) \to L^2(S^{2d-1})$ is the orthogonal projector on the Hardy space, i.e. the space of functions which have holomorphic extension on the unit ball of $\mathbb{C}^d$

| equation | Klein-Gordon | Schrödinger | Schrödinger |
|---|---|---|---|
| manifold $X$ | $S^d$ | $\mathbb{R}^d$ | $\mathbb{R}$ |
| pdo $T$ | $\sqrt{-\Delta}$ | $-\Delta + \|\|x\|\|^2$ | $(-\partial_x^2 + x^4)^{3/4}$ |
| $\mathrm{Sp}(T) : \lambda_k$ | $\sqrt{k(k+d-1)}$ | $2k+d$ | $\simeq k$ |
| bicar. flow on $T^\star X$ | geodesics | circles centred in $(0,0)$ | curves $\xi^2 + x^4 = C$ |

- the manifold $T^\star X$ is naturally symplectic
- the bicaracteristic flow is the hamiltonian flow of the principal symbol of $T$ on $T^\star X$
- for instance $\sigma(-\partial_x^2 + x^2)(x,\xi) = \xi^2 + x^2$

# Bicaracteristic principle

If $T$ is a pdo of order 1, self-adjoint and elliptic on a manifold $X$ and $\sigma(T) : T^\star X \to \mathbb{C}$ is the principal symbol of $T$, then we have the equivalence

- the hamiltonian flow of $\sigma(T)$ is simply periodic of period $\tau$
- the spectrum of $T$ approximate an arithmetic sequence in the following sense

$$\mathrm{sp}(T) \subset \bigcup_{k\geq 1} \left[\frac{2\pi}{\tau}k + \alpha - \frac{\beta}{k}, \frac{2\pi}{\tau}k + \alpha + \frac{\beta}{k}\right]$$

1. the number of eigenvalues in the packet of order $k$ is a polynomial of degree $\dim(X) - 1$ if $k \gg 1$
2. General ref : Colin de Verdière, Guillemin, Weinstein, Helffer-Robert,...

The Toeplitz pseudo-differential theory in the sense of Boutet de Monvel and Guillemin is well adapted to study equations

$$(\partial_t^2 - \Delta_{S^{2n-1}} + m^2)v \quad = \quad \pi(|v|^2 v)$$

because it takes care about Szegö projectors and involves a bicaracteristic principle.

Consider $X$ a compact manifold (not necessarily riemannian), and $\Sigma \subset T^\star X$ a symplectic cone

- if $(x, \xi^\star) \in \Sigma$ then $(x, t\xi^\star) \in \Sigma$ for all $t > 0$
- the natural symplectic form of $T^\star X$ is non-degenerate on $\Sigma$

Boutet de Monvel and Guillemin prove that there is an orthogonal projector, called the Szegö projector,

$$\pi : L^2(X, \mathbb{C}) \to L^2(X, \mathbb{C})$$

which satisfies some microlocally features about $\Sigma$ (see the book “The spectral theory of Toeplitz Operators”).
The triple $(X, \Sigma, \pi)$ is called a Toeplitz structure in the sense of Boutet de Monvel and Guillemin.

The exact definition of $\pi$ will not be necessary. Here are the two simplest examples :

1. if $\Sigma = T^\star X$ then $\pi = \mathrm{Id}$
2. if $X = S^{2d-1}$, then there is $\Sigma \subset T^\star X$ such that $\pi$ is the usual Szegö projector on the Hardy space of $L^2(S^{2d-1}, \mathbb{C})$

The first case allows us to generalize Bambusi-Delort-Grébert-Szeftel's result on Zoll manifold. Whereas the second case allows us to deal with Klein-Gordon equations involving Szegö projectors in nonlinearities.

A Toeplitz pdo $T$ on $(X, \Sigma, \pi)$ is a linear operator of the form $T = \pi Q \pi$ where $Q$ is a classic pdo on $X$ :

$$\forall \phi \in \pi(\mathcal{C}^\infty(X, \mathbb{C})) \qquad T(\phi) := \pi(Q(\phi))$$

Examples :

- $T = \sqrt{-\Delta}$ on $(X, T^\star X, \mathrm{Id})$
- $T = \sqrt{-\Delta}$ on $(S^{2d-1}, \Sigma, \pi)$

One can define ellipticity, self-adjointness, principal symbol,...

Example of principal symbol. If $Q_1$ and $Q_2$ are pdo on $X$ such that $T = \pi Q_1 \pi = \pi Q_2 \pi$ then $\sigma(Q_1)_{|\Sigma} = \sigma(Q_2)_{|\Sigma}$. This allows to define $\sigma(T) = \sigma(Q_i)_{|\Sigma}$.

Let $T$ be a Toeplitz pdo of order 1, self-adjoint, elliptic, on $(X, \Sigma, \pi)$, if the Hamiltonian flow of $\sigma(T) : \Sigma \to \mathbb{C}$ is simply periodic then

1. $$\mathrm{sp}(T) \subset \bigcup_{k \geq 1} \left[ \frac{2\pi}{\tau} k + \alpha - \frac{\beta}{k}, \frac{2\pi}{\tau} k + \alpha + \frac{\beta}{k} \right]$$

2. for $k \gg 1$, the number of eigenvalues in the $k$th packet is polynomial of degree $\frac{1}{2}(\dim \Sigma) - 1$

Remark : this generalizes the classic situation when $\Sigma = T^{\star}X$ and $\pi = \mathrm{Id}$.

- Sobolev spaces on $(X, \Sigma, \pi)$ are given by

$$H^s_\pi(X, \mathbb{C}) := \pi(H^s(X, \mathbb{C}))$$

- the Szegö projector $\pi$ satisfies $H^s_\pi(X, \mathbb{C}) \subset H^s(X, \mathbb{C})$
- one defines $||\phi||_{H^s_\pi} := ||\phi||_{H^s}$ for all $\phi \in \pi(H^s(X, \mathbb{C}))$
- for instance, in the case $(S^1, \Sigma, \pi)$ one has

$$H^s_\pi(S^1, \mathbb{C}) = \left\{ \sum_{n \geq 0} a_n e^{in\bullet}, \quad \sum (1+n)^{2s} |a_n|^2 < +\infty \right\}$$

# General theorem

Consider a Toeplitz pdo $T$ of order 1, self-adjoint, elliptic on $(X, \Sigma, \pi)$, we add the assumption that the Hamiltonian flow of $\sigma(T) : \Sigma \to \mathbb{C}$ is simply periodic. The Klein-Gordon equation is

$$(\partial_t^2 + T^2 + m^2)v = \pi(|v|^2 v), \qquad (t, x) \in \mathbb{R} \times X \qquad (1)$$

$$(v(0,\cdot), \partial_t v(0,\cdot)) = (\varepsilon v_0, \varepsilon v_1) \in H^s_\pi(X, \mathbb{C}) \oplus H^{s-1}_\pi(X, \mathbb{C})$$

$$||v_0||_{H^s} + ||v_1||_{H^{s-1}} = 1$$

For generic $m > 0$, one has

$$\forall r \geq 1, \quad \forall s \gg 1, \quad \exists C, K \geq 1, \quad \forall \varepsilon \ll 1$$

$$\begin{array}{rcl} \exists! v & \in & \mathcal{C}^0([-C\varepsilon^{-r}, C\varepsilon^{-r}], H^s_\pi(X, \mathbb{C})) \\ & \in & \mathcal{C}^1([-C\varepsilon^{-r}, C\varepsilon^{-r}], H^{s-1}_\pi(X, \mathbb{C})) \end{array}$$

$$\forall t \in [-C\varepsilon^{-r}, +C\varepsilon^{-r}] \quad ||v(t,\cdot)||_{H^s} + ||\partial_t v(t,\cdot)||_{H^{s-1}} \leq K\varepsilon$$

One endows $H^s_\pi(X,\mathbb{C})^2$ with the "natural" symplectic form

$$\omega((p_1,q_1),(p_2,q_2)) = \; = \; \mathrm{Re}\left(\int_X q_1\overline{p_2} - p_1\overline{q_2}dx\right)$$
$$= \; \mathrm{Re}(\langle q_1,p_2\rangle - \langle p_1,q_2\rangle)$$

This leads to define **symplectic gradient** and Poisson Bracket.

$H : H^s_\pi(X,\mathbb{C}^2) \to \mathbb{R}$ admits a **symplectic gradient** if

$$\forall (p,q),(h,k) \in H^s_\pi(X,\mathbb{C}^2)$$

$$H(p+h,q+k) - H(p,q) = \omega\left(X_H(p,q),(h,k)\right) + \text{Remainder}$$

# Hamiltonian form 1/2

Setting $\Lambda = \sqrt{T^2 + m^2}$, the equation becomes

$$(\partial_t^2 + \Lambda^2)v = \pi(|v|^2 v) \qquad (2)$$

Let us introduce the following notations

$$u(t) = (p(t), q(t)), \qquad (p(t), q(t)) = (\Lambda^{-1/2}\partial_t v, \Lambda^{1/2} v)$$

Consider the following maps on $H^s_\pi(X,\mathbb{C})^2$

$$\begin{aligned} H_0(p,q) &= \frac{1}{2}\int_X |\Lambda^{1/2}p|^2 + |\Lambda^{1/2}q|^2 dx \\ H_{NL}(p,q) &= -\frac{1}{4}\int_X |\Lambda^{-1/2}q(x)|^4 dx \end{aligned}$$

One checks that

$$(\partial_t^2+\Lambda^2)v = \pi(|v|^2 v)$$

is equivalent to

$$u'(t) = X_{H_0+H_{NL}}(u(t))$$

After the bicaracteristic principle, one has

$$\mathrm{sp}(T) \subset \bigcup_{k\geq 1} \left[\frac{2\pi}{\tau}k + \alpha - \frac{\beta}{k}, \frac{2\pi}{\tau}k + \alpha + \frac{\beta}{k}\right]$$

For the purpose of the talk, we suppose that the $k$-th spectral packet has only one eigenvalue $\lambda_k$. Le $\Pi_k$ be the $k$-th spectral projector. Remind that $\Lambda = \sqrt{T^2 + m^2}$, thus

$$\begin{aligned} H_0(p,q) &= \frac{1}{2}\int_X |\Lambda^{1/2}p|^2 + |\Lambda^{1/2}q|^2 dx \\ &= \frac{1}{2}\sum_{k\geq 1} \sqrt{\lambda_k^2 + m^2} \underbrace{\left(||\Pi_k p||_{L^2}^2 + ||\Pi_k q||_{L^2}^2\right)}_{J_k(p,q)} \end{aligned}$$

Remark : $\{J_k, J_\ell\} = 0$ for all $k, \ell$

We will say that the spectrum $(\lambda_k)$ is nonresonant if

$$\forall n \geq 3 \quad \forall r \in [[1, n-1]] \quad \forall k_1, \dots, k_n \geq 1$$
$$|\lambda_{k_1} + \dots + \lambda_{k_r} - \lambda_{k_{r+1}} - \dots - \lambda_{k_n}| \neq 0$$

To be honnest, the definition is more complicated but implies the previous relation (see works of Bambusi, Grébert, Paturel, Delort, Szeftel,...).
After a Delort-Szeftel's "Tour de force", the spectrum of $\Lambda = \sqrt{T^2 + m^2}$ is nonresonant for generic $m > 0$.

$$H_0(p, q) = \frac{1}{2} \sum_{k \geq 1} \sqrt{\lambda_k^2 + m^2} J_k(p, q)$$

For generic $m > 0$, for all $r \geq 3$ and $s \gg 1$, there is a canonic map $\Phi$ in the neighborhood of the origin of $H^s_\pi(X, \mathbb{C})^2$ such that

i) $(H_0 + H_{NL}) \circ \Phi = H_0 + Z + R$

ii) $\{Z, J_k\} = 0$ for all $k \geq 1$

iii) $||X_R(p, q)||_{H^s} \leq C||(p, q)||^r_{H^s}$

Consider $w(t) = \Phi^{-1}(u(t))$, as $\Phi$ is canonic we have

$$u'(t) = X_{H_0+H_{NL}}(u(t)) \quad \Leftrightarrow \quad w'(t) = X_{H_0+Z+R}(w(t))$$

$\Rightarrow$ we will study the almost global existence of $w(t)$

Remark $(H_0 + H_{NL})(u(t)) = (H_0 + Z + R)(w(t)) = CST$

Remind $(H_0 + H_{NL}) \circ \Phi = H_0 + Z + R$

An equivalent norm on $H^s_\pi(X,\mathbb{C})^2$ is

$$E_s(w) := \sum_{k\geq 1} \lambda_k^{2s} ||\Pi_k(w)||^2_{L^2} = \sum_{k\geq 1} \lambda_k^{2s} J_k(w) = ||T^s w||^2_{L^2}$$

because $T$ is a pdo of order 1

We want to prove that $E_s(w(t))$ has a weak growth. As

$$w'(t) = X_{H_0+Z+R}(w(t))$$

we have

$$\frac{d}{dt} E_s(w(t)) = \{E_s, H_0 + Z + R\} w(t)$$

$$\{E_s, H_0\} = \frac{1}{2}\sum_{k,\ell\geq 1}\sum_{k\geq 1}\lambda_k^{2s}\sqrt{\lambda_\ell^2+m^2}\,\{J_k, J_\ell\} = 0$$

$$\{E_s, Z\} = \sum_{k\geq 1}\lambda_k^{2s}\{J_k, Z\} = 0$$

Thus

$$\begin{aligned}\frac{d}{dt}E_s(w(t)) &= \{E_s, H_0+Z+R\}w(t)\\ &= \{E_s, H_0\}w(t)+\{E_s, Z\}w(t)+\{E_s, R\}w(t)\\ &= \{E_s, R\}w(t)\\ &\leq C||w(t)||_{H^s}^{r+1}\end{aligned}$$

Then, by integration

$$E_s(w(t)) - E_s(w(0)) \leq Ct \sup_{0\leq\tau\leq t} ||w(t)||_{H^s}^{r+1}$$

$$||w(t)||_{H^s}^2 - ||w(0)||_{H^s}^2 \leq Ct \sup_{0\leq\tau\leq t} ||w(t)||_{H^s}^{r+1}$$

That gives $||w(t)||_{H^s} \leq K\varepsilon$ if $||w(0)||_{H^s} = \varepsilon$ and $t \leq C\varepsilon^{-r+1}$.

On a Toeplitz structure $(X, \Sigma, \pi)$ with hypothesis on the periodicity of bicaracteristic curves, one consider

$$(\partial_t^2 + \Lambda^2)v = \pi(|v|^2 v) \tag{3}$$

The new variable $u = (\Lambda^{-1/2}\partial_t v, \Lambda^{1/2} v)$ satisfies the Hamiltonian equation

$$u'(t) = X_{H_0+H_{NL}}(u(t)) \qquad u(t) \in H^s_\pi(X, \mathbb{C}^2)$$

For almost all $m > 0$, spectrum of $\Lambda = \sqrt{T^2 + m^2}$ is nonresonant. A normal form procedure is possible

$$H_0 + H_{NL} \longrightarrow H_0 + Z + R$$

Finally, weak growth of Sobolev norms for long time.

Thank you.