A natural quantization for chaotic maps

General objective: relate spectral properties of classical chaos and quantum chaos.

F. Faure (Grenoble) with M. Tsujii (Kyushu univ.)

February 8, 2012

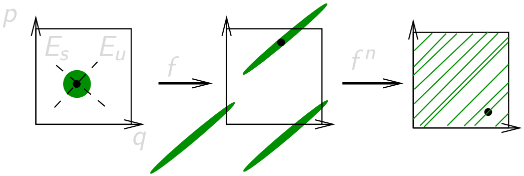
Anosov map

Let (M,ω) a **symplectic compact** manifold, $\dim M = 2d$,
$f : M \to M$ a smooth **symplectic Anosov** diffeom.: there exists a Df-invariant decomposition $TM = E_s \oplus E_u$, $\lambda > 1$,

$$\left\| Df_{/E_s} \right\| \leq \frac{1}{\lambda}, \qquad \left\| Df^{-1}_{/E_u} \right\| \leq \frac{1}{\lambda},$$

Examples

Arnold Cat map: $f_0 : \begin{pmatrix} q \\ p \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$ on $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$
and its perturbations $f = g \circ f_0$, $\|g - \mathrm{Id}\|_{C^1} \ll 1$, preserving $\omega = dq \wedge dp$.



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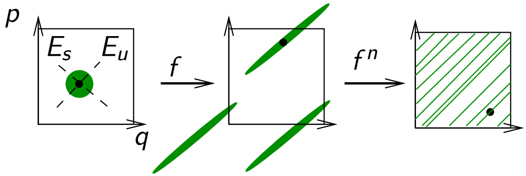
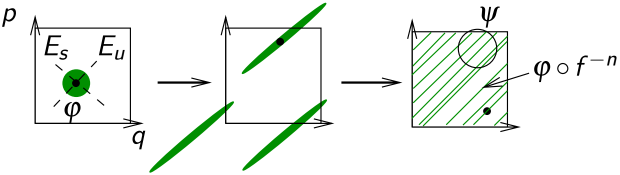


Mixing



Theorem (Anosov,Ruelle,..)

*f is **mixing** with exponential decay of correlations:*

$$\exists \alpha < 1, \forall \psi, \varphi \in C^\infty(M), \quad \langle \psi, \varphi \circ f^{-n} \rangle_{L^2(M)} = \left(\int \psi \right) \left(\int \varphi \right) + O(\alpha^n)$$

- This implies Central limit theorem, random aspects, irreversible convergence towards equilibrium, ...
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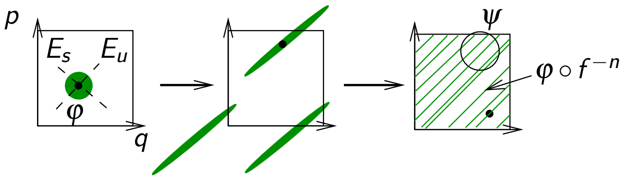

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Question of a natural quantization

Question (*): Existence of a sequence $\hbar = \hbar_j \underset{j\to\infty}{\to} 0$ and operators and spaces $\hat{F}_\hbar : \mathscr{H}_\hbar \to \mathscr{H}_\hbar$, with $\dim \mathscr{H}_\hbar < \infty$, s.t.

1. $\hat{F}_\hbar$ **is "almost unitary"**:
   $\exists \varepsilon_\hbar \to 0, \ \forall \hbar, \ \forall u \in \mathscr{H}_\hbar, \quad (1-\varepsilon_\hbar)\|u\| \leq \left\| \hat{F}_\hbar u \right\| \leq (1+\varepsilon_\hbar)\|u\|, \ ,$
2. **"Gutzwiller formula"**: , $\exists \theta < 1, \forall n, \forall \hbar,$

$$\mathrm{Tr}\left(\hat{F}_\hbar^n\right) = \sum_{x=f^n(x)} \frac{e^{iS_{n,x}/\hbar}}{\sqrt{|\mathrm{Det}(1-Df_x^n)|}} + C_\hbar \theta^n$$

with action $S_{n,x} = " \oint pdq - Hdt"$ defined later.

Proposition (Unicity)

*If $\hat{F}_\hbar$ exists, then its spectrum (with multiplicity) is **unique**. In particular $\dim \mathscr{H}_\hbar$ is unique.*

- **proof:** use: if F, F' are matrices with $\forall n, |\mathrm{Tr}(F^n) - \mathrm{Tr}(F'^n)| < C\theta^n$ then F, F' have same spectrum on $|z| > \theta$. So above, $\theta < 1$ **is important**.

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Existence

What is known for Anosov non linear map $f : \mathbb{T}^2 \to \mathbb{T}^2$ (Keating, S. Debievre, 90'):

- **Weyl quantization** (or **geometric quantization**) gives $\hat{F}_\hbar : \mathscr{H}_\hbar \to \mathscr{H}_\hbar$ unitary, $\dim \mathscr{H}_\hbar = \frac{1}{2\pi\hbar}$,
- Gutzwiller formula has a remainder $\hbar\theta^n$, but $\theta = e^{h_0} > 1$, with **"topological entropy"** $h_0 > 0$.
- Exception: the linear cat map map f_0, for which $\theta = 0$ (then answer to (*) is yes).

Theorem ((*) M.Tsujii, F.F.)

With assumptions:

1. $[\omega] \in H^2(M, \mathbb{Z})$,
2. 1 *is not eigenvalue of the linear map* $f_* : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$

then ***answer to Question (*) is yes****, for* $N = \frac{1}{2\pi\hbar} \in \mathbb{N}$ *large enough. The construction uses "****geometric pre-quantization****" of* f*. We have*

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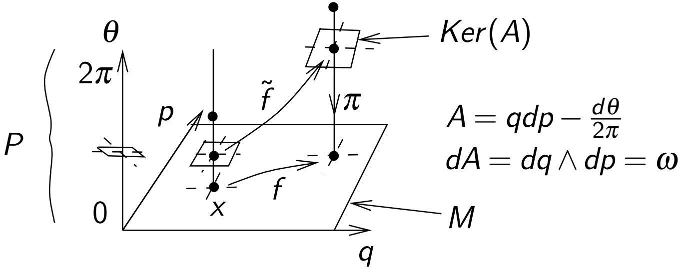
Geometric pre-quantization of f

Theorem (Kostant, Souriau, Kirillov 70', Zelditch 05)

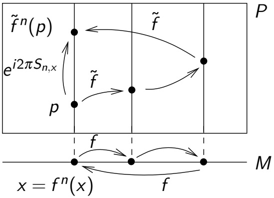
With assumptions 1,2 above, there exists (almost unique):

1. *$U(1)$-principal bundle $P \xrightarrow{\pi} M$ with connection 1-form A, curvature $dA = (\pi^* \omega)$*
2. *a **"prequantum map"** $\tilde{f} : P \to P$ with $f \circ \pi = \pi \circ \tilde{f}$, $\quad \tilde{f}^* A = A$, $\quad \tilde{f}\left(e^{i\theta} p\right) = e^{i\theta} \tilde{f}(p)$*

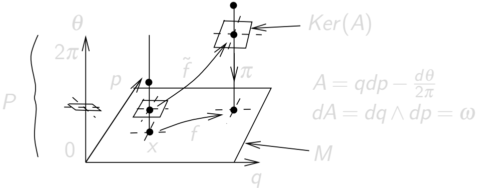


Remarks

- The **action** $e^{i2\pi S_{n,x}}$ of a periodic point is:



- $\tilde{f}$ is a **partially hyperbolic map** with neutral direction θ preserving the contact 1-form A. It is not obvious that $\tilde{f}$ is mixing with exp. decay of correlations.



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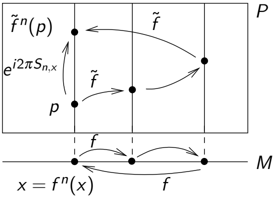

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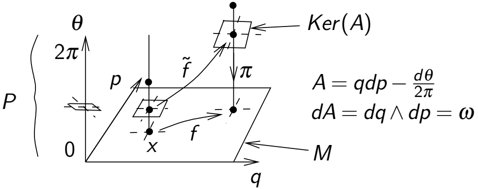

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Prequantum transfer operator

Definition

The **prequantum transfer operator** with **potential** $V \in C^\infty(M)$ is

$$\hat{F} : \begin{cases} C^\infty(P) & \to C^\infty(P) \\ u & \to e^{V \circ \pi} \left(u \circ \tilde{f}^{-1} \right) \end{cases}$$

- For every $N \in \mathbb{Z}$, $\hat{F}$ preserves the space:

$$C_N^\infty(P) := \left\{ u \in C^\infty(P), \quad u\left(e^{i\theta} p\right) = e^{iN\theta} u(p) \right\},$$

called “**equivariant functions**” or “Fourier mode N in the fiber”.

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$$\hat{F}_N := \hat{F}_{/C_N^\infty(P)}$$

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Spectrum of "Ruelle resonances"

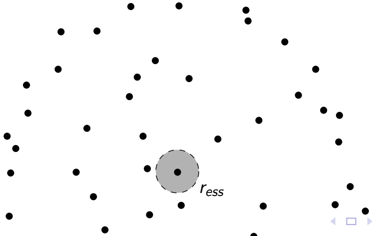
Theorem (V.Baladi, M.Tsujii 07, N.Roy, J.Sjöstrand, F.F., 08)

For every $N \in \mathbb{Z}$, in ***anisotropic Sobolev spaces*** *$H_N^m(P)$ of variable order $m(x,\xi)$, with $m(x,\xi) = \pm M$ along $E_{s/u}^*$, then*

$$\hat{F}_N : H_N^m(P) \to H_N^m(P), \qquad C_N^\infty(P) \subset H_N^m(P) \subset \mathscr{D}'_N(P)$$

is ***bounded*** *and $r_{ess.}\left(\hat{F}_N\right) \underset{M\to\infty}{\to} 0$. The eigenvalues outside r_{ess} do not depend on $H_N^m(P)$ and are called* ***Ruelle resonances*** ***$Res\left(\hat{F}_N\right)$.***

Remark : the only obvious resonance is for the "equilibrium" $u = cste$, $\lambda = 1$ for $N = 0$, .



Spectrum in annuli

Theorem (M.Tsujii, F.F.)

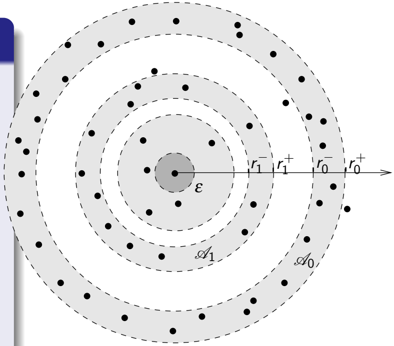
In the semiclassical limit $N = \frac{1}{2\pi\hbar} \to \infty$,

$$Res\left(\hat{F}_N\right) \subset \left\{ |z| \in \bigcup_{k\geq 0} \underbrace{\left[r_k^- - \delta_N, r_k^+ + \delta_N\right]}_{annulus \quad \mathscr{A}_k} \right\}$$

with $\delta_N = \frac{C}{N^\varepsilon} \to 0$, $\varepsilon > 0, C > 0$ *indept of* f,

$$r_k^- = \lim_{n\to\infty} \left| \inf_{x\in M} \left(e^{V_n(x)} \left(\underbrace{\left\| Df^n_{/E_u}(x) \right\|}_{>1} \right)^{-k} \left| \underbrace{\det Df^n_{/E_u}(x)}_{>1} \right|^{-1/2} \right) \right|^{1/n},$$

$$r_k^+ = \ldots \sup_{x\in M} \ldots \left(\left\| Df^n_{/E_u}(x) \right\|^{-1} \right)^{+k} \ldots, \quad V_n(x) = \sum_{j=1}^{n} V\left(f^j(x)\right)$$



Rem: $r_{k+1}^- < r_k^-$, $r_{k+1}^+ < r_k^+$. Choice $V = V_0 = \frac{1}{2}\log\left|\det Df_{/E_u}(x)\right|$ gives $e^{V_{0,n}} = \left|\det Df^n_{/E_u}(x)\right|^{1/2}$ and $r_0^- = r_0^+ = 1$.

Theorem

Let $\hat{F}_\hbar : \mathscr{H}_\hbar \to \mathscr{H}_\hbar$: spectral restriction of $\hat{F}_N$ on $\mathscr{A}_0$, then

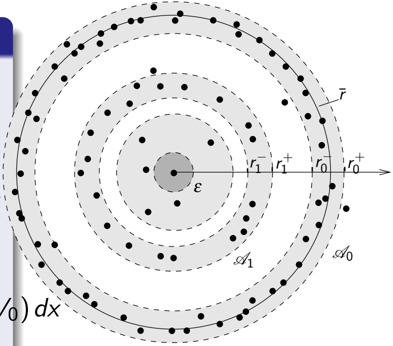
1. $\dim \mathscr{H}_\hbar = \frac{1}{(2\pi\hbar)} Vol(M) + o.$
2. *Most of eigenvalues of $\hat{F}_\hbar$* ***concentrate and equidistribute*** *on the circle of radius*

$$\bar{r} = e^{\langle V - V_0 \rangle}, \quad \langle V - V_0 \rangle := \frac{1}{Vol(M)} \int (V - V_0)\, dx$$

3. ***Gutzwiller trace formula:***

$$Tr\left(\hat{F}_\hbar^n\right) = \sum_{x = f^n(x)} \frac{e^{(V - V_{0,})_n} e^{iS_{n,x}/\hbar}}{\sqrt{|Det(1 - Df_x^n)|}} + \hbar^\varepsilon \left(\frac{1}{\lambda}\right)^n$$

$\rightarrow$ *We get initial Thm (*) if $V = V_0$.*



Corollary

The map $\tilde{f} : P \to P$ is ***mixing*** *with* ***exponential decay of correlations.***
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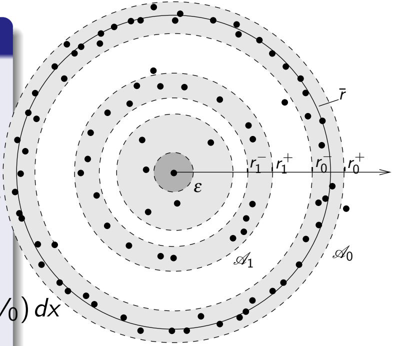

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Summary. Is it a “natural quantization”?

Construction:

- We start from $f : M \to M$ **symplectic and Anosov**.
- Then the **prequantum map** $\tilde{f} : P \to P$ and **prequantum operator** $\hat{F}_N : C_N^\infty(P) \to C_N^\infty(P)$ are (almost) uniquely defined.
  Take the potential $V_0 = \frac{1}{2}\log\left|\det Df_{/E_u}(x)\right|$.
- We consider $\Pi_N : H_N^m(P) \to \mathscr{H}_\hbar$: spectral (finite rank) **projector on the external annulus**. Let $\hat{F}_\hbar = \Pi_N \hat{F}_N \Pi_N$. (set $\hbar = \frac{1}{2\pi N}$).

Properties:

- $\hat{F}_h$ is **almost unitary** (in the limit $\hbar \to 0$) and satisfies **Gutzwiller formula** with exp. small remainder in large time.
- If $a \in C^\infty(M)$, define $\mathrm{Op}_\hbar(a) := \Pi_N a \Pi_N$. We have (obviously) **exact Egorov theorem**:

$$F_\hbar \mathrm{Op}_\hbar(a) = \mathrm{Op}_\hbar\left(a \circ f^{-1}\right) F_\hbar$$

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$$\mathbf{Op}_\hbar(a)\,\mathbf{Op}_\hbar(b) = \mathbf{Op}_\hbar(ab) + O(\hbar^\varepsilon) \quad (1)$$

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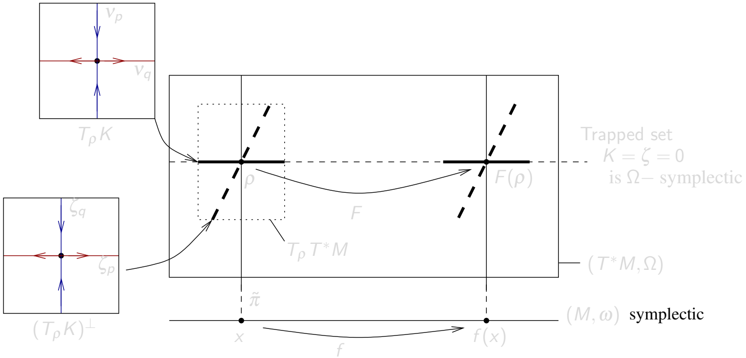
- Atiyah-Bott, **exact trace formula** (66).

The operator $\hat{F}_N : C_N^\infty(P) \to C_N^\infty(P)$ is a F.I.O. with **canonical map** on (T^*M, Ω):

$$F : (x,\zeta) \to \left(f(x), {}^t Df_x^{-1}.\zeta\right), \qquad \Omega = \sum_{j=1}^{2d} dx^j \wedge d\zeta^j + \omega$$



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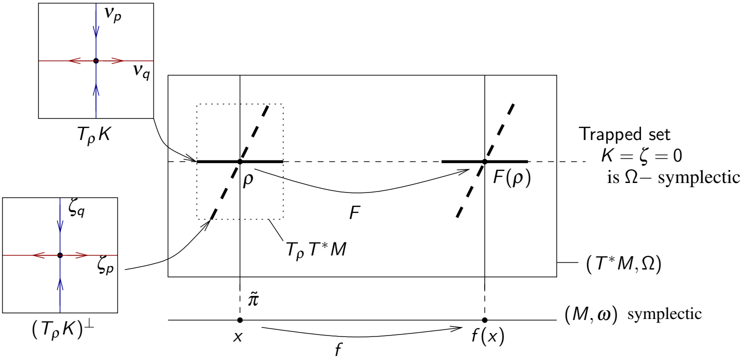

Some key results for linear maps

- Let $f = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ symplectic, hyperbolic with $A : \mathbb{R}^d \to \mathbb{R}^d$ expanding.
- Let $\hat{F}_f u := u \circ f^{-1}$: unitary in $L^2\left(\mathbb{R}^{2d}\right)$.
- Then
$$\hat{F}_f = \hat{F}_A \otimes \hat{F}_{{}^tA^{-1}}$$
with $\left(\hat{F}_A v\right) := \frac{1}{\sqrt{|\det A|}} v \circ A^{-1}$ unitary.
- In Sobolev space $H^{m(x,\xi)}$, $\hat{F}_A$ has discrete spectrum (in polyn. space) and $r\left(\hat{F}_A\right) = \frac{1}{\sqrt{|\det A|}}$.
- Also (Atiyah-Bott): $\mathrm{Tr}^\flat \hat{F}_f = \int_{\mathbb{R}^{2d}} \langle \delta_x | \hat{F}_f \delta_x \rangle = \int_{\mathbb{R}^{2d}} \delta\left(x - f^{-1}(x)\right) = \frac{1}{|\det(1-f)|}$ hence
$$\mathrm{Tr}^\flat \hat{F}_A = \mathrm{Tr}^\flat \hat{F}_{{}^tA^{-1}} = \frac{1}{\sqrt{|\det(1-f)|}},$$