Propagation estimates for photons in non-relativistic QED

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Part I

Introduction

The model

Standard model of non-relativistic QED

Non-relativistic atomic system (N charged non-relativistic quantum particles) interacting with the quantized electromagnetic field

Atomic system

To simplify, we assume that the atomic system consists of a hydrogen atom with infinitely heavy nucleus. The Schrödinger operator associated to it is written as

$$H_{\text{el}} = -\Delta_x + V(x),$$

where V is real-valued and $\|V(x)\psi\| \leq \varepsilon\|\Delta_x\psi\| + C_\varepsilon\|\psi\|$. For instance

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Fock space

HIbert space for the photon field

- Hilbert space for 1 photon: $\mathrm{L}^2(\mathbb{R}^3 \times \{1,2\})$
- Symmetric Fock space for the photon field:

$$\mathcal{F}_s = \mathbb{C} \oplus \bigoplus_{n \geq 1} S_n \mathrm{L}^2((\mathbb{R}^3 \times \{1,2\})^n) = \bigoplus_{n \geq 0} \mathcal{F}_s^n$$

where S_n is the symmetrization operator

Creation and annihilation operators

- $a^*_\lambda(k)$ and $a_\lambda(k)$:

$$a^*_\lambda(k) : \mathcal{F}_s^n \to \mathcal{F}_s^{n+1}$$
$$a_\lambda(k) : \mathcal{F}_s^n \to \mathcal{F}_s^{n-1}$$

- Canonical commutation rules:

$$[a^*_\lambda(k), a^*_{\lambda'}(k')] = [a_\lambda(k), a_{\lambda'}(k')] = 0$$
$$[a_\lambda(k), a^*_{\lambda'}(k')] = \delta_{\lambda\lambda'}\delta(k-k')$$

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Standard model in non-relativistic QED

Hilbert space for the non-relativistic electron and the photon field

$$\mathcal{H} = \mathrm{L}^2(\mathbb{R}^3) \otimes \mathcal{F}_s \simeq \mathrm{L}^2(\mathbb{R}^3; \mathcal{F}_s)$$

Pauli-Fierz Hamiltonian acting on $\mathcal{H}$

$$H = (-i\nabla_x + A(x))^2 + V(x) + H_f$$

where

$$A(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\kappa(k)}{\sqrt{|k|}} \varepsilon_\lambda(k) \left(a_\lambda^*(k) e^{-ik\cdot x} + a_\lambda(k) e^{ik\cdot x} \right) dk$$

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Ionization threshold, localization of photons

Ionization threshold

Let Σ denote the ionization threshold defined by

$$\Sigma := \lim_{R \to \infty} \inf_{\varphi \in D_R, \|\varphi\|=1} \langle \varphi, H\varphi \rangle,$$

where $D_R = \{\varphi \in D(H);\ \varphi(x) = 0 \text{ if } |x| < R\}$

Theorem (Bach, Fröhlich, Sigal), (Griesemer)

For all $\delta, \xi \in \mathbb{R}$ such that $\xi + \delta^2 < \Sigma$,

$$\left\| e^{\delta |x|} \mathbb{1}_{(-\infty, \xi]}(H) \right\| < \infty$$

Localization of photons

Let $f \in \mathrm{C}_0^\infty(\mathbb{R}; [0,1])$ be such that $\mathrm{supp}(f) \subset [1,2]$ and define $F(s) = \int_{-\infty}^{s} f(\tau)\, d\tau$. Let $y := i\nabla_k$. We localize the photon position using the operator $F(|y| \geq ct) := F(|y|/ct)$

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Maximal velocity of photons

Theorem (Bony, F., Sigal)

Let $\chi \in \mathrm{C}_0^\infty((-\infty, \Sigma))$ and $\mathrm{c} > 1$. For all $u \in D(\mathrm{d}\Gamma(\langle y \rangle)^{\frac{1}{2}})$, we have

$$\Big\| \mathrm{d}\Gamma\big(F(|y| \geq \mathrm{c}t)\big)^{\frac{1}{2}} e^{-itH} \chi(H) u \Big\| \lesssim t^{-\gamma} \big\| \big(\mathrm{d}\Gamma(\langle y \rangle) + 1\big)^{\frac{1}{2}} u \big\|,$$

where

$$\gamma < \min\Big(\frac{1}{2}\Big(1 - \frac{1}{\mathrm{c}}\Big), \frac{1}{10}\Big).$$

Remark

The operator $\mathrm{d}\Gamma\big(F(|y| \geq ct)\big)$ represents the number of photons in the region $\{|y| \geq ct\}$. Hence the theorem means that, asymptotically (as $t \to \infty$), the probability to find photons in the region $\{|y| \geq ct\}$ vanishes. In other words, photons do not propagate faster that the speed of light ($= 1$ in our units)

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Part II

The method of propagation observables

Propagation observables

Heisenberg derivative

Let $\psi_t := e^{-itH}\psi_0$. Given a family of operators Φ_t on $\mathcal{H}$, the Heisenberg derivative is defined by

$$D\Phi_t := \partial_t \Phi_t + i[H, \Phi_t], \quad \text{so that} \quad \partial_t \langle \psi_t, \Phi_t \psi_t \rangle = \langle \psi_t, D\Phi_t \psi_t \rangle$$

Definition

A family of operators Φ_t on a subspace $\mathcal{H}_1 \subset \mathcal{H}$ is called a *weak propagation observable*, if for all $\psi_0 \in \mathcal{H}_1$, it has the following properties

- $\sup_t \langle \psi_t, \Phi_t \psi_t \rangle \lesssim \|\psi_0\|_*^2$;
- $D\Phi_t \geq G_t + \mathrm{Rem}$, where $G_t \geq 0$ and $\int_0^\infty dt |\langle \psi_t, \mathrm{Rem}\, \psi_t \rangle| \lesssim \|\psi_0\|_\diamond^2$,

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Remark

For the strong propagation estimate, Φ_t does *not* need to be uniformly bounded in time. On the other hand, Φ_t must be ≥ 0, with ≤ 0 Heisenberg derivative (up to integrable remainder terms)

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Idea of the proof of the main theorem

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for some $\gamma > 0$

Idea of the proof

It suffices to show that $\Phi_t := t^{2\gamma} \mathrm{d}\Gamma\big(F(|y| \geq \mathrm{c}t)\big)$ is a strong propagation observable on $\mathrm{Ran}\chi(H) \cap D(\mathrm{d}\Gamma(\langle y \rangle)^{\frac{1}{2}})$

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Part III

Sketch of the proof

Field operators

Simplification for this talk

We consider scalar bosons on the Fock space

$$\mathcal{F}_s = \mathbb{C} \oplus \bigoplus_{n\geq 1} S_n \mathrm{L}^2(\mathbb{R}^{3n}) = \bigoplus_{n\geq 0} \mathcal{F}_s^n$$

Creation and annihilation operators

For $f \in \mathrm{L}^2(\mathbb{R}^3)$,

$$(a^*(f)\Phi)^{(n)}(k_1, \cdots, k_n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(k_i) \Phi^{(n-1)}(k_1, \cdots, \hat{k}_i, \cdots, k_n)$$

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Second quantization and commutation properties

Second quantized operators

Let b be an operator on $\mathrm{L}^2(\mathbb{R}^3)$. The second quantization of b is defined on $\mathcal{F}_s$ by

$$\mathrm{d}\Gamma(b)|_{\mathcal{F}_s^n} = \sum_{j=1}^{n} \mathbb{1} \otimes \cdots \otimes b \otimes \cdots \otimes \mathbb{1}$$
$$\mathrm{d}\Gamma(b)\Omega = 0$$

where $\Omega = (1, 0, 0, \cdots)$ (Fock vacuum)

Commutation properties

Let b, c be operators on $\mathrm{L}^2(\mathbb{R}^3)$ and $f \in \mathrm{L}^2(\mathbb{R}^3)$. We have

$$[\mathrm{d}\Gamma(b), \mathrm{d}\Gamma(c)] = \mathrm{d}\Gamma([b, c]),$$
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Proof of the theorem (I)

Nelson model

To simplify the presentation, we consider the Nelson model

$$H = -\Delta_x + V(x) + \mathrm{d}\Gamma(|k|) + \Phi(h_x), \quad h_x(k) = |k|^{\mu} e^{ik\cdot x}\kappa(k), \quad \mu > 0$$

Propagation observable

Let $v = |y|/ct$. The choice $\Phi_t = t^{2\gamma}\mathrm{d}\Gamma\big(F(v)\big)$ does *not* work. Let instead $\Phi_t = t^{2\gamma}\mathrm{d}\Gamma(J_\beta(v^2))$, where $J_\beta(s) = s^\beta F(s^{1/2})$. Then Φ_t is well-defined on $\chi(H)D\big(\mathrm{d}\Gamma(\langle y\rangle^{2\beta})\big)$ (not trivial) and satisfies $\Phi_t \geq t^{2\gamma}\mathrm{d}\Gamma\big(F(v)\big)$ (trivial)

Heisenberg derivative

$$\begin{aligned} D\Phi_t = \partial_t\Phi_t + i[H,\Phi_t] &= 2\gamma t^{2\gamma-1}\mathrm{d}\Gamma\big(J_\beta(v^2)\big) \leftarrow \text{has a sign} \\ &- 2t^{2\gamma-1}\mathrm{d}\Gamma\big(v^2 J_\beta'(v^2)\big) \leftarrow \text{has a sign} \\ &+ t^{2\gamma}\mathrm{d}\Gamma\big(i\big[|k|, J_\beta(v^2)\big]\big) \leftarrow ?? \\ &- t^{2\gamma}\Phi(iJ_\beta(v^2)h_x) \leftarrow \text{remainder term} \end{aligned}$$

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Proof of the theorem (II)

The field remainder term

Recall $h_x(k) = |k|^\mu e^{ik\cdot x}\kappa(k)$

- Use that

$$\|\Phi(iJ_\beta(v^2)h_x)(H_f+1)^{-1/2}\| \lesssim \||k|^{-1/2}J_\beta(v^2)h_x\|_{L^2} + \|J_\beta(v^2)h_x\|_{L^2},$$

- Use that $J_\beta(\cdot)$ is supported away from 0: $\||y|^{-\rho}J_\beta(v^2)\| \lesssim t^{-\rho}$
- Use that $\||k|^{-1/2}|y|^\rho h_x\|_{L^2} \lesssim \langle x\rangle^\rho$ (for ρ not too large, depending on the value of μ)
- Use exponential decay in the electron position variable below the ionization threshold to bound $\|\langle x\rangle^\rho\chi(H)\| < \infty$
- It follows that

$$t^{2\gamma}\|\chi(H)\Phi(iJ_\beta(v^2)h_x)\chi(H)\| \lesssim t^{-1-\varepsilon}$$

for $2\gamma + 1 + \varepsilon + 1/2 + 2\beta - \mu < 3/2$ (requires $\mu > 0$)

Proof of the theorem (III)

Commutator expansion

Consider the term $t^{2\gamma} \mathrm{d}\Gamma\big(i\big[|k|, J_\beta(v^2)\big]\big)$

- By the Helffer-Sjöstrand formula,

$$i\big[|k|, J_\beta(v^2)\big] = (J'_\beta)^{\frac{1}{2}}(v^2) i[|k|, v^2](J'_\beta)^{\frac{1}{2}}(v^2) + \mathrm{Rem}$$
$$= -\frac{1}{\mathrm{c}t}(J'_\beta)^{\frac{1}{2}}(v^2)\big(v \cdot \frac{k}{|k|} + \frac{k}{|k|} \cdot v\big)(J'_\beta)^{\frac{1}{2}}(v^2) + \mathrm{Rem}$$

- The main term: since $\mathrm{supp}(J'_\beta) \subset [1, \infty)$,

$$-\frac{1}{\mathrm{c}t}(J'_\beta)^{\frac{1}{2}}(v^2)\big(v \cdot \frac{k}{|k|} + \frac{k}{|k|} \cdot v\big)(J'_\beta)^{\frac{1}{2}}(v^2) \le \frac{2}{\mathrm{c}t} v^2 J'_\beta(v^2)$$

- The remainder term: use Hardy's inequality in $\mathbb{R}^3$ ($\| |k|^{-\delta} u \|_{L^2} \lesssim \| |y|^\delta u \|_{L^2}$ for $0 \le \delta < 3/2$) to show that

$$\big\| |k|^{\frac{\delta}{2}} \mathrm{Rem} |k|^{\frac{\delta}{2}} \big\| \lesssim t^{-1-\delta}, \quad 0 \le \delta \le 1$$

Proof of the theorem (IV)

Control of small momenta

- After second quantization, the previous estimate give

$$t^{2\gamma}\mathrm{d}\Gamma\big(i\big[|k|,J_\beta(v^2)\big]\big) \le 2\mathrm{c}^{-1}t^{2\gamma-1}\mathrm{d}\Gamma\big(v^2 J_\beta'(v^2)\big) + Ct^{-1-\delta}\mathrm{d}\Gamma(|k|^{-\delta})$$

- Control the growth of $\mathrm{d}\Gamma(|k|^{-\delta})$ along the evolution: Assume $\psi_0 \in D(|H|^{\frac12}) \cap D(\mathrm{d}\Gamma(|k|^{-\delta})^{\frac12})$. Then for any $\delta \in [-1,1]$,

$$\langle \psi_t, \mathrm{d}\Gamma(|k|^{-\delta})\psi_t\rangle \lesssim t^{\frac{1+\delta}{2+\mu}}\big(\langle \psi_0, (|H|+1)\psi_0\rangle + \langle \psi_0, \mathrm{d}\Gamma(|k|^{-\delta})\psi_0\rangle\big)$$

- Therefore

$$t^{-1-\delta}\langle \psi_t, \mathrm{d}\Gamma(|k|^{-\delta})\psi_t\rangle \lesssim t^{-1-\varepsilon},$$

with $\varepsilon = \delta - \frac{1+\delta}{2+\mu}$, and hence $\varepsilon > 0$ provided that $\mu > \delta^{-1} - 1$

Proof of the theorem (V)

Conclusion

- Combine the previous estimates to obtain

$$\begin{aligned} D\Phi_t &= 2\gamma t^{2\gamma-1}\mathrm{d}\Gamma\big(J_\beta(v^2)\big) - 2t^{2\gamma-1}\mathrm{d}\Gamma\big(v^2 J'_\beta(v^2)\big) \\ &\quad + t^{2\gamma}\mathrm{d}\Gamma\big(i\big[|k|, J_\beta(v^2)\big]\big) - t^{2\gamma}\Phi(iJ_\beta(v^2)h_x) \\ &\le 2t^{2\gamma-1}\mathrm{d}\Gamma(\gamma J_\beta(v^2) - (1-\mathrm{c}^{-1})v^2 J'_\beta(v^2)) + \mathrm{Int.\,term} \end{aligned}$$

- Use that $v^2 J'_\beta(v^2) \ge \beta J_\beta(v^2)$ which implies

$$D\Phi_t \le 2t^{2\gamma-1}(\gamma - (1-\mathrm{c}^{-1})\beta)\mathrm{d}\Gamma(J_\beta(v^2)) + \mathrm{Int.\,term}$$

- Choose β such that $\gamma - (1-\mathrm{c}^{-1})\beta < 0$

Standard model of non-relativistic QED

- Infrared singularity is of order $|k|^\mu$ with $\mu = -1/2$ (while we used the condition $\mu > 0$ for the Nelson model)
- Use a (generalized) Pauli-Fierz transformation to improve the infrared behavior

Proof of the theorem (V)

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Part IV

Minimal velocity estimate?

Minimal velocity of photons

Conjecture

Since physically, photons propagate at the speed of light ($\mathrm{c}=1$), we expect that, for $F \in \mathrm{C}_0^\infty((0,1))$ and $\chi \in \mathrm{C}_0^\infty((-\infty,\Sigma) \setminus \sigma_p(H))$

$$\int_1^\infty \left\| \mathrm{d}\Gamma\big(F(|y|/t)\big)^{\frac{1}{2}} e^{-itH} \chi(H) u \right\|^2 \frac{dt}{t} < \infty$$

for u in some dense set

Problems

- Let $\Phi_t = \mathrm{d}\Gamma\big(G(v^2)\big)$. We have

$$D\Phi_t = \partial_t \mathrm{d}\Gamma(v^2) + i[H, G(v^2)]$$

  Now the second term should dominate the first one
- To prove a weak propagation estimate, we need that $\langle \psi_t, \Phi_t \psi_t \rangle$ is uniformly bounded. However, it is *not* known that $\langle \psi_t, \mathrm{d}\Gamma(1)\psi_t \rangle$ is uniformly bounded

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Local decay

Mourre estimate (Fröhlich, Griesemer, Sigal)

Let $E := \inf \sigma(H)$, $e_{\rm gap} = e_1 - e_0$ with $e_0 = \inf \sigma(H_{\rm el})$, $e_1 = \inf \sigma(H_{\rm el}) \setminus \{e_0\}$, and

$$B_\sigma := \mathrm{d}\Gamma(b_\sigma), \quad b_\sigma := \frac{1}{2}\eta_\sigma(k)(k \cdot y + y \cdot k)\eta_\sigma(k)$$

Let $I \Subset (0,1)$ be an open interval. There exist $\mathrm{c}_0 > 0$ such that, for sufficiently small coupling and $0 < \sigma \leq e_{\rm gap}/2$,

$$\mathbb{1}_{\sigma I}(H-E)[H, iB_\sigma]\mathbb{1}_{\sigma I}(H-E) \geq \mathrm{c}_0 \sigma \mathbb{1}_{\sigma I}(H-E)$$

(Non uniform) Local decay (Hunziker, Sigal, Soffer)

Let $\varphi \in \mathrm{C}_0^\infty((0,1);\mathbb{R})$, $\varphi_\sigma(\cdot) := \varphi(\cdot/\sigma)$. For all $s \geq 0$, there exits $\mathrm{C}_{s,\varphi} > 0$ such that, for sufficiently small coupling and for all $0 < \sigma \leq e_{\rm gap}/2$ and $t \in \mathbb{R}$,

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Minimal velocity of at least one photon

(Uniform) Local decay (Bony, F.)

For sufficiently small coupling, for all $\chi \in C_0^\infty((E, E + e_{\rm gap}/4); \mathbb{R})$ and $0 \leq s < 2$, we have

$$\| \langle \mathrm{d}\Gamma(|y|) \rangle^{-s} e^{-itH} \chi(H) \langle \mathrm{d}\Gamma(|y|) \rangle^{-s} \| \lesssim \langle t \rangle^{-s}$$

Consequence

Let $\Gamma(b) = b \otimes \cdots \otimes b$ on the n-particles subspace. For sufficiently small coupling, for all $\chi \in C_0^\infty((E, E + e_{\rm gap}/4); \mathbb{R})$

$$\left\| \Gamma(F(|y| \leq c t^\alpha)) e^{-itH} \chi(H) \psi \right\|^2 \lesssim t^{\alpha - \frac{9}{20}},$$

for all $c > 0$ and $\alpha < 9/20$. In other words, asymptotically as $t \to \infty$, at least one particle is in the region $\{|y| \geq c t^\alpha\}$

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