Resolvent estimates for the Laplacian on asymptotically hyperbolic manifolds

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Abstract

Combining results of Cardoso-Vodev [6] and Froese-Hislop [9], we use Mourre's theory to prove high energy estimates for the boundary values of the weighted resolvent of the Laplacian on an asymptotically hyperbolic manifold. We derive estimates involving a class of pseudo-differential weights which are more natural in the asymptotically hyperbolic geometry than the weights $\langle r \rangle^{-1/2-\epsilon}$ used in [6].

1 Introduction, results and notations

The purpose of this paper is to prove resolvent estimates for the Laplace operator Δ_g on a non compact Riemannian manifold $(\mathcal{M}, g)$ of asymptotically hyperbolic type. The latter means that $\mathcal{M}$ is a connected manifold of dimension n with or without boundary such that, for some relatively compact open subset $\mathcal{K}$, some closed manifold Y (i.e. compact, without boundary) and some $r_0 > 0$, $(\mathcal{M} \setminus \mathcal{K}, g)$ is isometric to $[r_0, +\infty) \times Y$ equipped with a metric of the form

$$dr^2 + e^{2r} h(r). \tag{1.1}$$

For each r, $h(r)$ is a Riemannian metric on Y which is a perturbation of a fixed metric h, meaning that, for all k and all semi-norm $|||.|||$ of the space of smooth sections of $T^*Y \otimes T^*Y$,

$$\sup_{r \geq r_0} \big|\big|\big| \langle r \rangle^2 \partial_r^k (h(r) - h) \big|\big|\big| < \infty, \tag{1.2}$$

with $\langle r \rangle = (1+r^2)^{1/2}$. Here, and in the sequel, r denotes a positive smooth function on $\mathcal{M}$ going to $+\infty$ at infinity and which is a coordinate near $\mathcal{M} \setminus \mathcal{K}$, i.e. such that dr doesn't vanish near $\mathcal{M} \setminus \mathcal{K}$. Such manifolds include the hyperbolic space $\mathbb{H}_n$ and some of its quotients by discrete isometry groups. More generally, we have typically in mind the context of the 0-geometry of Melrose [15].

Let G be the Dirichlet or Neumann realization of Δ_g (or the standard one if ∂M is empty) on $L^2(\mathcal{M}, d\mathrm{Vol}_g)$. Then, according to [6], it is known that the limits $\langle r \rangle^{-s}(G - \lambda \pm i0)^{-1} \langle r \rangle^{-s} := \lim_{\varepsilon \to 0^+} \langle r \rangle^{-s}(G - \lambda \pm i\varepsilon)^{-1} \langle r \rangle^{-s}$ exist, for all $s > 1/2$, and satisfy

$$\big|\big| \langle r \rangle^{-s}(G - \lambda \pm i0)^{-1} \langle r \rangle^{-s} \big|\big|_{L^2(\mathcal{M}, d\mathrm{Vol}_g)} \leq C e^{C_G \lambda^{1/2}}, \qquad \lambda \gg 1. \tag{1.3}$$

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In [23], it is shown that the right hand side can be replaced by $C\lambda^{-1/2}$, under a non trapping condition.

In the present paper, we will mainly prove that, up to logarithmic terms in λ, such estimates still hold if one replaces $\langle r\rangle^{-s}$ by a class of operators which are, in some sense, weaker than $\langle r\rangle^{-s}$ and more adapted to the framework of the asymptotically hyperbolic scattering.

Let us fix the notations used in this article.

Throughout the paper, $C_c^\infty(\mathcal{M})$ denotes the space of smooth functions with compact support. If $\mathcal{M}$ has a boundary, $C_0^\infty(\mathcal{M})$ is the subspace of $C_c^\infty(\mathcal{M})$ of functions vanishing near $\partial\mathcal{M}$ and if B denotes the boundary conditions associated to G (if any), $C_B^\infty(\mathcal{M})$ is the subspace of $\varphi \in C_c^\infty(\mathcal{M})$ such that $B\varphi = 0$ (e.g. $B\varphi = \varphi_{|\partial M}$ for the Dirichlet condition).

We set $I = (r_0, +\infty)$ and call ι the isometry from $\mathcal{M} \setminus \mathcal{K}$ to $\bar{I} \times Y$. If $\Psi : U_Y \subset Y \ni \omega \mapsto (y_1, \cdots, y_{n-1}) \in U \subset \mathbb{R}^{n-1}$ is a coordinate chart and $\mathcal{M} \setminus \mathcal{K} \ni m \mapsto \omega(m) \in Y$ is the natural projection induced by ι, we define the chart $\tilde{\Psi} : \iota^{-1}(I \times U_Y) \subset \mathcal{M} \to I \times U$ by

$$\tilde{\Psi}(m) = (r(m), \Psi(\omega(m))) . \tag{1.4}$$

There clearly exists a finite atlas on $\mathcal{M}$ composed of such charts and compactly supported ones. For any diffeomorphism $f : M \to N$, between open subsets of two manifolds, we use the standard notations f^* and f_* for the maps defined by $f^*u = u \circ f^{-1}$ and $f_*u = u \circ f$, respectively on $C^\infty(M)$ and $C^\infty(N)$ (and more generally on differential forms or sections of density bundles).

By (1.1) and (1.2), we have $\iota^*(d\mathrm{Vol}_g) = \tilde{\Theta} e^{(n-1)r} dr d\mathrm{Vol}_h$ on $\mathcal{M} \setminus \mathcal{K}$, with $\tilde{\Theta} = d\mathrm{Vol}_{h(r)}/d\mathrm{Vol}_h$ satisfying $\sup_I |||\langle r\rangle^2 \partial_r^k(\tilde{\Theta}(r, .) - 1)||| < \infty$ for all k and all seminorm $|||.|||$ of $C^\infty(Y)$. We choose a positive function $\Theta \in C^\infty(\mathcal{M})$ such that $\iota^*\Theta = e^{(n-1)r}\tilde{\Theta}$ on $\mathcal{M} \setminus \mathcal{K}$ and we define a new measure $d\mathrm{Vol}_{\mathcal{M}} = \Theta^{-1} d\mathrm{Vol}_g$. This is convenient since we now have $\iota^*(d\mathrm{Vol}_{\mathcal{M}}) = dr d\mathrm{Vol}_h$ on $I \times Y$ hence, if we set $L^2(\mathcal{M}) = L^2(\mathcal{M}, d\mathrm{Vol}_{\mathcal{M}})$, we get natural unitary isomorphisms

$$L^2(\mathcal{K}) \oplus L^2(\mathcal{M} \setminus \mathcal{K}) \approx L^2(\mathcal{K}) \oplus L^2(I, dr) \otimes L^2(Y, d\mathrm{Vol}_h) \approx L^2(\mathcal{K}) \oplus \bigoplus_{k=0}^{\infty} L^2(I, dr), \tag{1.5}$$

using, for the last one, an orthonormal basis $(\psi_k)_{k\geq 0}$ of eigenfunctions of Δ_h. More explicitely, the isomorphism between $L^2(I, dr) \otimes L^2(Y, d\mathrm{Vol}_h)$ and $\bigoplus_{k=0}^{\infty} L^2(I, dr)$ is given by $\varphi \mapsto (\varphi_k)_{k\geq 0}$ with

$$\varphi_k(r) = \int_Y \varphi(r, \omega)\overline{\psi_k(\omega)}\ d\mathrm{Vol}_h(\omega). \tag{1.6}$$

In what follows, we will consider the self-adjoint operator

$$H = \Theta^{1/2} G \Theta^{-1/2}$$

on $L^2(\mathcal{M})$, with domain $\Theta^{1/2} D(G)$. If $\partial\mathcal{M}$ is non empty, we furthermore assume that $\Theta \equiv 1$ near $\partial\mathcal{M}$ in order to preserve the boundary condition. This is an elliptic differential operator, unitarily equivalent to G, which takes the form, on $\mathcal{M} \setminus \mathcal{K}$,

$$H = D_r^2 + e^{-2r}\Delta_h + V + (n-1)^2/4, \tag{1.7}$$

with Δ_h the Laplace operator on Y associated to the r-independent metric h and V a second order differential operator of the following form in local coordinates

$$\tilde{\Psi}^* V \tilde{\Psi}_* = \sum_{|\beta|\leq 2} \langle r\rangle^{-2} v_\beta(r, y)(e^{-r} D_y)^\beta, \tag{1.8}$$

with $\partial_r^k \partial_y^\alpha v_\beta$ bounded on $I \times U_0$ for all $U_0 \Subset U$ and all k, α. Here U is associated to the chart Ψ (see above (1.4)). Without loss of generality, by possibly increasing r_0, we may assume that

$$H = H_0 + V$$

with V of the same form as above, with coefficients supported in $\mathcal{M} \setminus \mathcal{K}$, which is H bounded with relative bound < 1 (see Lemma 1.4 of [9] or Lemma 3.5 below), and H_0 another self-adjoint operator (with the same domain as H) such that

$$H_0 = D_r^2 + e^{-2r}\Delta_h + (n-1)^2/4, \tag{1.9}$$

on $\iota^{-1}((r_0+1,\infty) \times Y)$.

We next choose a positive function $w \in C^\infty(\mathbb{R})$ such that

$$w(x) = \begin{cases} 1, & x \leq 0, \\ x, & x \geq 1 \end{cases}. \tag{1.10}$$

If $\mathrm{spec}(\Delta_h) = (\mu_k)_{k \geq 0}$ and $s \geq 0$, we define a bounded operator $\widetilde{W}_{-s}$ on $L^2(I) \otimes L^2(Y, d\mathrm{Vol}_h)$ by

$$(\widetilde{W}_{-s}\varphi)(r,\omega) = \sum_{k \geq 0} w^{-s}(r - \log\sqrt{\langle \mu_k \rangle})\varphi_k(r)\psi_k(\omega). \tag{1.11}$$

Using (1.5), we pull $\widetilde{W}_{-s}$ back as an operator W_{-s} on $L^2(\mathcal{M})$, assigning W_{-s} to be the identity on $L^2(\mathcal{K})$. We can now state our main result.

Theorem 1.1. *Assume that, for some function $\varrho(\lambda) \geq c\lambda^{-1/2}$ and some real number $0 < s_0 \leq 1$,*

$$||\langle r \rangle^{-s_0}(H - \lambda \pm i0)^{-1}\langle r \rangle^{-s_0}|| \leq C\varrho(\lambda), \qquad \lambda \gg 1. \tag{1.12}$$

Then, for all $s > 1/2$, there exists C_s such that

$$||W_{-s}(H - \lambda \pm i0)^{-1}W_{-s}|| \leq C_s(\log\lambda)^{2s_0+2s}\varrho(\lambda), \qquad \lambda \gg 1. \tag{1.13}$$

Using the results of [6, 23], i.e. the estimates (1.3), we obtain

Corollary 1.2. *Let $W_{-s}^\Theta = \Theta^{-1/2}W_{-s}\Theta^{1/2}$ with $s > 1/2$. On any asymptotically hyperbolic manifold, we have*

$$\left|\left|W_{-s}^\Theta(G - \lambda \pm i0)^{-1}W_{-s}^\Theta\right|\right|_{L^2(\mathcal{M}, d\mathrm{Vol}_g)} \leq C_s(\log\lambda)^{4s}e^{C_G\lambda^{1/2}}, \qquad \lambda \gg 1,$$

with the same C_G as in (1.3). If the manifold is non trapping (in the sense of [23]), we have

$$\left|\left|W_{-s}^\Theta(G - \lambda \pm i0)^{-1}W_{-s}^\Theta\right|\right|_{L^2(\mathcal{M}, d\mathrm{Vol}_g)} \leq C(\log\lambda)^{4s}\lambda^{-1/2}, \qquad \lambda \gg 1.$$

These results improve the estimate (1.3) to the extent that W_{-s} and W_{-s}^Θ are "weaker" than $\langle r \rangle^{-s}$ in the sense that $W_{-s}\langle r \rangle^s$ is not bounded. The latter is easily verified using (1.11) by choosing a sequence $(\varphi_k)_{k \geq 0} \in L^2(I)$ such that $\sum_k ||\varphi_k||^2 = 1$ with φ_k supported close to $\log\sqrt{\langle \mu_k \rangle}$.

A result similar to Theorem 1.1 has already been proved by Bruneau-Petkov in [2] for Euclidean scattering (on $\mathbb{R}^n$). They essentially show that, if P is a long range perturbation of $-\Delta_{\mathbb{R}^n}$ such that $||\chi(P - \lambda \pm i0)^{-1}\chi|| = \mathcal{O}(e^{C\lambda})$ for all $\chi \in C_0^\infty(\mathbb{R}^n)$, then $||\langle x \rangle^{-s}(P - \lambda \pm i0)^{-1}\langle x \rangle^{-s}|| = \mathcal{O}(e^{C_1\lambda})$, with $s > 1/2$. In other words, one can replace compactly supported weights by polynomially decaying ones.

Weighted resolvent estimates can be used for various applications among which are spectral asymptotics, analysis of scattering matrices, of scattering amplitudes or non linear problems. In particular, they are known to be useful to obtain Weyl formulas for scattering phases in Euclidean scattering [20, 21, 2, 3] and the present paper was motivated by similar considerations in the hyperbolic context [4, 5]. Actually, high energy estimates are important tools to get semiclassical approximations of the Schrödinger group by the techniques of Isozaki-Kitada [13, 14]. This is well known on $\mathbb{R}^n$ [20, 21, 3] and is being developed for asymptotically hyperbolic manifolds [4, 5]. These applications will be published elsewhere (they would otherwise lead to a paper of unreasonnable length).

We now introduce a class of pseudo-differential operators associated with the scale of weights defined by the operators W_{-s}. For $s \in \mathbb{R}$, we set

$$w_s(r,\eta) = w^s(r - \log\langle\eta\rangle)$$

and define the space $\mathcal{S}(w_s) \subset C^\infty(\mathbb{R}_r \times \mathbb{R}_y^{n-1} \times \mathbb{R}_\rho \times \mathbb{R}_\eta^{n-1})$ as the set of symbols satisfying

$$\left|\partial_r^j \partial_y^\alpha \partial_\rho^k \partial_\eta^\beta a(r,y,\rho,\eta)\right| \leq C_{j\alpha k\beta} w_s(r,\eta), \qquad r,\rho \in \mathbb{R}, \quad y,\eta \in \mathbb{R}^{n-1}.$$

Note that, w_s is a temperate weight in the sense of [12] (see Lemma 4.2 of the present paper). Note also that $\mathcal{S}(w_{s_1}) \subset \mathcal{S}(w_{s_2})$ if $s_1 \leq s_2$.

To construct operators on the manifold $\mathcal{M}$, we consider a chart $\Psi : U \to U_Y$ (we keep the notations above (1.4)) and we choose open sets $U_0 \Subset U_1 \Subset U_2 \Subset U$. We pick cutoff functions $\kappa, \tilde{\kappa} \in C^\infty(\mathbb{R}_r \times \mathbb{R}_y^{n-1})$ which are respectively supported in $I \times U_1$ and $I \times U_2$, with bounded derivatives and such that $\tilde{\kappa} \equiv 1$ near supp κ, $\kappa \equiv 1$ on $(r_1, +\infty) \times U_0$ for some $r_1 > r_0$. For bounded symbols a, we can then define

$$\tilde{\Psi}_* \kappa Op(a) \tilde{\kappa} \tilde{\Psi}^* = \tilde{\Psi}_* \kappa a(r,y,D_r,D_y) \tilde{\kappa} \tilde{\Psi}^*,$$

on $L^2(\mathcal{M})$.

Theorem 1.3. *Assume that $a \in \mathcal{S}(w_{-s})$ for some $s \in [0,1]$. Then, there exist bounded operators $B_{1,s}$ and $B_{2,s}$ on $L^2(\mathcal{M})$ such that*

$$\tilde{\Psi}_* \kappa Op(a) \tilde{\kappa} \tilde{\Psi}^* = B_{1,s} W_{-s} = W_{-s} B_{2,s}.$$

The interest of this theorem is that Theorem 1.1 still holds if one replaces W_{-s} by pseudo-differential operators with symbols in $\mathcal{S}(w_{-s})$, $s > 1/2$. This is important since the classes $\mathcal{S}(w_{-s})$, with $s > 0$, are naturally associated with the functional calculus of asymptotically hyperbolic Laplacians as we shall see below.

Let us explain why polynomial weights $\langle r\rangle^{-s}$ are more natural for Euclidean scattering than for the asymptotically hyperbolic one. In polar coordinates on $\mathbb{R}^n$, the principal symbol of the flat Laplacian is $\rho^2 + r^{-2} q_0$ (with $q_0 = q_0(y,\eta)$ the principal symbol of the Laplacian on the sphere) and since $dr^{-2}/dr = -2r^{-2} \times r^{-1}$, it is easy to check that, for all $k \in \mathbb{N}$, $\gamma \in \mathbb{N}^{n-1}$ and $z \notin [0,+\infty)$, one has

$$\left|\partial_r^k \partial_\eta^\gamma (\rho^2 + r^{-2} q_0 - z)^{-1}\right| \leq C_{z,k,\gamma} |\rho^2 + r^{-2} q_0 - z|^{-1} r^{-k-|\gamma|}, \qquad r \gg 1. \tag{1.14}$$

Here we consider the function $(\rho^2 + r^{-2} q_0 - z)^{-1}$ for it is the principal symbol of $(-\Delta_{\mathbb{R}^n} - z)^{-1}$ (in polar coordinates) and hence the prototype of the symbols involved in the functional calculus of perturbations of $-\Delta_{\mathbb{R}^n}$. Besides, we note that when one considers a perturbation of $-\Delta_{\mathbb{R}^n}$ by a long range potential V_L, one usually assumes that, for some $\varepsilon > 0$,

$$|\partial_x^\alpha V_L(x)| \leq C_\alpha \langle x \rangle^{-\varepsilon-|\alpha|}.$$

Hence, powers of r^{-1} are naturally involved in the symbol classes for Euclidean scattering. This is compatible with the fact that the weights needed to get resolvent estimates in this context are also powers of r^{-1}.

In hyperbolic scattering, the situation is different. The principal symbol of H_0 (see (1.9)) takes the form $\rho^2 + e^{-2r} q_h$ (with $q_h = q_h(y, \eta)$ the principal symbol of Δ_h on Y) and since $de^{-2r}/dr = -2e^{-2r}$ we cannot hope to gain any extra decay of the symbols with respect to r, unlike in the Euclidean case. However, remarking that

$$\left| \frac{e^{-2r} q_h}{\rho^2 + e^{-2r} q_h + 1} \right| \leq C_s w_{-s}(r, \eta), \qquad \forall \ s \geq 0,$$

it is easy to check that, if $k + |\gamma| \geq 1$,

$$\left| \partial_r^k \partial_\eta^\gamma (\rho^2 + e^{-2r} q_h - z)^{-1} \right| \leq C_{z,k,\gamma,s} |\rho^2 + e^{-2r} q_h - z|^{-1} w_{-s}(r, \eta), \qquad \forall \ s \geq 0. \tag{1.15}$$

Here again, we have chosen $(\rho^2 + e^{-2r} q_h - z)^{-1}$ since it is the principal symbol of the pseudo-differential approximation of $(H_0 - z)^{-1}$ (see [4, 5]). The estimate (1.15) reflects the fact that the weights w_{-s} are more natural than $\langle r \rangle^{-s}$ in hyperbolic scattering: we do not gain any power of r^{-1} by differentiating but we gain powers of w_{-1} and these weights are naturally associated with the resolvent estimates as shown by Theorems 1.1 and 1.3.

Let us now say a few words about the simple idea on which Theorem 1.1 is based. The proof uses Mourre's theory and relies on two remarks. The first one is roughly the following: assume that, for $\lambda \gg 1$, we can find $f_\lambda \in C_0^\infty(\mathbb{R})$ and some self-adjoint operator A such that the (formal) commutator $i[H, A]$ has a bounded closure $i[H, A]^0$ on $D(H)$ and

$$f_\lambda(H) i[H, A]^0 f_\lambda(H) \geq \lambda f_\lambda^2(H) \tag{1.16}$$

with $f_\lambda = 1$ on $(\lambda - \delta_\lambda, \lambda + \delta_\lambda)$. Then, one has

$$||\langle A \rangle^{-s} (H - \lambda \pm i0)^{-1} \langle A \rangle^{-s}|| = \mathcal{O}(\delta_\lambda^{-1}).$$

This essentially follows from the techniques of [16] (thought our assumptions on A and H won't fit the framework of [16]) and is the purpose of the next section. We emphasize that, instead of (1.16), a Mourre estimate usually looks like

$$E_{I(\lambda)}(H) i[H, A]^0 E_{I(\lambda)}(H) \geq 2\lambda E_{I(\lambda)}(H) + E_{I(\lambda)}(H) K_\lambda E_{I(\lambda)}(H) \tag{1.17}$$

with $E_{I(\lambda)}(H)$ the spectral projector of H on some interval $I(\lambda) \ni \lambda$, and K_λ a compact operator. As explained in [16], (1.17) implies (1.16) provided f_λ is supported away from the point spectrum of H and δ_λ is small enough, since $f_\lambda(H) K_\lambda \to 0$ as $\delta_\lambda \to 0$. But *we don't have any control on* δ_λ in general and here comes our second remark. If one already knows some a priori estimates on $(H - \lambda \pm i0)^{-1}$, we can hope to control δ_λ from below by mean of the following easy lemma which links explicitly the size of the support of the function, i.e. δ_λ, to estimates on the resolvent.

Lemma 1.4. *Let $(L, D(L))$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and J an interval. Assume that, for some bounded operator K,*

$$\sup_{\lambda \in J, \ 0 < \epsilon < 1} \left| \left| K^* (L - \lambda \pm i\epsilon)^{-1} K \right| \right| < \infty. \tag{1.18}$$

Then, for all $f \in C_0^\infty(J)$, one has

$$||f(L)K|| \leq \pi^{-1/2}|J|^{1/2}||f||_\infty \sup_{\lambda\in J} \left|\left|K^*(L-\lambda \pm i0)^{-1}K\right|\right|^{1/2},$$

with $|J|$ the Lebesgue measure of J, provided the right hand side is well defined.

Proof. This is a direct consequence of the Spectral Theorem which shows that, for all $\varphi \in \mathcal{H}$,

$$||f(L)K\varphi||^2 = (2i\pi)^{-1} \lim_{\epsilon\downarrow 0} \int_J |f(E)|^2 \left(\left((L-E-i\epsilon)^{-1} - (L-E+i\epsilon)^{-1}\right)K\varphi, K\varphi\right) \, \mathrm{d}E. \qquad \Box$$

Remark. If $L = H$ and $J \Subset ((n-1)^2/4, +\infty)$, the condition (1.18) is known to hold by [6, 9], choosing for instance $K = \langle r\rangle^{-s}$ with $s > 1/2$.

We shall apply this strategy, i.e. deduce (1.16) from an estimate of the type (1.17) using the above trick with the a priori estimates of Cardoso-Vodev proved in [6]. The conjugate operator A (which will actually depend on λ) is essentially the one constructed by Froese-Hislop in [9].

We note in passing that we actually prove a stronger result than Theorem 1.1, namely a Mourre estimate (see Theorem 3.12) which implies Theorem 1.1. Thus, using the techniques of [17], we could also get other propagation estimates involving "incoming" or "outgoing" spectral cutoffs.

This method is rather general and could certainly be adapted to other settings than the asymptotically hyperbolic one. For instance, we could consider manifolds with Euclidean ends or both asymptotically hyperbolic and Euclidean ends, using the standard generator of dilations $rD_r + D_r r$ (cut off near infinity) as a conjugate operator in Euclidean ends, as in [9].

The organization of the paper is the following. In Section 2, we review Mourre's theory with a class of operators adapted to our purpose and give a rather explicit dependence of the estimates with respect to the different parameters. We point out that some of our technical assumptions on A and H will not be the same as those of [16]. For this reason and also to take the parameters into account, we need to provide some details. In Section 3, we review the construction of the conjugate operator A introduced in [9]. For the same reasons as for Section 2, we cannot use directly the results of [9] and we need again to review some proofs. We also give a pseudo-differential approximation for A. In Section 4, we prove Theorems 1.1 and 1.3.

2 Mourre's theory

2.1 Algebraic results

In what follows, $(H, D(H))$ and $(A, D(A))$ are self-adjoint operators on a Hilbert space $\mathcal{H}$ that will eventually satisfy the assumptions *(a)*, *(b)* and *(c)* below. These assumptions are slightly different from the ones used in [16] but, taking into account some minor modifications, they allow to follow the original proof of Mourre to get estimates on $\langle A\rangle^{-s}(H-\lambda \pm i0)^{-1}\langle A\rangle^{-s}$. In this subsection, we record results allowing to justify the algebraic manipulations needed for that purpose. Differential inequalities and related estimates are given in Subsection 2.2.

(a) Assumptions on domains: there exists a subspace $\mathcal{D} \subset D(H) \cap D(A)$ dense in $\mathcal{H}$, such that

$$\mathcal{D} \text{ is a core for } A, \tag{2.1}$$

i.e. is dense in $D(A)$ equipped with the graph norm. We also assume the existence of a sequence ζ_n of bounded operators satisfying, for all $n \in \mathbb{N}$,

$$\zeta_n D(H) \subset D(H), \qquad \zeta_n D(A) \subset D(A), \tag{2.2}$$
$$\zeta_n (H-z)^{-1}\mathcal{D} \subset \mathcal{D}, \qquad \forall\, z \notin \mathrm{spec}(H), \tag{2.3}$$
$$\zeta_n g(H)\mathcal{H} \subset \mathcal{D}, \qquad \forall\, g \in C_0^\infty(\mathbb{R}), \tag{2.4}$$

and furthermore, as $n \to \infty$,

$$\zeta_n \varphi \;\to\; \varphi, \qquad \forall\, \varphi \in \mathcal{H}, \tag{2.5}$$
$$A\zeta_n \varphi \;\to\; A\varphi, \qquad \forall\, \varphi \in D(A), \tag{2.6}$$
$$H\zeta_n \varphi \;\to\; H\varphi, \qquad \forall\, \varphi \in D(H). \tag{2.7}$$

The last condition regarding the domains is the following important one

$$(H-z)^{-1}D(A) \subset D(A), \qquad \forall\, z \notin \mathrm{spec}(H). \tag{2.8}$$

Remark. When A and H are pseudo-differential operators on manifolds, most of these conditions are easily verified. The hardest is to check (2.8). We point out that sufficient conditions ensuring (2.8) are given in [16] (see also [1, 10]), namely conditions on e^{itA}, but they don't seem to be satisfied by the operators considered in Section 3. We thus rather set (2.8) as an assumption in this part; in the next section, the explicit forms of A and H will allow us to check it directly (see Proposition 3.9).

Note also the following easy result.

Lemma 2.1. *Conditions* (2.2), (2.5), (2.6) *and* (2.7) *imply that* $A\zeta_n(A+i)^{-1}$, $H\zeta_n(H+i)^{-1}$ *are bounded operators on* $\mathcal{H}$*, uniformly with respect to* n*. In addition,* (2.3) *implies that* $\mathcal{D}$ *is a core for* H*.*

Proof. We only consider H. For all $\epsilon > 0$, $H(\epsilon H + i)^{-1}\zeta_n(H+i)^{-1}$ is bounded and converges strongly on $\mathcal{H}$ as $\epsilon \to 0$, since $D(H)$ is stable by ζ_n. This proves that $H\zeta_n(H+i)^{-1}$ is bounded, by uniform boundedness principle. Then, by (2.7), $H\zeta_n(H+i)^{-1}$ converges strongly on $\mathcal{H}$ to $H(H+i)^{-1}$ and hence is uniformly bounded by the same principle. Thus, if $\psi \in D(H)$ and $\mathcal{D} \ni \varphi_n \to (H+i)\psi$ in $\mathcal{H}$, then $\psi_n := \zeta_n(H+i)^{-1}\varphi_n$ is clearly a sequence of $\mathcal{D}$ such that $\psi_n \to \psi$ and $H\psi_n \to H\psi$ in $\mathcal{H}$. ☐

(b) Commutators assumptions. There exists a bounded operator $[H,A]^0$ from $D(H)$ (equipped with the graph norm) to $\mathcal{H}$, and $C_{H,A} > 0$ such that, for all $\varphi, \psi \in \mathcal{D}$,

$$(A\varphi, H\psi) - (H\varphi, A\psi) = \left([H,A]^0\varphi, \psi\right), \tag{2.9}$$
$$\left|\left(A\varphi, i[H,A]^0\psi\right) - \left(i[H,A]^0\varphi, A\psi\right)\right| \le C_{H,A}\|\psi\|\, \|(H+i)\varphi\|. \tag{2.10}$$

Note that we only require that $\varphi, \psi \in \mathcal{D}$ in (2.9) and (2.10) (instead of $D(A) \cap D(H)$ in the original paper [16]). Note also that $i[H,A]^0$ is automatically *symmetric* on $\mathcal{D}$, hence on $D(H)$ by Lemma 2.1.

We now state the main assumption.

(c) Positive commutator estimate at $\lambda \in \mathbb{R}$. There exists $\delta > 0$ and $f \in C_0^\infty(\mathbb{R}, \mathbb{R})$ with $0 \le f \le 1$, such that,

$$f(E) = \begin{cases} 1 & \text{if } |E-\lambda| < 2\delta, \\ 0 & \text{if } |E-\lambda| > 3\delta, \end{cases}$$

and satisfying, for some $\alpha > 0$,

$$f(H)i[H,A]^0 f(H) \geq \alpha f(H)^2. \tag{2.11}$$

Remark that (2.11) makes perfectly sense, for $f(H)i[H,A]^0 f(H)$ is bounded and self-adjoint in view of the symmetry of $i[H,A]^0$ on $D(H)$.

The main condition among *(a)*, *(b)* and *(c)* is the *Mourre estimate* (2.11). We include the parameters α and δ to emphasize their important roles in the estimates given in the next subsection.

We now record the main algebraic tools needed to repeat Mourre's strategy.

Proposition 2.2. *Assume that all the conditions* (2.1), $\cdots$, (2.9) *but* (2.4) *hold. Then, on* $D(A)$,

$$[(H-z)^{-1}, A] = -(H-z)^{-1}[H,A]^0(H-z)^{-1}, \qquad z \notin \mathrm{spec}(H). \tag{2.12}$$

Furthermore, $(A \pm i\Lambda)^{-1}D(H) \subset D(H)$ *for all* $\Lambda \gg 1$ *and, by setting* $A(\Lambda) = i\Lambda A(A+i\Lambda)^{-1}$, *we have*

$$[H, A(\Lambda)]\varphi \to [H,A]^0\varphi, \qquad \Lambda \to \infty, \tag{2.13}$$

in $\mathcal{H}$, *for all* $\varphi \in D(H)$.

Proof. We apply (2.9) to $\varphi_n = \zeta_n(H-z)^{-1}\tilde{\varphi}$ and $\psi_n = \zeta_n(H-\bar{z})^{-1}\tilde{\psi}$ with $\tilde{\varphi}, \tilde{\psi} \in \mathcal{D}$. Since $[H,A]^0$ is bounded on $D(H)$, (2.7) implies that $[H,A]^0\varphi_n \to [H,A]^0(H-z)^{-1}\tilde{\varphi}$. Furthermore, $\zeta_n(H-z)^{-1}\tilde{\varphi} \to (H-z)^{-1}\tilde{\varphi}$ in $D(A)$ by (2.6) and (2.8) (the same holds for $\tilde{\psi}$) and hence

$$\left((H-z)^{-1}\tilde{\varphi}, A\tilde{\psi}\right) - \left(A\tilde{\varphi}, (H-\bar{z})^{-1}\tilde{\psi}\right) = \left([H,A]^0(H-z)^{-1}\tilde{\varphi}, (H-\bar{z})^{-1}\tilde{\psi}\right).$$

Since $\mathcal{D}$ is a core for A, the above equality actually holds for all $\tilde{\varphi}, \tilde{\psi} \in D(A)$. This shows (2.12). The proof of (2.13) follows as in [16]. Indeed (2.12) yields

$$[(H-z)^{-1}, (A-Z)^{-1}] = -(A-Z)^{-1}(H-z)^{-1}[H,A]^0(H-z)^{-1}(A-Z)^{-1}, \tag{2.14}$$

which implies that $(H+i)^{-1}(A \pm i\Lambda)^{-1} = (A \pm i\Lambda)^{-1}(H+i)^{-1}(1+\mathcal{O}(\Lambda^{-1}))$, where $\mathcal{O}(\Lambda^{-1})$ holds in the operator sense. This clearly implies that $(A \pm i\Lambda)^{-1}D(H) \subset D(H)$ for $\Lambda \gg 1$ and that $B(\Lambda) := (H+i)i\Lambda(A+i\Lambda)^{-1}(H+i)^{-1} \to 1$, in the strong sense on $\mathcal{H}$. The latter leads to (2.13) since, on $D(H)$,

$$[H, A(\Lambda)] = i\Lambda(A+i\Lambda)^{-1}[H,A]^0(H+i)^{-1}B(\Lambda)(H+i). \tag{2.15}$$

The proof is complete. □

The next proposition is important for several reasons. Firstly, it will allow to justify the manipulation of some commutators and secondly, it gives an explicit estimate for the norm of (the closure of) $[g(H), A](H+i)^{-1}$. It is also a key to the proof of the useful Proposition 2.4 below. We include the proof of Proposition 2.3, essentially taken from [16], to convince the reader that our assumptions are sufficient to get it.

Proposition 2.3. *Under the assumptions of Proposition* 2.2, *the following holds: for any bounded Borel function* g *such that* $\int |t\hat{g}(t)|\mathrm{d}t < \infty$, *we have* $g(H)(D(A) \cap D(H)) \subset D(A)$ *and*

$$||[g(H), A]\varphi|| \leq (2\pi)^{-1}\int |t\hat{g}(t)|\mathrm{d}t \; ||[H,A]^0(H+i)^{-1}|| \; ||(H+i)\varphi||, \qquad \forall\, \varphi \in D(A) \cap D(H).$$

Before proving this proposition, we quote the following important consequence.

Proposition 2.4. *In addition to the assumptions of Proposition* 2.2*, suppose that* (2.4) *holds. Then, for any* $\varphi \in D(A) \cap D(H)$*, there exists a sequence* $\varphi_n \in \mathcal{D}$ *such that, as* $n \to \infty$*,*

$$\varphi_n \to \varphi, \qquad A\varphi_n \to A\varphi \qquad \text{and} \qquad H\varphi_n \to H\varphi.$$

In particular, (2.9) *and* (2.10) *hold for all* $\varphi, \psi \in D(A) \cap D(H)$*.*

Proof. We choose $g \in C_0^\infty(\mathbb{R})$, $g = 1$ near 0, and set $\varphi_n = \zeta_n g_n(H)\varphi$, with $g_n(E) = g(E/n)$. It belongs to $\mathcal{D}$ by (2.4) and clearly converges to φ in $\mathcal{H}$. Furthermore, $(H+i)\zeta_n(H+i)^{-1}$ converges strongly on $\mathcal{H}$ by (2.7) and this easily shows that $H\varphi_n \to H\varphi$. Regarding $A\varphi_n$, we write

$$A\varphi_n = A\zeta_n(A+i)^{-1}g_n(H)(A+i)\varphi - A\zeta_n(A+i)^{-1}[g_n(H), A]\varphi$$

where $A\zeta_n(A+i)^{-1}$ converges strongly on $\mathcal{H}$ by (2.6) and $||[A, g_n(H)]\varphi|| \leq Cn^{-1}||(H+i)\varphi||$ since $\int |t\hat{g}_n(t)|\mathrm{d}t = \mathcal{O}(n^{-1})$. □

As a consequence of this proposition, we can define, for further use, the form $[[H, A]^0, A]$ by

$$\big([[H, A]^0, A]\varphi, \psi\big) := \big(A\varphi, i[H, A]^0\psi\big) - \big(i[H, A]^0\varphi, A\psi\big), \qquad \varphi, \psi \in D(A) \cap D(H). \qquad (2.16)$$

Proof of Proposition 2.3. We first observe that, if $\varphi \in D(A) \cap D(H)$ and $\Lambda \gg 1$, then for all t

$$e^{itH}A(\Lambda)e^{-itH}\varphi = A(\Lambda)\varphi + i\int_0^t e^{isH}[H, A(\Lambda)]e^{-isH}\varphi \; \mathrm{d}s.$$

This can be easily seen by weakly differentiating both sides with respect to t, testing them against an arbitrary element of $D(H)$. This equality shows that, for any $\psi \in \mathcal{H}$,

$$([A(\Lambda), g(H)]\varphi, \psi) = \frac{i}{2\pi}\int \hat{g}(t)\int_0^t \Big(e^{-i(t-s)H}[H, A(\Lambda)](H+i)^{-1}e^{-isH}(H+i)\varphi, \psi\Big) \, \mathrm{d}s\mathrm{d}t. \quad (2.17)$$

By (2.15), $[H, A(\Lambda)](H+i)^{-1}$ is uniformly bounded, so the modulus of right hand side is dominated by $C||\psi||$, for some C independent of Λ. In particular, if $\psi \in D(A)$,

$$(g(H)\varphi, A\psi) = \lim_{\Lambda\to\infty}(g(H), A(-\Lambda)\psi) = \lim_{\Lambda\to\infty}(g(H)A(\Lambda)\varphi, \psi) - ([g(H), A(\Lambda)]\varphi, \psi)$$

proves that $|(g(H)\varphi, A\psi)| \leq C||\psi||$, with C independent of $\psi \in D(A)$. This implies that $g(H)\varphi \in D(A^*) = D(A)$. Then, letting $\Lambda \to \infty$ in (2.17) clearly leads to the estimate on $||[g(H), A]\varphi||$. □

We now quote a crucial result which is directly taken from [16].

Proposition 2.5. *Assume that* B *is a bounded operator on* $\mathcal{H}$*. Then for any* $z \notin \mathbb{R}$ *and any* $\varepsilon \in \mathbb{R}$ *such that* $\mathrm{Im}(z)\varepsilon \geq 0$*, the operator* $H - z - i\varepsilon B^*B$ *is a bounded isomorphism from* $D(H)$ *(with the graph norm) onto* $\mathcal{H}$*. If we set*

$$G_z(\varepsilon) = (H - z - i\varepsilon B^*B)^{-1}$$

we have, provided $\mathrm{Im}(z)\varepsilon \geq 0$ *and* $\mathrm{Im}(z)\varepsilon_0 \geq 0$*,*

$$G_z(\varepsilon) - G_z(\varepsilon_0) = G_z(\varepsilon)i(\varepsilon - \varepsilon_0)B^*BG_z(\varepsilon_0),$$
$$G_z(\varepsilon)^* = G_{\bar{z}}(-\varepsilon), \qquad ||G_z(\varepsilon)|| \leq |\mathrm{Im}(z)|^{-1},$$

in the sense of bounded operators on $\mathcal{H}$*. Furthermore, if* B' *and* C *are bounded operators, with* C *self-adjoint, and if* $\mathrm{Im}(z)\varepsilon > 0$*, then*

$$B'^*B' \leq B^*B \qquad \Rightarrow \qquad ||B'G_z(\varepsilon)C|| \leq |\varepsilon|^{-1/2}\,||CG_z(\varepsilon)C||^{1/2}.$$

This result, which is one of the keys of the differential inequality technique of Mourre, will of course be used with $B^*B = f(H)i[H,A]^0 f(H)$, but it doesn't depend on any of the assumptions quoted in the beginning of this section. We refer to [16] for the proof and rather put emphasize on the following result.

Proposition 2.6. *Assume that all the conditions from* (2.1) *to* (2.11) *hold and define* $G_z(\varepsilon)$ *as above with* $B^*B = f(H)i[H,A]^0 f(H)$. *Then* $G_z(\varepsilon)D(A) \subset D(A) \cap D(H)$.

Proof. It suffices to show that $G_z(\varepsilon)\varphi$ belongs to $D(A)$ for any $\varphi \in D(A)$. As in the proof of Proposition 2.3, this is implied by the fact that $\sup_{\Lambda \geq \Lambda_0} ||[G_z(\varepsilon), A(\Lambda)]|| < \infty$, for Λ_0 large enough. To prove this, we remark that

$$[A(\Lambda), G_z(\varepsilon)] = G_z(\varepsilon)[H, A(\Lambda)]G_z(\varepsilon) - i\varepsilon G_z(\varepsilon)[B^*B, A(\Lambda)]G_z(\varepsilon)$$

where the first term of the right hand side is uniformly bounded by (2.13) and the uniform boundedness principle. We are thus left with the study of the second term for which we observe that

$$\begin{aligned}\big([(A+i\Lambda)^{-1}, B^*B]\psi_1, \psi_2\big) \quad = \quad & \big(i[H,A]^0 f(H)\psi_1, [f(H), (A-i\Lambda)^{-1}]\psi_2\big) + \\ & \big([(A+i\Lambda)^{-1}, f(H)]\psi_1, i[H,A]^0 f(H)\psi_2\big) + \\ & \big([(A+i\Lambda)^{-1}, i[H,A]^0]f(H)\psi_1, f(H)\psi_2\big),\end{aligned}$$

for all $\psi_1, \psi_2 \in D(H)$. Since $A(\Lambda) = i\Lambda + \Lambda^2(A+i\Lambda)^{-1}$, multiplying this equality by Λ^2 allows to replace $(A \pm i\Lambda)^{-1}$ by $A(\pm\Lambda)$. By (2.15) and (2.17) , $[f(H), A(\pm\Lambda)](H+i)^{-1}$ is uniformly bounded which reduces the proof of the proposition to the study of $\big([A(\Lambda), i[H,A]^0]f(H)\psi_1, f(H)\psi_2\big)$. To that end, we note that, if $\tilde{\psi}_1, \tilde{\psi}_2$ belong to $D(H)$, then $([A(\Lambda), i[H,A]^0]\tilde{\psi}_1, \tilde{\psi}_2)$ can be written

$$\Lambda^2 \left(A(A+i\Lambda)^{-1}\tilde{\psi}_1, i[H,A]^0(A-i\Lambda)^{-1}\tilde{\psi}_2\right) - \Lambda^2 \left(i[H,A]^0(A+i\Lambda)^{-1}\tilde{\psi}_1, A(A-i\Lambda)^{-1}\tilde{\psi}_2\right).$$

Using (2.10) and Proposition 2.4, combined with the fact that $\Lambda(H+i)(A \pm i\Lambda)^{-1}(H+i)^{-1}$ is uniformly bounded (see the proof of Proposition 2.2), we obtain the existence of $C > 0$ such that

$$\left|\left([A(\Lambda), i[H,A]^0]\tilde{\psi}_1, \tilde{\psi}_2\right)\right| \leq C||(H+i)\tilde{\psi}_1|| \, ||\tilde{\psi}_2||, \qquad \tilde{\psi}_1, \tilde{\psi}_2 \in D(H)$$

for $\Lambda \gg 1$. The conclusion follows. □

Note that we have chosen to include this proof, thought it is essentially the one of [16], since our assumptions on A are not the same as those of [16].

2.2 The limiting absorption principle

In this part, we repeat the method of differential inequalities of Mourre [16] to get estimates on the boundary values of $(H-z)^{-1}$. Our main goal is an explicit control of the different estimates in terms of the parameters, namely $A, H, f, \lambda, \alpha, \delta$ and $C_{H,A}$ (see (2.10)). As we shall see, the following quantities will play a great role

$$N_{[H,A]} := \left|\left|[H,A]^0(H+i)^{-1}\right|\right| \tag{2.18}$$

$$S_{H,A}^{f,\alpha} := \left(1 + \alpha^{-1}\left|\left|[H,A]^0 f(H)\right|\right|\right)^2 \tag{2.19}$$

$$\Delta_f := (2\pi)^{-1}\int_{\mathbb{R}} |t\hat{f}(t)| \, \mathrm{d}t. \tag{2.20}$$

We assume that all the conditions from (2.1) to (2.11) hold and that $G_z(\varepsilon)$ is defined by Proposition 2.5 with $B^*B = f(H)i[H,A]^0 f(H)$.

As a direct consequence of Proposition 2.5, we first get the estimate

$$\big|\big|f(H)(H+i)^k G_z(\varepsilon)w(A)\big|\big| \le (1+|\lambda|+3\delta)^k \alpha^{-1/2}|\varepsilon|^{-1/2}\,||w(A)G_z(\varepsilon)w(A)||^{1/2}$$

which holds for any bounded and real valued Borel function w. We also obtain immediately

$$||f(H)G_z(\varepsilon)f(H)|| \le \alpha^{-1}|\varepsilon|^{-1}. \tag{2.21}$$

On the other hand, by the resolvent identity given in Proposition 2.5, we see that

$$G_z(\varepsilon)f(H) = G_z(0)\left(f(H) - \varepsilon f(H)[H,A]^0 f(H)G_z(\varepsilon)f(H)\right)$$

where the bracket is uniformly bounded with respect to ε by (2.21) and we obtain

$$\big|\big|(H+i)^k(1-f)(H)G_z(\varepsilon)f(H)\big|\big| \le \sup_{|E-\lambda|\ge 2\delta}\frac{|E+i|^k}{|E-z|}\left(1+\alpha^{-1}\,\big|\big|[H,A]^0 f(H)\big|\big|\right), \tag{2.22}$$

for $k=0,1$. Here we used the fact that $f(H)[H,A]^0$ has a bounded closure whose norm equals $||[H,A]^0 f(H)||$. Another application of the resolvent identity also gives

$$G_z(\varepsilon)(1-f)(H) = G_z(0)\left((1-f)(H) - \varepsilon f(H)[H,A]^0 f(H)G_z(\varepsilon)(1-f)(H)\right) \tag{2.23}$$

in which $f(H)G_z(\varepsilon)(1-f)(H)$ can be estimated (independently of ε) using (2.22).

Summing up, all this leads to

Proposition 2.7. *Assume that λ,δ,α satisfy condition (c) of Subsection 2.1 and that*

$$\varepsilon \mathrm{Im}\, z > 0, \qquad |\mathrm{Re}\, z - \lambda| \le \delta, \qquad \delta \le \alpha \qquad \text{and} \qquad |\varepsilon| \le \delta\alpha^{-1}. \tag{2.24}$$

Then, for $k=0,1$ and all bounded Borel function w such that $||w||_\infty \le 1$, we have

$$\big|\big|(H+i)^k(1-f)(H)G_z(\varepsilon)\big|\big| \le (1+|\lambda|+2\delta)^k\delta^{-1}\left(1+S^{f,\alpha}_{H,A}\right), \tag{2.25}$$

$$\big|\big|(H+i)^k f(H)G_z(\varepsilon)w(A)\big|\big| \le (1+|\lambda|+3\delta)^k\alpha^{-1/2}|\varepsilon|^{-1/2}\,||w(A)G_z(\varepsilon)w(A)||^{1/2}, \tag{2.26}$$

$$||w(A)G_z(\varepsilon)w(A)|| \le \alpha^{-1}|\varepsilon|^{-1}\left(2+S^{f,\alpha}_{H,A}\right). \tag{2.27}$$

Note that the right hand side of (2.25) is independent of ε. Note that we also get estimates on $G_z(\varepsilon)(1-f)(H)$ and $w(A)G_z(\varepsilon)f(H)$ for free, by taking the adjoints, since $G_z(\varepsilon)^* = G_{\bar z}(-\varepsilon)$.

We then need to get an estimate on $dG_z(\varepsilon)/d\varepsilon$. To that end, we simply repeat the proof of Mourre [16], observing that the algebraic manipulations are valid in our context thanks to the results of Subsection 2.5. In the sense of quadratic forms on $D(A)$, using in particular $[[H,A]^0,A]$ defined by (2.16), we thus obtain

$$\begin{aligned}\frac{dG_z(\varepsilon)}{d\varepsilon} &= G_z(\varepsilon)(1-f)(H)[H,A]^0 f(H)G_z(\varepsilon) + G_z(\varepsilon)[H,A]^0(1-f)(H)G_z(\varepsilon) - \\ &\quad \varepsilon\left\{G_z(\varepsilon)f(H)[H,A]^0[f(H),A]G_z(\varepsilon) + G_z(\varepsilon)[f(H),A][H,A]^0 f(H)G_z(\varepsilon)\right. \\ &\quad \left.+\, G_z(\varepsilon)f(H)[[H,A]^0,A]f(H)G_z(\varepsilon)\right\} + G_z(\varepsilon)A - AG_z(\varepsilon).\end{aligned} \tag{2.28}$$

Let us set $F_z(\varepsilon) := w(A)G_z(\varepsilon)w(A)$. By Proposition 2.7, (2.28) leads to the differential inequality

$$\begin{aligned}\left|\left|w(A)\frac{dG_z(\varepsilon)}{d\varepsilon}w(A)\right|\right| &\leq C_1||F_z(\varepsilon)|| + C_{1/2}|\varepsilon|^{-1/2}||F_z(\varepsilon)||^{1/2} + C_0 \\ &\quad + 2||Aw(A)||\left(\alpha^{-1/2}|\varepsilon|^{-1/2}||F_z(\varepsilon)||^{1/2} + \delta^{-1}\left(1 + S^{f,\alpha}_{H,A}\right)\right) \qquad (2.29)\end{aligned}$$

where, by Proposition 2.3, the constants $C_0, C_{1/2}$ and C_1 can be chosen as follows

$$\begin{aligned} C_0 &= \delta^{-2}(1 + |\lambda| + 2\delta)\left(1 + S^{f,\alpha}_{H,A}\right)^2 N_{[H,A]}, \\ C_{1/2} &= 2\alpha^{-1/2}\delta^{-1}(1 + |\lambda| + 3\delta)S^{f,\alpha}_{H,A}N_{[H,A]}\left(1 + \delta\alpha^{-1}\Delta_f N_{[H,A]}(1 + |\lambda| + 3\delta)\right), \\ C_1 &= \alpha^{-1}(1 + |\lambda| + 3\delta)\left(C_{H,A} + 2\Delta_f N^2_{[H,A]}(1 + |\lambda| + 3\delta)\right). \end{aligned}$$

The second line of (2.29) suggests that $Aw(A)$ must be bounded. Of course, this holds if $w(E) = \langle E\rangle^{-1}$ (which was the original choice of weight in [16]) however a trick of Mourre, which is reproduced in [18], allows to consider

$$w(E) = \langle E\rangle_\varepsilon^{-s} := \langle E\rangle^{-s}\langle\varepsilon E\rangle^{s-1}, \qquad 1/2 < s \leq 1.$$

It is indeed not hard to check that the following inequality holds for all $\varepsilon \neq 0$ and $E \in \mathbb{R}$

$$\left|\frac{\partial}{\partial\varepsilon}\langle E\rangle_\varepsilon^{-s}\right| = (1-s)\langle E\rangle_\varepsilon^{-s}\frac{|\varepsilon|E^2}{1 + \varepsilon^2E^2} \leq (1-s)|\varepsilon|^{s-1},$$

and this implies that

$$\left|\left|d\langle A\rangle_\varepsilon^{-s}/d\varepsilon G_z(\varepsilon)\langle A\rangle_\varepsilon^{-s}\right|\right| \leq (1-s)|\varepsilon|^{s-1}\left(\alpha^{-1/2}|\varepsilon|^{-1/2}||F_z(\varepsilon)||^{1/2} + \delta^{-1}\left(1 + S^{f,\alpha}_{H,A}\right)\right). \quad (2.30)$$

Using (2.29), (2.30) and the fact that $\langle E\rangle_\varepsilon^{-s}\langle E\rangle \leq |\varepsilon|^{s-1}$ for $0 < |\varepsilon| \leq 1$, we get the final differential inequality

$$\begin{aligned}||dF_z(\varepsilon)/d\varepsilon|| &\leq C_1||F_z(\varepsilon)|| + C_{1/2}|\varepsilon|^{-1/2}||F_z(\varepsilon)||^{1/2} + C_0 \\ &\quad + 2(2-s)|\varepsilon|^{s-1}\left(2\alpha^{-1/2}|\varepsilon|^{-1/2}||F_z(\varepsilon)||^{1/2} + \delta^{-1}\left(1 + S^{f,\alpha}_{H,A}\right)\right) \quad (2.31)\end{aligned}$$

which is valid if $0 < |\varepsilon| \leq 1$ and if (2.24) holds.

Starting from (2.27) and using (2.31), a finite number of integrations leads to a uniform bound on $||F_z(\varepsilon)||$ for $0 < |\varepsilon| \leq \min(1, \delta\alpha^{-1})$ and thus on $||F_z(0)||$. Such estimates depend of course on $A, H, f, \alpha, \lambda, \delta, C_0, C_{1/2}$ and C_1, but there is no reasonable way to express this dependence in general. We thus rather consider a particular case in the following theorem, which lightens the role of α, λ, δ.

Theorem 2.8. *Consider families of operators H_ν, A_ν, of numbers $\lambda_\nu, \alpha_\nu, \delta_\nu$ and of functions f_ν satisfying conditions (a),(b),(c) for all ν describing some set Σ. Denote by $C_{0,\nu}, C_{1/2,\nu}$ and $C_{1,\nu}$ the corresponding constants defined on page 12. Assume that $\varepsilon_\nu := \delta_\nu\alpha_\nu^{-1} \leq 1$ and that there exists $C > 0$ such that, for all $\nu \in \Sigma$,*

$$C_{0,\nu} \leq C\varepsilon_\nu^{-1}\delta_\nu^{-1}, \qquad C_{1/2,\nu} \leq C\varepsilon_\nu^{-1/2}\delta_\nu^{-1/2}, \qquad C_{1,\nu} \leq C\varepsilon_\nu^{-1}, \qquad ||[H_\nu, A_\nu]^0 f_\nu(H_\nu)|| \leq C\alpha_\nu \quad (2.32)$$

with f_ν of the form $f_\nu(E) = f((E-\lambda_\nu)/\delta_\nu)$, for some fixed $f \in C_0^\infty(\mathbb{R})$. Then, for all $1/2 < s \leq 1$, there exists $C_s > 0$ such that, for all $\nu \in \Sigma$,

$$||\langle A_\nu\rangle^{-s}(H_\nu - z)^{-1}\langle A_\nu\rangle^{-s}|| \leq C_s\delta_\nu^{-1}, \tag{2.33}$$

provided $|\mathrm{Re}\ z - \lambda_\nu| \leq \delta_\nu$. Furthermore, for any $\mu \in (\lambda_\nu - \delta_\nu, \lambda_\nu + \delta_\nu)$, the limits

$$\langle A_\nu\rangle^{-s}(H_\nu - \mu \pm i0)^{-1}\langle A_\nu\rangle^{-s} := \lim_{\varepsilon\to 0^+}\langle A_\nu\rangle^{-s}(H_\nu - \mu \pm i\varepsilon)^{-1}\langle A_\nu\rangle^{-s}$$

exist and are continuous, with respect to μ, in the operator topology.

In practice, the conditions (2.32) can be checked using the explicit forms of $C_0, C_{1/2}$ and C_1 given on page 12. We shall use this extensively in the next section.

Proof. We only consider the case where $\varepsilon \in (0, \varepsilon_\nu]$, i.e. the situation where Im z is positive, since the one of $\varepsilon \in [-\varepsilon_\nu, 0)$ is similar. By the assumption on $||[H_\nu, A_\nu]^0 f_\nu(H_\nu)||$, the estimate (2.27) takes the form $||F_z(\varepsilon)|| \leq C\alpha_\nu^{-1}\varepsilon^{-1}$, thus (2.31) implies that

$$||F_z(\varepsilon) - F_z(\varepsilon_\nu)|| \leq C_s\left(\delta_\nu^{-1} + \delta_\nu^{-1}\log(\varepsilon_\nu/\varepsilon) + \alpha_\nu^{-1}\varepsilon^{s-1}\right), \qquad \forall\ \nu \in \Sigma,$$

if $1/2 < s < 1$. If $s = 1$, the term ε^{s-1} must be replaced by $\log(\varepsilon_\nu/\varepsilon)$ which can be absorbed by the second term of the bracket, for we assume that $\alpha_\nu^{-1} \leq \delta_\nu^{-1}$. Since $||F_z(\varepsilon_\nu)|| \leq C\delta_\nu^{-1}$, a finite number of iterations of Lemma 2.9 below completes the proof of (2.33). For the existence of the boundary values of the resolvent, which are purely local, we refer to [18] (Theorem 8.1). □

Lemma 2.9. *Let $0 \leq \sigma < 1$ and assume the existence of C such that, for all $\nu \in \Sigma$ and all $\varepsilon \in (0, \varepsilon_\nu]$,*

$$||F_z(\varepsilon)|| \leq C\left(\delta_\nu^{-1} + \delta_\nu^{-1}\log(\varepsilon_\nu/\varepsilon) + \alpha_\nu^{-1}\varepsilon^{-\sigma}\right).$$

Then, there exists $C_{s,\sigma}$ such that, for all $\nu \in \Sigma$ and all $\varepsilon \in (0, \varepsilon_\nu]$

$$||F_z(\varepsilon)|| \leq C_{s,\sigma}\begin{cases} \delta_\nu^{-1} + \alpha_\nu^{-1}\varepsilon^{s-1/2-\sigma/2}, & \text{if } s - 1/2 < \sigma/2, \\ \delta_\nu^{-1} + \delta_\nu^{-1}\log(\varepsilon_\nu/\varepsilon), & \text{if } s - 1/2 = \sigma/2, \\ \delta_\nu^{-1}, & \text{if } s - 1/2 > \sigma/2. \end{cases}$$

Proof. It simply follows from (2.31) and the fact that $||F_z(\varepsilon_\nu)|| \leq C\delta_\nu^{-1}$, by studying separately the three cases and using the trivial inequality

$$\left(\delta_\nu^{-1} + \delta_\nu^{-1}\log(\varepsilon_\nu/\varepsilon) + \alpha_\nu^{-1}\varepsilon^{-\sigma}\right)^{1/2} \leq\ \delta_\nu^{-1/2} + \delta_\nu^{-1/2}\log^{1/2}(\varepsilon_\nu/\varepsilon) + \alpha_\nu^{-1/2}\varepsilon^{-\sigma/2}$$

to control the terms involving $||F_z(\varepsilon)||^{1/2}$. □

3 Applications to asymptotically hyperbolic manifolds

3.1 The conjugate operator

In this part, we recall the construction of the conjugate operator defined by Froese-Hislop in [9]. We emphasize that the main ideas, namely the form of the conjugate operator and the existence of a positive commutator estimate, are taken from [9]. However, since some of our assumptions (especially *(a), (b)* in subsection 2.1) differ from those of [9] and since we need to control estimates with respect to the spectral parameter, we will give a rather detailed construction.

Let $\chi, \xi \in C^\infty(\mathbb{R})$ be non negative and non decreasing functions such that

$$\chi(r) = \begin{cases} 0, & r \leq 1, \\ 1, & r \geq 2 \end{cases}, \qquad \xi(r) = \begin{cases} 0, & r \leq -1, \\ 1, & r \geq -\frac{1}{2} \end{cases}.$$

By possibly replacing χ and ξ by χ^2 and ξ^2, we may assume that $\chi^{1/2}$ and $\xi^{1/2}$ are smooth. For $R > r_0$ and $S > R$, we set $\chi_R(r) = \chi(r/R)$ and $\xi_S(r) = \xi(r/S)$. Then, recalling that $(\mu_k)_{k\geq 0} = \mathrm{spec}(\Delta_h)$ and setting $\nu_k = (1+\mu_k)^{1/2}$, we define the sequence of smooth functions

$$a_k(r) = (r + 2S - \log \nu_k)\chi_R(r)\xi_S(r - \log \nu_k).$$

They are real valued and it is easy to check that their derivatives satisfy, for all $j \geq 1$ and $k \in \mathbb{N}$,

$$||a_k^{(j)}||_\infty \leq C_j S^{1-j}, \qquad ||a_k a_k^{(j+1)}||_\infty \leq C_j, \tag{3.1}$$

uniformly with respect to $R > S > r_0$. Further on, R and S will depend on the large spectral parameter λ but till then we won't mention the dependence of a_k (nor of the related objects) on R, S.

According to the results recalled in Appendix A, there exists, for each k, a strongly continuous unitary group e^{itA_k} on $L^2(\mathbb{R})$ whose self-adjoint generator A_k is

$$A_k = a_k D_r - i a_k'/2, \tag{3.2}$$

i.e. a self-adjoint realization of the r.h.s. Furthermore, we can consider e^{itA_k} as a group on $L^2(I)$, since e^{itA_k} acts as the identity on functions supported in $(-\infty, R)$ hence maps functions supported in I into functions supported in I (see Appendix A). Therefore, using the notation (1.6) for φ_k, the linear map

$$\varphi \mapsto \sum_{k\geq 0} e^{itA_k}\varphi_k \otimes \psi_k \tag{3.3}$$

clearly defines a strongly continuous unitary group on $L^2(I) \otimes L^2(Y, d\mathrm{Vol}_h)$. The pull back on $L^2(\mathcal{M} \setminus \mathcal{K})$ of the operator (3.3), extended as the identity on $L^2(\mathcal{K})$, is also a strongly continuous unitary group on $L^2(\mathcal{M})$ which we denote by $U(t)$. (Here again we omit the R, S dependence in the notation). Using Stone's Theorem [19], we can state the

Definition 3.1. *We call A the self-adjoint generator of $U(t)$. In particular, its domain is*

$$D(A) = \{\varphi \in L^2(\mathcal{M}) \mid U(t)\varphi \text{ is strongly differentiable at } t = 0\},$$

and $A\varphi = i^{-1}dU(t)\varphi/dt_{|t=0}$ for all $\varphi \in D(A)$.

Remark. Note that this definition clearly implies that $L^2(\mathcal{K}) \subset D(A)$ and that $A_{|L^2(\mathcal{K})} \equiv 0$.

Now we choose a sequence of functions $\zeta_n \in C_c^\infty(\mathcal{M})$ such that $\zeta_n \to 1$ strongly on $L^2(\mathcal{M})$. More precisely, we choose ζ_n of the form $\zeta_n = \zeta(2^{-n}r)$ for some $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta = 1$ on a large enough compact set (containing 0) to ensure that $\zeta_n = 1$ near $\overline{\mathcal{K}}$.

Proposition 3.2. *i) For all n, $\zeta_n D(A) \subset D(A)$.*
ii) For all $\varphi \in D(A)$, $A\zeta_n\varphi \to A\varphi$ as $n \to \infty$.
iii) $C_B^\infty(\mathcal{M})$ is a core for A and $AC_B^\infty(\mathcal{M}) \subset C_0^\infty(\mathcal{M})$.

Proof. In view of the remark above, we only have to consider $\varphi \in L^2(\mathcal{M} \setminus \mathcal{K})$ (i.e. supported in $\mathcal{M} \setminus \mathcal{K}$). Furthermore, to simplify the notations, we shall denote indifferently by φ an element of $L^2(\mathcal{M} \setminus \mathcal{K})$ and the corresponding element in $L^2(I) \otimes L^2(Y, d\mathrm{Vol}_h)$ via (1.5).

Let us first observe that, for all such $\varphi, \tilde{\varphi}$, Parseval's identity yields

$$||(U(t)\varphi - \varphi)/it - \tilde{\varphi}||^2 = \sum_k \left|\left|\left(e^{itA_k}\varphi_k - \varphi_k\right)/it - \tilde{\varphi}_k\right|\right|^2 .$$

Thus, by dominated convergence, this easily implies that $\varphi \in D(A)$ if and only if $\varphi_k \in D(A_k)$ for all k and $\sum_k ||A_k\varphi_k||^2 < \infty$, in which case $(A\varphi)_k = A_k\varphi_k$ for all k. Combining this characterization with $(A.3)$, and using the fact that $(a_k\zeta_n')(r) = 2^{-n}a_k(r)\zeta'(2^{-n}r)$ is uniformly bounded with respect to $k, n \in \mathbb{N}$ on I, which is due to the fact that $a_k(r)/r$ is bounded with respect r and k, we get *i)*. This also shows that

$$||A\zeta_n\varphi - \zeta_n A\varphi||^2 = \sum_k ||a_k\zeta_n'\varphi_k||^2$$

where the right hand side goes to 0 as $n \to \infty$ by dominated convergence, and hence implies *ii)*. We now prove *iii)*. Since $A\varphi \equiv 0$ for any function supported outside $\iota^{-1}([R,\infty) \times Y)$, and since any element of $D(A)$ can be approached by compactly supported ones by *ii)*, it is clearly enough to show that for any $\varphi \in D(A)$, compactly supported in $\iota^{-1}([R',\infty) \times Y)$ with $r_0 < R' < R$, and any $\epsilon > 0$ small enough, there exists $\varphi^\epsilon \in C_0^\infty(\mathcal{M} \setminus \mathcal{K})$ such that $||\varphi - \varphi^\epsilon|| + ||A\varphi - A\varphi^\epsilon|| < \epsilon$. Using the function θ_ϵ defined in Appendix A, we set

$$\varphi^\epsilon = \sum_k \varphi_k * \theta_\epsilon \otimes e^{-|\epsilon|\mu_k}\psi_k.$$

It is clearly compactly supported in $I \times Y$ if ϵ is small enough and smooth since $\partial_r^j \Delta_h^l \varphi^\epsilon \in L^2$ for all $j, l \in \mathbb{N}$. Then, by Parseval's identity, we have $\varphi^\epsilon \to \varphi$ and using $(A.9)$ we also have $A\varphi^\epsilon \to A\varphi$. For the last statement, we first observe that, if φ is compactly supported, so is $A\varphi$. We are thus left with the regularity for which we observe that $[\partial_r^j, A_k] = \sum_{m\leq j} b_{k,m}(r)\partial_r^m$, with $b_{k,m}$ uniformly bounded by (3.1), and hence

$$||\partial_r^j \mu_k^l A_k\varphi_k|| \;\leq\; ||A_k(\partial_r^j\Delta_h^l\varphi)_k|| + C\sum_{m\leq j} ||(\partial_r^m\Delta_h^l\varphi)_k|| \;\in\; l^2(\mathbb{N}_k)$$

yields the result. □

Note that the choice of $C_B^\infty(\mathcal{M})$ is dictated by the following proposition.

Proposition 3.3. *For all $n \in \mathbb{N}$, $z \notin \mathrm{spec}(H)$ and $g \in C_0^\infty(\mathbb{R})$, we have*

$$\zeta_n(H - z)^{-1}C_B^\infty(\mathcal{M}) \subset C_B^\infty(\mathcal{M}), \qquad \zeta_n g(H)L^2(\mathcal{M}) \subset C_B^\infty(\mathcal{M}).$$

Proof. This is a direct consequence of standard elliptic regularity results (see for instance [7, 12]), taking into account the fact that $\zeta_n = 1$ near $\partial\mathcal{M}$ (if non empty). □

We now consider the calculations of $[H, A]$ and $[[H, A], A]$. Note that these commutators make perfectly sense on $C_B^\infty(\mathcal{M})$ by Propositions 3.2 and the fact that $C_B^\infty(\mathcal{M}) \subset D(H)$.

We first consider the "free parts", i.e. the commutators involving H_0 defined by (1.9).

Proposition 3.4. *There exists C such that for all $R > S > r_0 + 1$ and all $\varphi \in C_B^\infty(\mathcal{M})$.*

$$||[H_0, A]\varphi|| + ||[[H_0, A], A]\varphi|| \leq C||(H + i)\varphi||. \tag{3.4}$$

Proof. Similarly to the proof of Proposition 3.2, we identify $L^2(\mathcal{M}\setminus\mathcal{K})$ and $L^2(I)\otimes L^2(Y,d\mathrm{Vol}_h)$ for notational simplicity. Straightforward calculations show that

$$i[H_0,A]\varphi = \sum_{k\geq 0}\Big(2a_k' D_r^2 + 2a_k\mu_k e^{-2r} - 2a_k''\partial_r - a_k^{(3)}/2\Big)\varphi_k\otimes\psi_k, \tag{3.5}$$

$$[[H_0,A],A]\varphi = \sum_{k\geq 0}\big(b_k D_r^2 + c_k D_r + d_k\big)\varphi_k\otimes\psi_k, \tag{3.6}$$

where the functions $b_k(r), c_k(r), d_k(r)$ are given by

$$b_k = 2(a_k a_k'' - 2{a_k'}^2), \qquad c_k = 5ia_k'a_k'' - ia_k a_k^{(3)},$$
$$d_k = 2a_k\mu_k e^{-2r}(a_k' - 2a_k) + a_k'a_k^{(3)} - (a_k a_k^{(4)} - {a_k''}^2)/2.$$

One easily checks that $a_k\mu_k e^{-2r}$ and $a_k^2\mu_k e^{-2r}$ are uniformly bounded with respect to $k\in\mathbb{N}$ and $R>S>r_0+1$, thus, using (3.1), the result is direct consequence of the following lemma. □

Lemma 3.5. *For all differential operator P with coefficients supported in $\mathcal{M}\setminus\mathcal{K}$ such that*

$$\tilde{\Psi}^* P\tilde{\Psi}_* = \sum_{j+|\beta|\leq 2} c_{j,\beta}(r,y)(e^{-r}D_y)^\beta D_r^j$$

with $c_{j,\beta}$ bounded on $I\times U_0$ for all $U_0\Subset U$ (with the notations of page 2), there exists C such that

$$||P\varphi|| \leq C||(H+i)\varphi||, \qquad \forall\ \varphi\in D(H).$$

Proof. It is a direct application of Lemma 1.3 of [9]. □

We will now give a pseudo-differential approximation of A which will be useful both for computing the "perturbed parts" $[A,V]$, $[A,[A,V]]$ and for the proof of Theorem 1.3.

Following [12], we say that, for $m\in\mathbb{R}$, $g\in S^m(\mathbb{R}_x^{d_1}\times\mathbb{R}_\varsigma^{d_2})$ if $|\partial_x^\alpha\partial_\varsigma^\beta g(x,\varsigma)|\leq C_{\alpha,\beta}\langle\varsigma\rangle^{m-|\beta|}$, for all α,β. If $g\in S^0(\mathbb{R}_r\times\mathbb{R}_\mu)$ is supported in $I\times\mathbb{R}$, we clearly define a bounded operator on $L^2(I\times Y)$ by

$$g(r,\Delta_h)\varphi = \sum_{k\geq 0} g(r,\mu_k)\varphi_k\otimes\psi_k.$$

Abusing the notation for convenience, we still denote by $g(r,\Delta_h)$ the pullback of this operator on $L^2(\mathcal{M}\setminus\mathcal{K})$, extended by 0 on $L^2(\mathcal{K})$. If $\theta\in C^\infty(Y)$, we also denote by θ (instead of $1\otimes\theta$) its natural extension to $I\times Y$ which is independent of r. Our pseudo-differential approximation of A will mainly follow from the following result.

Proposition 3.6. *Let $g\in S^0(\mathbb{R}_r\times\mathbb{R}_\mu)$ be supported in $I\times Y$. For all coordinate patch $U_Y\subset Y$, all $\theta,\tilde{\theta}\in C_0^\infty(U_Y)$ such that $\tilde{\theta}\equiv 1$ near the support of θ and all N large enough, there exists $g_N\in S^0(\mathbb{R}_{r,y}^n\times\mathbb{R}_\eta^{n-1})$ and an operator $\mathcal{R}_N^\theta : L^2(\mathcal{M})\to L^2(\mathcal{M})$ such that*

$$\theta g(r,\Delta_h) = G_N^\theta + \mathcal{R}_N^\theta \tag{3.7}$$

where $G_N^\theta = \tilde{\Psi}_*\Big((\Psi^*\theta)(y)g_N(r,y,D_y)(\Psi^*\tilde{\theta})(y)\Big)\tilde{\Psi}^*$ (with the notation (1.4)) and

$$\Big|\Big|\Delta_h^j\mathcal{R}_N^\theta\Delta_h^k\varphi\Big|\Big| \leq C_{j,k}||\varphi||, \qquad \varphi\in C_c^\infty(\mathcal{M}), \tag{3.8}$$

$$\Big|\Big|\Delta_h^j[D_r,\mathcal{R}_N^\theta]\Delta_h^k\varphi\Big|\Big| \leq C_{j,k}||\varphi||, \qquad \varphi\in C_c^\infty(\mathcal{M}), \tag{3.9}$$

for all $j,k \le N$. If p_h is the principal symbol of Δ_h, we actually have

$$g_N(r,y,\eta) = g(r,p_h(y,\eta)) + \sum_{1\le j\le j_N} \sum_l d_{jl}(y,\eta)\partial_\mu^j g(r,p_h(y,\eta))$$

where d_{jl} are polynomials of degree $2j-l$ in η, obtained as universal sums of products of the full symbol of Δ_h in coordinates (y,η).

More generally, if $(g_\lambda)_{\lambda\in\Lambda}$ is a bounded family in $S^0(\mathbb{R}_r\times\mathbb{R}_\mu)$ with support in $I\times\mathbb{R}$, the associated family $(g_{\lambda,N})_{\lambda\in\Lambda}$ is bounded in S^0 and the constant $C_{j,k}$ in (3.8) can be chosen independent of $\lambda\in\Lambda$.

The proof is given in Appendix B. Note that, strictly speaking, this proposition is not a direct consequence of the standard functional calculus for elliptic pseudo-differential operators on closed manifolds [22] since g depends on the extra variable r. However, the proof follows from minor adaptations of the techniques of [11, 22].

Remark 1. The operators $g(r,\Delta_h)$ and G_N^θ commute with operators of multiplication by functions of r, hence so does $\mathcal{R}_N^\theta$.

Remark 2. In (3.8), we have abused the notation by identifying Δ_h, which acts on functions on Y, with its natural extension acting on functions on $\mathcal{M}$ which are supported in $\mathcal{M}\setminus\mathcal{K}$.

The previous proposition is motivated by the fact that we can write

$$A = g_{R,S}(r,\Delta_h)rD_r + \tilde{g}_{R,S}(r,\Delta_h). \tag{3.10}$$

with functions $g_{R,S}$ and $\tilde{g}_{R,S}$ belonging to $S^0(\mathbb{R}_r\times\mathbb{R}_\mu)$ as explained by the following lemma.

Lemma 3.7. *There exist two families $g_{R,S}, h_{R,S} \in S^0(\mathbb{R}_r\times\mathbb{R}_\mu)$, bounded for $R > S > r_0+1$, supported in $r > R$ and such that*

$$g_{R,S}(r,\mu_k) = a_k(r)/r, \qquad \tilde{g}_{R,S}(r,\mu_k) = a_k'(r)/2i$$

for all $k\ge 0$.

Proof. With $\gamma\in C^\infty(\mathbb{R}_\mu)$ such that $\gamma=1$ on $\mathbb{R}^+$ and $\operatorname{supp}\gamma\in[-1/2,\infty)$, we may choose

$$g_{R,S}(r,\mu) = \gamma(\mu)\chi_R(r)\left(1+2\frac{S}{r}-\frac{1}{2r}\log(1+\mu)\right)\xi_S\left(r-\frac{1}{2}\log(1+\mu)\right).$$

It is easily seen to belong to $S^0(\mathbb{R}^2)$ and the boundedness with respect to R,S follows from

$$\partial_\mu^j\xi_S\left(r-\frac{1}{2}\log(1+\mu)\right) = \sum_{1\le k\le j} c_{jk}S^{-k}\xi^{(k)}\left(\frac{r}{S}-\frac{1}{2S}\log(1+\mu)\right)(1+\mu)^{-j},$$

the fact that $-S/2\le r-\frac{1}{2}\log(1+\mu)\le r+\log 2^{1/2}$ on the support of $\gamma(\mu)\xi_S(r-\frac{1}{2}\log(1+\mu))$ and the fact that S/r is bounded on the support of $\chi_R(r)$. Then, we may choose $\tilde{g}_{R,S} = g_{R,S}+r\partial_r g_{R,S}$ since one checks similarly that $r\partial_r g_{R,S}$ is bounded in S^0. □

We are now ready to study the contribution of the perturbation V for the commutators.

Proposition 3.8. *There exists $C>0$ such that, for all $R>S>r_0$ and all $\varphi\in C_B^\infty(\mathcal{M})$*

$$||\langle r\rangle^j[A,V]\varphi|| \le CR^{j-1}||(H+i)\varphi||, \qquad j=0,1, \tag{3.11}$$

$$||[A,[A,V]]\varphi|| \le C||(H+i)\varphi||. \tag{3.12}$$

Proof. Dropping the subscripts R, S on g and $\tilde{g}$, we have

$$A = g(r, \Delta_h) r D_r + \tilde{g}(r, \Delta_h) = (G_N + \mathcal{R}_N)\, r D_r + \tilde{G}_N + \tilde{\mathcal{R}}_N$$

with $G_N = \sum_l G_N^{\theta_l}$ and $\mathcal{R}_N = \sum_l \mathcal{R}_N^{\theta_l}$ associated to g by mean of proposition 3.6 and of a partition of unit $\sum_l \theta_l = 1$ on Y. Of course, $\tilde{G}_N$ and $\tilde{\mathcal{R}}_N$ are similarly associated to $\tilde{g}$. Note that $g(r, \Delta_h)$ and G_N map $C_c^\infty(\mathcal{M})$ into $C_0^\infty(\mathcal{M} \setminus \mathcal{K})$ and thus so does $\mathcal{R}_N$. Therefore, on $C_B^\infty(\mathcal{M})$, we have

$$[A, V] = ([G_N, V] + [\mathcal{R}_N, V])\, r D_r + g(r, \Delta_h)[r D_r, V] + [\tilde{G}_N, V] + [\tilde{\mathcal{R}}_N, V].$$

We study the terms one by one. Note first that $[G_N, V] r D_r = r \langle r \rangle^{-2} [G_N, \langle r \rangle^2 V] D_r$. If $\tilde{\Psi}_l$ is associated to a coordinate chart Ψ_l defined in a neighborhood of supp θ_l by (1.4), we have

$$\tilde{\Psi}_l^* [G_N^{\theta_l}, \langle r \rangle^2 V] \tilde{\Psi}_{l*} = \sum_{|\beta| \leq 1} q_\beta(r, y, D_y)(e^{-r} D_y)^\beta$$

with $q_\beta \in S^0$ which depends, in a bounded way, on $R > S > r_0$ and is supported in $r \geq R$. This follows by standard pseudo-differential calculus and thus, by Lemma 3.5, we have

$$\big|\big| \langle r \rangle^j [G_N, V] r D_r \varphi \big|\big| \leq C R^{j-1} ||(H+i)\varphi||, \qquad \varphi \in C_B^\infty(\mathcal{M})$$

with C independent of $R > S > r_0$. Similarly, we get the same estimate for $[\mathcal{R}_N, V] r D_r$ since $\langle r \rangle^2 [\mathcal{R}_N, V]$ is a bounded operator, uniformly with respect to $R > S > r_0$, with range contained in the space of functions supported in $r \geq R$. The same holds for $[\tilde{G}_N, V]$ and $[\tilde{\mathcal{R}}_N, V]$. Finally, $\langle r \rangle [V, r D_r]$ is an operator of the form considered in Lemma 3.5, whereas $||\langle r \rangle^{-1} g(r, \Delta_h)|| \leq C R^{-1}$, so (3.11) follows.

We now consider $[A, [A, V]]$. We only study $[g(r, \Delta_h) r D_r, [g(r, \Delta_y) r D_r, V]]$, since the other terms can be studied similarly and involve less powers of $r D_r$. This double commutator reads

$$\begin{aligned} &[g(r, \Delta_h), [g(r, \Delta_y) r D_r, V]]\, r D_r + g(r, \Delta_h)\, [r D_r, [g(r, \Delta_y) r D_r, V]] = \\ &\qquad [G_N, [G_N r D_r, V]]\, r D_r + G_N\, [r D_r, [G_N r D_r, V]] + I_N D_r^2 + J_N D_r + K_N \end{aligned} \tag{3.13}$$

where I_N, J_N, K_N are bounded operator on $L^2(\mathcal{M})$, uniformly with respect to $R > S > r_0 + 1$. This clearly follows from Proposition 3.6 and the fact that $1 \otimes (\Delta_h + 1)^{-j} (r^2 V) 1 \otimes (\Delta_h + 1)^{-k}$ is bounded if $j + k \geq 1$. Precisely, $1 \otimes (\Delta_h + 1)^{-1}$ is actually defined on $L^2(I \otimes Y)$ but, here, it is identified with its pullback on $L^2(\mathcal{M} \setminus \mathcal{K})$. By Lemma 3.5, $||(I_N D_r^2 + J_N D_r + K_N)\varphi|| \leq C ||(H+i)\varphi||$. On the other hand, for all θ_{l_1} and θ_{l_2} associated with overlapping coordinate patches, we have

$$\tilde{\Psi}_{l_1}^* \left[G_N^{\theta_{l_1}}, [G_N^{\theta_{l_2}} r D_r, V] \right] r D_r \tilde{\Psi}_{l_1 *} = \sum_{|\beta| + k \leq 2} \tilde{q}_\beta(r, y, D_y)(e^{-r} D_y)^\beta D_r^k$$

with $\tilde{q}_\beta$ bounded in S^0 for $R > S > r_0$. This follows again from the usual composition rules of pseudo-differential operators and it clearly implies that

$$||[G_N, [G_N r D_r, V]]\, r D_r \varphi|| \leq C ||(H+i)\varphi||, \qquad \varphi \in C_B^\infty(\mathcal{M}),$$

with C independent of R, S. Similarly, the same holds for $G_N\, [r D_r, [G_N r D_r, V]]$ and the result follows. □

We conclude this subsection with the following proposition which summarizes what we know so far on A and H.

Proposition 3.9. *With $\mathcal{D} = C_B^\infty(\mathcal{M})$, all the conditions from* (2.1) *to* (2.10) *hold. Furthermore, in* (2.10)*, $C_{H,A}$ can be chosen independently of $R > S > r_0 + 1$.*

Proof. Using Lemma 3.5, it is clear that $[H, \zeta_n] \to 0$ strongly on $D(H)$ as $n \to \infty$. Therefore, all the conditions from (2.1) to (2.7) are fulfilled. In particular, $C_B^\infty(\mathcal{M})$ is a core for H, hence Propositions 3.4 and 3.8 yield the existence of $[H, A]^0$, and thus (2.9) and (2.10) hold. It only remains to prove (2.8). Assume for a while that, for all $\varphi, \psi \in C_B^\infty(\mathcal{M})$,

$$((H-z)^{-1}\varphi, A\psi) - (A\varphi, (H-\bar{z})^{-1}\psi) = \left((H-z)^{-1}[H,A]^0(H-z)^{-1}\varphi, \psi\right). \tag{3.14}$$

Then this holds for all $\varphi, \psi \in D(A)$. Since $(H-z)^{-1}[H,A]^0(H-z)^{-1}$ is bounded, (3.14) yields

$$\left|((H-z)^{-1}\varphi, A\psi)\right| \le C(||A\varphi|| + ||\varphi||)||\psi||,$$

which shows that $(H-z)^{-1}\varphi \in D(A^*) = D(A)$ for all $\varphi \in D(A)$ and hence (2.8). Let us show (3.14). By (2.3), the right hand side of (3.14) can be written as the limit, as $n \to \infty$, of $([H,A]\zeta_n(H-z)^{-1}\varphi, \zeta_n(H-\bar{z})^{-1}\psi)$ i.e. the limit of

$$\left(\zeta_n(H-z)^{-1}\varphi, A\zeta_n\psi + A[H,\zeta_n](H-\bar{z})^{-1}\psi\right) - \left(A\rho_n\varphi + A[H,\zeta_n](H-z)^{-1}\varphi, \zeta_n(H-\bar{z})^{-1}\psi\right).$$

By (3.10), Lemma 3.5 and the fact that $2^{-n}r\zeta'(2^{-n}r) \to 0$, it is clear that $A[H,\zeta_n](H-z)^{-1}\varphi \to 0$. The same holds for ψ of course and thus $([H,A]\zeta_n(H-z)^{-1}\varphi, \zeta_n(H-\bar{z})^{-1}\psi)$ converges to the left hand side of (3.14). This completes the proof. □

3.2 Positive commutator estimate

This subsection is devoted to the proof of a positive commutator estimate of the form (2.11) at large energies λ (with control on δ with respect to λ).

We start with some notation. Let $\Xi_{R,S}$ be the pullback on $L^2(\mathcal{M}\backslash\mathcal{K})$ (extended by 0 on $L^2(\mathcal{K})$) of the operator defined on $L^2(I \times Y)$ by

$$\varphi \mapsto \sum_{k\ge 0} \chi_R^{1/2}(r)(1-\xi_S^{1/2})(r - \log\nu_k)\varphi_k \otimes \psi_k$$

with the notation (1.6). We also set $\widetilde{\Xi}_{R,S} = \chi_R^{1/2} - \Xi_{R,S}$. Similarly, $\Xi'_{R,S}, \Xi''_{R,S}$ are the operators respectively defined by $\partial_r\left(\chi_R^{1/2}(r)(1-\xi_S^{1/2})(r-\log\nu_k)\right)$ and $\partial_r^2\left(\chi_R^{1/2}(r)(1-\xi_S^{1/2})(r-\log\nu_k)\right)$.

Proposition 3.10. *There exists C such that, for all $\lambda \gg 1$, all $F \in C_0^\infty([0,2\lambda],[0,1])$ and all $R > S > r_0 + 1$, one has*

$$\begin{aligned} F(H)i[H,A]^0F(H) - 2HF(H)^2 \quad &\ge \\ -C\lambda\Big(||F(H)\langle r\rangle^{-1}|| + ||F(H)(\chi_R - 1)|| &+ ||F(H)(1-\widetilde{\Xi}^2_{R,S})|| + S^{-1} + \lambda^{-1}\Big). \end{aligned} \tag{3.15}$$

Proof. We first note that the right hand side of (3.5) is nothing but $2D_r a_k' D_r + 2a_k\mu_k e^{-2r} - a_k^{(3)}/2$. Since $a_k'(r) \ge \chi_R(r)\xi_S(r-\log\nu_k)$ and $a_k(r) \ge S\chi_R(r)\xi_S(r-\log\nu_k)$, we get

$$i[H_0, A] \ge 2D_R\widetilde{\Xi}^2_{R,S}D_R + 2\widetilde{\Xi}_{R,S}e^{-2r}\Delta_h\widetilde{\Xi}_{R,S} - CS^{-2}.$$

This estimate, as well as the following, holds when tested against elements of $\mathcal{D} = C_B^\infty(\mathcal{M})$. For any $a \in C^\infty(\mathbb{R})$, one has $D_r a^2 D_r = aD_r^2 a + aa''$, so the above inequality yields

$$i[H_0, A] \ge 2\widetilde{\Xi}_{R,S}H_0\widetilde{\Xi}_{R,S} - (n-1)^2\widetilde{\Xi}^2_{R,S}/4 - CS^{-2},$$

for $R > S > r_0 + 1$. We then write

$$\widetilde{\Xi}_{R,S} H_0 \widetilde{\Xi}_{R,S} = H_0 + (\chi_R - 1)H_0 + (1 - \widetilde{\Xi}^2_{R,S})H_0 - \widetilde{\Xi}'_{R,S}\widetilde{\Xi}_{R,S}\partial_r - \widetilde{\Xi}_{R,S}\widetilde{\Xi}''_{R,S}$$

and this implies that, on $D(H)$, $i[H,A]^0 \geq 2H + Q_{R,S} - C$ with C independent of R,S and

$$Q_{R,S} = i[V,A]^0 - V + (\chi_R - 1)H_0 + (1 - \widetilde{\Xi}^2_{R,S})H_0 - \widetilde{\Xi}'_{R,S}\widetilde{\Xi}_{R,S}\partial_r - \widetilde{\Xi}_{R,S}\widetilde{\Xi}''_{R,S},$$

where $[V,A]^0$ is the closure of $[V,A]$ (defined on $C_B^\infty(\mathcal{M})$) on $D(H)$. Then, using Lemma 3.5, we have $||H_0F(H)|| + ||\chi_R^{1/2}\partial_r F(H)|| + ||\langle r\rangle^2 VF(H)|| \leq C\lambda$, and using Proposition 3.8, the result follows. $\square$

Note that, if F is supported close enough to λ, $2HF(H)^2 \geq 3\lambda F^2(H)/2$ and thus we will get (2.11) by making the bracket of the right hand side of (3.15) small enough.

Using the technique of [9], we are able to estimate $||F(H)(1 - \widetilde{\Xi}^2_{R,S})||$ for suitable F. Let us recall the proof of this fact. For $R > S > r_0 + 1$, a direct calculation yields

$$\Xi^2_{R,S}H_0 + H_0\Xi^2_{R,S} = 2\Xi_{R,S}H_0\Xi_{R,S} - 2\left(\Xi'_{R,S}\right)^2.$$

On the other hand, $e^{-2r}\mu_k\chi_R(r) \geq e^S - e^{-2R}$ on the support of $\chi_R(r)\xi_S(r - \log\nu_k)$ so we also have $\Xi_{R,S}H_0\Xi_{R,S} \geq \left(e^S - e^{-2R}\right)\Xi^2_{R,S}$, and we obtain

$$\Xi^2_{R,S}\left(\tau(H_0 - \lambda) - z\right) + \left(\tau(H_0 - \lambda) - \bar{z}\right)\Xi^2_{R,S} \geq 2\tau\left(e^S - e^{-2R} - \lambda - \frac{\mathrm{Re}z}{\tau}\right)\Xi^2_{R,S} - 2\tau\left(\Xi'_{R,S}\right)^2,$$

for all real $\tau \neq 0$, $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$. Testing this inequality against $\left(\tau(H_0 - \lambda) - z\right)^{-1}\psi$, we get

$$\begin{aligned} 2\left(\psi, \Xi^2_{R,S}\left(\tau(H_0 - \lambda) - z\right)^{-1}\psi\right) + 2\tau\left|\left|\Xi'_{R,S}\left(\tau(H_0 - \lambda) - z\right)^{-1}\psi\right|\right|^2 \\ \geq 2\tau\left(e^S - e^{-2R} - \lambda - \frac{\mathrm{Re}z}{\tau}\right)\left|\left|\Xi_{R,S}\left(\tau(H_0 - \lambda) - z\right)^{-1}\psi\right|\right|^2 \end{aligned}$$

and this clearly implies, provided $e^S - e^{-2R} - \lambda - \mathrm{Re}z/\tau > 0$, $\tau > 0$ and $R > S > r_0 + 1$, that

$$\left|\left|\Xi_{R,S}\left(\tau(H_0 - \lambda) - z\right)^{-1}\right|\right| \leq \frac{1}{|\mathrm{Im}z|}\left(e^S - e^{-2R} - \lambda - \frac{\mathrm{Re}z}{\tau}\right)^{-1/2}\left(\frac{C_{\chi,\xi}}{S} + \frac{|\mathrm{Im}z|^{1/2}}{\tau^{1/2}}\right). \quad (3.16)$$

This estimate is essentially taken from [9] and is the main tool of the proof of

Proposition 3.11. *Let $F_0 \in C_0^\infty([-1,1],\mathbb{R})$ such that $0 \leq F_0 \leq 1$. There exists C such that, with*

$$R = \log 5\lambda, \qquad S = \log 4\lambda, \qquad \tau = \lambda^{-1},$$

we have

$$\left|\left|F_0\left(\lambda^{-1}H - 1\right)(1 - \widetilde{\Xi}^2_{R,S})\right|\right| \leq C\lambda^{-1/2}(\log\lambda)^{-1}, \qquad \lambda \gg 1.$$

Proof. We shall use Helffer-Sjöstrand formula (see for instance [8]) , i.e.

$$F_0\left(\tau(H_0 - \lambda)\right) = \frac{1}{2\pi}\iint_{\mathbb{R}^2}\overline{\partial}\tilde{F}_0(u + iv)\left(\tau(H_0 - \lambda) - u - iv\right)^{-1}\mathrm{d}u\mathrm{d}v,$$

where $\overline{\partial} = \partial_u + i\partial_v$, $\tilde{F_0} \in C_0^\infty(\mathbb{C})$ is such that $\tilde{F}_{0|\mathbb{R}} = F_0$ and $\overline{\partial}\tilde{F_0} = \mathcal{O}(|v|^\infty)$ near $v = 0$. As a direct consequence of (3.16) with $R = \log 5\lambda$, $S = \log 4\lambda$, $\tau = \lambda^{-1}$, and assuming that $|\mathrm{Re} z| \leq 2$ on the support of $\tilde{F_0}$, Helffer-Sjöstrand formula gives

$$\left|\left| F_0\left(\lambda^{-1}H_0 - 1\right)(1 - \widetilde{\Xi}^2_{R,S})\right|\right| \leq C_{F_0}\lambda^{-1/2}(\log\lambda)^{-1}, \qquad \lambda \gg 1.$$

We are thus left with the study of $|| \left(F_0\left(\lambda^{-1}H - 1\right) - F_0\left(\lambda^{-1}H_0 - 1\right)\right)(1 - \widetilde{\Xi}^2_{R,S})||$ or, equivalently, with $||(1 - \widetilde{\Xi}^2_{R,S})\left(F_0\left(\lambda^{-1}H - 1\right) - F_0\left(\lambda^{-1}H_0 - 1\right)\right)||$. Using the resolvent identity

$$\left(\lambda^{-1}H - 1 - z\right)^{-1} - \left(\lambda^{-1}H_0 - 1 - z\right)^{-1} = \left(\lambda^{-1}H_0 - 1 - z\right)^{-1}\lambda^{-1}V\left(\lambda^{-1}H - 1 - z\right)^{-1},$$

and the fact that V is H bounded with relative bound < 1, which implies that, for some C independent of $\lambda \gg 1$ and $z \in \operatorname{supp}\tilde{F_0}$, $||\lambda^{-1}V\left(\lambda^{-1}H - 1 - z\right)^{-1}|| \leq C|\mathrm{Im} z|^{-1}$, another application of Helffer-Sjostrand formula implies that

$$\left|\left|(1 - \widetilde{\Xi}^2_{R,S})\left(F_0\left(\lambda^{-1}H - 1\right) - F_0\left(\lambda^{-1}H_0 - 1\right)\right)\right|\right| \leq C_{F_0}\lambda^{-1/2}(\log\lambda)^{-1}, \qquad \lambda \gg 1.$$

The result follows. □

We can now explain how to get an estimate of the form (2.11). For any fixed $0 < \epsilon < 1$, one can clearly choose $F_0 \in C_0^\infty(\mathbb{R}, \mathbb{R})$ as above such that for all $F_1 \in C_0^\infty(\mathbb{R}, \mathbb{R})$ supported in $[(1-\epsilon)\lambda, (1+\epsilon)\lambda]$, we have $F_1(E) = F_0(\lambda^{-1}E - 1)F_1(E)$ for all $E \in \mathbb{R}$. Thus, for all such F_1 satisfying $0 \leq F_1 \leq 1$, Propositions 3.10 and 3.11 imply that, for $\lambda \gg 1$,

$$\begin{aligned} F_1(H)i[H,A]^0F_1(H) \geq{}& (2 - 2\epsilon)\lambda F_1(H)^2 - \\ & C\lambda\left(||F_1(H)\langle r\rangle^{-1}|| + ||F_1(H)(\chi_R - 1)|| + (\log\lambda)^{-1}\right), \end{aligned} \tag{3.17}$$

if $R = \log 5\lambda$ and $S = \log 4\lambda$. Then, if we assume that there exists $0 < s_0 \leq 1$ such that

$$||\langle r\rangle^{-s_0}(H - \lambda \pm i0)^{-1}\langle r\rangle^{-s_0}|| \leq \varrho(\lambda), \qquad \lambda \gg 1, \tag{3.18}$$

with $\varrho(\lambda) > \lambda^{-1}/C$, we can choose F_1 in view of Lemma 1.4. Indeed, $\chi_R - 1$ is supported in $|r| \leq C\log\lambda$, so we have $||F_1(H)(\chi_R - 1)|| \leq C||F_1(H)\langle r\rangle^{-s_0}||(\log\lambda)^{s_0}$, and thus (3.17) reads

$$F_1(H)i[H,A]^0F_1(H) \geq \frac{3}{2}\lambda F_1(H)^2 - C\lambda\left((\log\lambda)^{s_0}||F_1(H)\langle r\rangle^{-s_0}|| + (\log\lambda)^{-1}\right).$$

Hence, if F_1 supported in $[\lambda - c\varrho(\lambda)^{-1}(\log\lambda)^{-2s_0}, \lambda + c\varrho(\lambda)^{-1}(\log\lambda)^{-2s_0}]$ with $c > 0$ small enough (independent of λ), Lemma 1.4 clearly shows that

$$F_1(H)i[H,A]^0F_1(H) \geq (2 - 2\epsilon)\lambda F_1(H)^2 - \lambda/2, \qquad \lambda \gg 1.$$

Note that the condition $[\lambda - c\varrho(\lambda)^{-1}(\log\lambda)^{-2s_0}, \lambda + c\varrho(\lambda)^{-1}(\log\lambda)^{-2s_0}] \subset [(1-\epsilon)\lambda, (1+\epsilon)\lambda]$ is ensured, for $\lambda \gg 1$, by the fact that $\varrho(\lambda) \geq \lambda^{-1}/C$. All this easily leads to the

Theorem 3.12. *Let A_λ be the operator given in Definition* 3.1*, with $R = \log(5\lambda)$ and $S = \log(4\lambda)$. Assume that* (3.18) *holds for some $0 < s_0 \leq 1$ and $\varrho(\lambda) > \lambda^{-1}/C$ and let*

$$f_\lambda(E) = f\left(\frac{E - \lambda}{\delta_\lambda}\right), \qquad \delta_\lambda = (\log\lambda)^{-2s_0}\varrho(\lambda)^{-1}/C,$$

with $f \in C_0^\infty(\mathbb{R}, [0,1])$, supported in $[-3,3]$ and $f = 1$ on $[-2,2]$. Then, for C large enough, we have

$$f_\lambda(H)i[H,A_\lambda]^0f_\lambda(H) \geq \lambda f_\lambda(H)^2, \qquad \lambda \gg 1.$$

4 Proofs of the main results

4.1 Proof of Theorem 1.1

By Proposition 3.9 and Theorem 3.12, we are in position to use Theorem 2.8. Here the parameter ν is λ and we consider

$$H_\nu = H, \qquad A_\nu = A_\lambda/\lambda^{1/2}, \qquad \alpha_\nu = \lambda^{1/2}, \qquad \delta_\nu = (\log\lambda)^{-2s_0}\varrho(\lambda)^{-1}/C,$$

with C large enough, independent of λ. Assuming that $\varrho(\lambda) \geq \lambda^{-1/2}/C$ ensures that $\delta_\nu\alpha_\nu^{-1} \leq 1$. Using the forms of $C_0, C_{1/2}, C_1$ given on page 12, it is easy to check that

$$C_{0,\nu} \leq C\alpha_\nu\delta_\nu^{-2}, \qquad C_{1/2,\nu} \leq C\alpha_\nu^{1/2}\delta_\nu^{-1}, \qquad C_{1,\nu} \leq C\alpha_\nu\delta_\nu^{-1}.$$

Furthermore, it is clear that, with $f_\nu = f_\lambda$, we have $||[H_\nu, A_\nu]f_\nu(H_\nu)|| \leq C\alpha_\nu$, so Theorem 2.8 yields

$$\left|\left|\langle A_\lambda/\lambda^{1/2}\rangle^{-s}(H-\lambda \pm i0)^{-1}\langle A_\lambda/\lambda^{1/2}\rangle^{-s}\right|\right| \leq C\varrho(\lambda)^{-1}(\log\lambda)^{2s_0}, \qquad \lambda \gg 1.$$

Then, by writing

$$(H-z)^{-1} = (H-Z)^{-1} + (z-Z)(H-Z)^{-2} + (z-Z)^2(H-Z)^{-1}(H-z)^{-1}(H-Z)^{-1}$$

with $Z = \lambda + i\lambda^{1/2}$, $z = \lambda \pm i\varepsilon$ and letting $\varepsilon \to 0$, Theorem 1.1 will be a consequence of the following lemma.

Lemma 4.1. *There exists $C_s > 0$ such that*

$$\left|\left|W_{-s}(H-\lambda - i\lambda^{1/2})^{-1}\langle A_\lambda/\lambda^{1/2}\rangle^s\varphi\right|\right| \leq C_s\lambda^{-1/2}(\log\lambda)^s||\varphi||, \qquad \varphi \in D(A_\lambda), \quad \lambda \gg 1.$$

Proof. We follow [18], i.e. argue by complex interpolation. We only have to consider the case $s=1$ and thus study $\lambda^{-1/2}W_{-1}(H-\lambda-i\lambda^{1/2})^{-1}A_\lambda$ which we can write, on $D(A_\lambda)$, as

$$\lambda^{-1/2}W_{-1}A_\lambda(H-\lambda-i\lambda^{1/2})^{-1} - \lambda^{-1/2}W_{-1}(H-\lambda-i\lambda^{1/2})^{-1}[H, A_\lambda]^0(H-\lambda-i\lambda^{1/2})^{-1}.$$

The second term is $\mathcal{O}(\lambda^{-1/2})$ since $[H, A_\lambda]^0(H+i)^{-1}$ is uniformly bounded by Propositions 3.4 and 3.8, and $||(H+i)(H-\lambda-i\lambda^{1/2})^{-1}|| = \mathcal{O}(\lambda^{-1/2})$. For the first term, it is easy to check that $||\chi_{r_0+1}D_r(H-\lambda-i\lambda^{1/2})^{-1}|| \leq C$, using Proposition 3.5 and thus

$$\left|\left|\lambda^{-1/2}W_{-1}A_\lambda(H-\lambda-i\lambda^{1/2})^{-1}\right|\right| \leq C\lambda^{-1/2}\sup_{k\geq 0,\ r\geq R}\left(1 + \frac{(r+2S-\log\nu_k)\chi_R(r)\xi_S(r-\log\nu_k)}{w(r-\log\nu_k)}\right)$$

with $R = \log(5\lambda)$ and $S = \log(4\lambda)$. It is not hard to check that the supremum is dominated by $C\log\lambda$ and the result follows. □

4.2 Proof of Theorem 1.3

We first prove that $w(r - \log\langle\eta\rangle)$ is a temperate weight, i.e. satisfies (4.1) below.

Lemma 4.2. *There exist $C, M > 0$ such that, for all $r, r_1 \in \mathbb{R}$ and all $\eta, \eta_1 \in \mathbb{R}^{n-1}$*

$$w(r-\log\langle\eta\rangle) \leq Cw(r_1 - \log\langle\eta_1\rangle)\left(1 + |r-r_1| + |\eta-\eta_1|\right)^M. \tag{4.1}$$

Proof. By Taylor's formula, $w(x) = w(x_1) + \int_0^1 w'(x_1 + t(x - x_1))\mathrm{d}t(x - x_1)$ and since

$$w'(x_1 + t(x - x_1)) \leq C_1 \leq C_2 w(x_1)$$

for all $x, x_1 \in \mathbb{R}$ and $t \in [0, 1]$, we have $w(x) \leq C w(x_1)(1 + |x - x_1|)$. The result then easily follows from the fact that $|\log\langle\eta\rangle - \log\langle\eta_1\rangle| \leq C(1 + |\eta - \eta_1|)$. □

As a consequence, for all $s \in \mathbb{R}$, $(w(r - \log\langle\eta\rangle))^s$ is also a temperate weight. Hence, by well known pseudo-differential calculus [12] on $\mathbb{R}^n$, for all $a \in \mathcal{S}(w_{-s})$ and $b \in \mathcal{S}(w_s)$

$$a(r, y, D_r, D_y)b(r, y, D_y) = c(r, y, D_r, D_y) \tag{4.2}$$

for some $c \in \mathcal{S}(w_0)$ (depending continuously on a and b). In particular, by the Calderòn-Vaillancourt theorem, $c(r, y, D_r, D_y)$ is a bounded operator on L^2. More generally, if a and b are respectively in bounded subsets of $\mathcal{S}(w_{-s})$ and $\mathcal{S}(w_s)$, then $c(r, y, D_r, D_y)$ stays in a bounded subset of the space of bounded operators on L^2 (the norm of $c(r, y, D_r, D_y)$ depends on finitely many semi-norms of c in $\mathcal{S}(w_0)$). Similarly, if $a \in \mathcal{S}(w_{-s})$, then

$$a(r, y, D_r, D_y)^* = a^{\#}(r, y, D_r, D_y) \tag{4.3}$$

for some $a^{\#} \in \mathcal{S}(w_{-s})$ depending continuously on a.

For $s \geq 0$, we introduce W_s as the inverse (unbounded if $s \neq 0$) of W_{-s}, i.e. $W_s \equiv 1$ on $L^2(\mathcal{K})$ and is defined on $L^2(\mathcal{M} \setminus \mathcal{K})$ as the pullback of the operator $\widetilde{W}_s$ defined on $L^2(I \times Y)$ by

$$(\widetilde{W}_s \varphi)(r, \omega) = \sum_{k \geq 0} w^s(r - \log\sqrt{\langle\mu_k\rangle})\varphi_k(r)\psi_k(\omega).$$

It is clearly well defined on the dense subspace of functions with fast decay with respect to r. Then, Theorem 1.3 will clearly follow from the fact that $W_s \kappa Op(a)\tilde{\kappa}$ and $\kappa Op(a)\tilde{\kappa}W_s$, defined on $C_c^\infty(\mathcal{M})$, have bounded closures on $L^2(\mathcal{M})$. We only consider $W_s \kappa Op(a)\tilde{\kappa}$, the other case follows by adjunction, using (4.3).

We will use a complex interpolation argument and thus we will need to consider $w_{s+i\sigma}(r, \eta) := (w(r - \log\langle\eta\rangle))^{s+i\sigma}$ for $s, \sigma \in \mathbb{R}$ (note that $w_{s+i\sigma} \in \mathcal{S}(w_s)$). Since any $a \in \mathcal{S}(w_{-s})$ can be written $w_{-s}\tilde{a}$ for some $\tilde{a} \in \mathcal{S}(w_0)$, it is clearly enough to show that, for all $b \in \mathcal{S}(w_0)$, there exists $C > 0$ and $N \geq 0$ such that

$$||W_1 \kappa Op(w_{-1+i\sigma}b)\tilde{\kappa}\varphi|| \leq C(1 + |\sigma|)^N ||\varphi||, \qquad \forall\, \varphi \in C_c^\infty(\mathcal{M}),\ \forall\, \sigma \in \mathbb{R}, \tag{4.4}$$

and that, for all $\varphi \in C_c^\infty(\mathcal{M})$, there exists C_φ such that

$$||W_s \kappa Op(w_{-s+i\sigma}b)\tilde{\kappa}\varphi|| \leq C_\varphi(1 + |\sigma|)^N, \qquad \forall\, \sigma \in \mathbb{R},\ \forall\, s \in [0, 1]. \tag{4.5}$$

Observing that $W_s\langle r\rangle^{-1}$ is bounded, this last estimate clearly follows from the fact that one can write

$$W_s \kappa Op(w_{-s+i\sigma}b)\tilde{\kappa} = W_s\langle r\rangle^{-1}\left(\langle r\rangle \kappa Op(w_{-s+i\sigma}b)\tilde{\kappa}\langle r\rangle^{-1}\right)\langle r\rangle$$

and the fact that $||\langle r\rangle \kappa Op(w_{-s+i\sigma}b)\tilde{\kappa}\langle r\rangle^{-1}|| \leq C(1 + |\sigma|)^N$, by the Calderòn-Vaillancourt theorem.

We thus have to focus on (4.4) which we shall prove by using a pseudo-differential approximation of W_1. To that end we observe that, if ξ is defined as in the beginning of Section 3, then

$$w(r - \log\langle\mu\rangle^{1/2}) = (r - \log\langle\mu\rangle^{1/2})\xi(r - \log\langle\mu\rangle^{1/2}) + c(r, \mu)$$

with $c \in L^\infty(\mathbb{R}_r \times \mathbb{R}_\mu)$. Thus, by choosing $\chi = \chi(r)$ supported in $(r_0 + 2, \infty)$ such that $\chi = 1$ near infinity, it is easy to check that, with the notations used in Proposition 3.6,

$$W_1 = (r - \log\langle \Delta_h \rangle^{1/2})\chi(r)\xi(r - \log\langle \Delta_h \rangle^{1/2}) + B$$

for some bounded operator B. Since $||\kappa Op(w_{-s+i\sigma}b)\tilde{\kappa}|| \leq C(1 + |\sigma|)^N$, the contribution of B to (4.4) is clear. It remains to prove the following

Proposition 4.3. *For all $b \in \mathcal{S}(w_0)$, there exist $C > 0$ and $N > 0$ such that, fopr all $\sigma \in \mathbb{R}$,*

$$\left|\left|(r - \log\langle \Delta_h \rangle^{1/2})\chi(r)\xi(r - \log\langle \Delta_h \rangle^{1/2})\kappa Op(w_{-1+i\sigma}b)\tilde{\kappa}\right|\right| \leq C(1 + |\sigma|)^N.$$

Proof. Observe first that $(r - \log\langle \Delta_h \rangle^{1/2})\kappa Op(w_{-1+i\sigma}b)\tilde{\kappa}$ reads

$$\kappa Op((r - \log\langle p_h \rangle^{1/2})w_{-1+i\sigma}b)\tilde{\kappa} + B_\sigma \tag{4.6}$$

for some bounded operator B_σ with norm bounded by $C(1+|\sigma|)^N$. This follows from the Calderòn-Vaillancourt theorem and the pseudo-differential expansion of $\log\langle \Delta_h \rangle^{1/2}$ given by Proposition 3.6. We next insert the partition of unit

$$1 = \xi(r - \log\langle p_h \rangle^{1/2}) + (1 - \xi)(r - \log\langle p_h \rangle^{1/2})$$

in front of the symbol of the first term of (4.6). Since $(r - \log\langle p_h \rangle^{1/2}) \times \xi(r - \log\langle p_h \rangle^{1/2}) \times w_{-1+i\sigma}$ belongs to $\mathcal{S}(w_0)$, the contribution of this term is clear. Thus we are left with the study of

$$\chi(r)\xi(r - \log\langle \Delta_h \rangle^{1/2})\kappa Op\left((r - \log\langle p_h \rangle^{1/2})(1 - \xi)(r - \log\langle p_h \rangle^{1/2})w_{-1+i\sigma}b\right)\tilde{\kappa}. \tag{4.7}$$

We observe that

$$\kappa \times (r - \log\langle p_h \rangle^{1/2})(1 - \xi)(r - \log\langle p_h \rangle^{1/2})w_{-1+i\sigma} \in S^\epsilon$$

for all $\epsilon > 0$, since $r \leq \log\langle p_h \rangle^{1/2} + C$ on the support of this symbol. Then, by using the pseudo-differential expansion of $\xi(r - \log\langle \Delta_h \rangle^{1/2})$, we see that (4.7) reads

$$\chi(r)\kappa Op\left((r - \log\langle p_h \rangle^{1/2})(1 - \xi)(r - \log\langle p_h \rangle^{1/2})\xi(r - \log\langle p_h \rangle^{1/2})w_{-1+i\sigma}b\right)\tilde{\kappa} + \widetilde{B}_\sigma$$

with $\widetilde{B}_\sigma$ similar to B_σ. The symbol of the first term belongs to $\mathcal{S}(w_0)$ since $r - \log\langle p_h \rangle^{1/2}$ must be bounded on its support and the Calderòn-Vaillancourt theorem completes the proof. ☐

A Operators on the real line

If we consider a function $a \in C^\infty(\mathbb{R}, \mathbb{R})$, with a' bounded, then the flow γ_t, i.e. the solution to

$$\dot{\gamma}_t = a(\gamma_t), \qquad \gamma_0(r) = r, \tag{A.1}$$

is well defined on $\mathbb{R}_t \times \mathbb{R}_r$. For each t, γ_t is a C^∞ diffeomorphism on $\mathbb{R}$ and it is easy to check that

$$U_t\varphi := (\partial_r\gamma_t)^{1/2}\varphi \circ \gamma_t \tag{A.2}$$

defines a strongly continuous unitary group $(U_t)_{t\in\mathbb{R}}$ on $L^2(\mathbb{R})$ whose generator, i.e. the operator A such that $U_t = e^{itA}$ for all t, is a selfadjoint realization of the differential operator

$$\frac{a(r)D_r + D_r a(r)}{2} = a(r)D_r + \frac{a'(r)}{2i},$$

meaning that, restricted to $C_0^\infty(\mathbb{R})$, A acts as the operator above. Indeed, according to Stone's Theorem [19], the domain of A, $D(A)$, is the set of $\varphi \in L^2(\mathbb{R})$ such that $U_t\varphi$ is strongly differentiable at $t = 0$, thus it clearly contains $C_0^\infty(\mathbb{R})$, which is moreover invariant by U_t. This also easily implies that A acts on elements of its domain in the distributions sense.

It is worth noticing as well that, if, for some R, $a(r) = 0$ for $r \leq R$, then $\gamma_t(r) = r$ for $r \leq R$ and thus U_t acts as the identity on $L^2(-\infty, R)$. Moreover, if $\zeta \in C_0^1(\mathbb{R})$ and $\varphi \in D(A)$, then $\zeta\varphi \in D(A)$ since $U_t(\zeta\varphi) = \zeta \circ \gamma_t U_t \varphi$ is easily seen to be strongly differentiable at $t = 0$ and we have

$$A(\zeta\varphi) = \zeta A\varphi - ia\zeta'\varphi. \tag{A.3}$$

Of course, it is not hard to deduce from this property that the subspace of $D(A)$ consisting of compactly supported elements is dense in $D(A)$ for the graph norm.

We want to show that $C_0^\infty(\mathbb{R})$ is also a core for A and thus consider $\theta_\epsilon(r) = \epsilon^{-1}\theta(r/\epsilon)$ with $\theta \in C_0^\infty(-1, 1)$ such that $\int_{\mathbb{R}} \theta = 1$. A simple calculation shows that

$$U_t(\varphi * \theta_\epsilon) = K_{t,\epsilon} U_t \varphi$$

where $K_{t,\epsilon}$ is the operator with kernel

$$\kappa_{t,\epsilon}(r, r') = (\partial_r \gamma_t(r))^{1/2} (\partial_r \gamma_t(r'))^{1/2} \theta_\epsilon(\gamma_t(r) - \gamma_t(r')).$$

Note that this operator is bounded on $L^2(\mathbb{R})$ in view of the following well known Schur's Lemma which we recall since we will use it extensively.

Lemma A.1 (Schur). *If $j(x, y)$ is a measurable function on $\mathbb{R}^{2d}$ such that*

$$\operatorname{ess-sup}_{y\in\mathbb{R}} \int |j(x, y)| \, \mathrm{d}x \leq C, \qquad \operatorname{ess-sup}_{x\in\mathbb{R}} \int |j(x, y)| \, \mathrm{d}y \leq C$$

then the operator J with kernel j is bounded on $L^2(\mathbb{R}^d)$ and $||J|| \leq C$.

Since $K_{0,\epsilon}\varphi = \varphi * \theta_\epsilon$, we have

$$\frac{U_t(\varphi * \theta_\epsilon) - \varphi * \theta_\epsilon}{it} - \left(\frac{U_t\varphi - \varphi}{it}\right) * \theta_\epsilon = \frac{K_{t,\epsilon} - K_{0,\epsilon}}{it}(U_t\varphi). \tag{A.4}$$

In order to estimate the right hand side, we start with a few remarks. Note first that we have

$$||\partial_r \gamma_t||_\infty \leq e^{||a'||_\infty |t|}, \qquad ||\partial_r \partial_t \gamma_t||_\infty \leq ||a'||_\infty e^{||a'||_\infty |t|}. \tag{A.5}$$

The first estimate is obtained by applying ∂_r to $(A.1)$ and using Gronwall's lemma. The second one then follows from the first one. This implies in particular the existence of some t_0, depending only on $||a'||_\infty$, such that $||\partial_r \gamma_t - 1||_\infty \leq 1/2$ for $|t| \leq t_0$. Differentiating $(A.1)$ twice with respect to r and t yields $\partial_t^2 \partial_r \gamma_t = a(\gamma_t) a''(\gamma_t) \partial_r \gamma_t + a'(\gamma_t)^2 \partial_r \gamma_t$ and thus, if aa'' is bounded,

$$||\partial_t^2 \partial_r \gamma_t||_\infty \leq \frac{3}{2} \left(||aa''||_\infty + ||a'||_\infty^2\right), \qquad |t| \leq t_0. \tag{A.6}$$

Thus, if J_ϵ denotes the operator with kernel $\partial_t \kappa_{t,\epsilon}|_{t=0}(r, r')$ that is

$$\frac{1}{2} (a'(r) + a'(r')) \theta_\epsilon(r - r') + (a(r) - a(r')) \theta'_\epsilon(r - r'), \tag{A.7}$$

then Taylor's formula combined with Schur's Lemma show that

$$||K_{t,\epsilon} - K_{0,\epsilon} - tJ_\epsilon|| \leq C_\epsilon t^2, \qquad |t| \leq t_0 \tag{A.8}$$

for some C_ϵ depending only on θ_ϵ, $||a'||_\infty$ and $||aa''||_\infty$ (recall that t_0 depends only on $||a'||_\infty$ as well). Since J_ϵ is a bounded operator (with norm uniformly bounded by $||a'||_\infty \int |r\theta'(r)|+|\theta(r)|\mathrm{d}r$), $(A.4)$ and $(A.8)$ show that if $\varphi \in D(A)$ then $\varphi * \theta_\epsilon \in D(A)$ and $A(\varphi * \theta_\epsilon) = (A\varphi) * \theta_\epsilon - iJ_\epsilon\varphi$. Furthermore $J_\epsilon \to 0$ strongly as $\epsilon \to 0$ for it is uniformly bounded and $J_\epsilon\psi \to 0$ for all $\psi \in C_0^\infty(\mathbb{R})$. All this shows that, for any $\varphi \in D(A)$,

$$||A(\varphi * \theta_\epsilon) - (A\varphi) * \theta_\epsilon|| \leq C||\varphi||, \qquad A(\varphi * \theta_\epsilon) - (A\varphi) * \theta_\epsilon \to 0, \qquad \epsilon \to 0, \tag{A.9}$$

with C independent of ϵ, depending only on $||a'||_\infty$. In particular, $(A.3)$ and $(A.9)$ imply easily that $C_0^\infty(\mathbb{R})$ is a core for A.

B Proof of Proposition 3.6

We start with some reductions. We may clearly write $g(r,\mu)$ as $g_1(r,\mu)(i+\mu)$ with $g_1 \in S^{-1}$ hence by studying $g_1(r,\Delta_h)$ instead of $g(r,\Delta_h)$ we can assume that $g \in S^m$ with $m < 0$. Note that the composition by $\Delta_h + i$ on the right of (3.7) doesn't cause any trouble in view of (3.8), (3.9) and of the standard composition rules for pseudo-differential operators. Furthermore, by positivity of Δ_h, we have $g(r,\Delta_h) = g_2(r,\Delta_h + 1)$ for some $g_2 \in S^m$ which we can assume to be supported in $[1/2,\infty)$. This support property will be useful to consider Mellin transforms below.

By the standard procedure for the calculus of a parametrix of the resolvent of an elliptic operator on a closed manifold [22], there are symbols $q_{-2}(y,\eta,z), q_{-3}(y,\eta,z),\cdots$ of the form

$$q_{-2} = (p_h - z)^{-1}, \qquad q_{-2-j} = \sum_{1\leq l\leq 2j} d_{jl}(p_h - z)^{-l-1}, \qquad j \geq 1 \tag{B.1}$$

such that, for all N large enough,

$$\theta(\Delta_h - z)^{-1} - \Psi^* \left((\Psi_*\theta) \sum_{j=0}^{N} q_{-2-j}(y, D_y, z) \right) \Psi_*\, \tilde{\theta} = M_N(z).$$

Here $M_N(z)$ is bounded from H^κ to $H^{\kappa+N}$ for all κ, $H^\kappa = H^\kappa(Y)$ being the standard Sobolev space on Y and d_{jl} are polynomials in η of degree $2j-l$, which are independent of z and linear combinations of products of derivatives of the full symbol of Δ_h in the chart we consider. Furthermore, for all κ and N, there exist C and γ such that

$$||M_N(z)||_{H^\kappa \to H^{\kappa+N}} \leq C \frac{\langle z\rangle^\gamma}{|\mathrm{Im} z|^{\gamma+1}}.$$

We now repeat the arguments of [11]. For each s such that $\mathrm{Re}s < 0$, we choose a contour Γ_s surrounding $[1/2,+\infty)$ on which $\langle z\rangle/|\mathrm{Im} z|$ is bounded, and by Cauchy formula we get

$$\theta(\Delta_h + 1)^s - \Psi^* \left((\Psi_*\theta) \sum_{j=0}^{N} a_j(y, D_y, s) \right) \Psi_*\, \tilde{\theta} = \frac{i}{(2\pi)} \int_{\Gamma_s} z^s M_N(z)\, \mathrm{d}z$$

with $a_j(s) = \sum_{1\le l\le 2j}(-1)^l d_{jl} s(s-1)\cdots(s-l+1)(p_h+1)^{s-l}/l!$ if $j \geq 1$ and $a_0(s) = (p_h+1)^s$. As in [11], we choose the contour so that, if Re $s < 0$ is fixed,

$$\left|\left|\int_{\Gamma_s} z^s M_N(z)\ \mathrm{d}z\right|\right|_{H^\kappa \to H^{\kappa+N}} \leq C_{\mathrm{Re}s,\kappa,N} \langle \mathrm{Im}s \rangle^\gamma .$$

We then consider the Mellin transform $\mathrm{M}[g_2](r,s) := \int_0^\infty \mu^{s-1} g_2(r,\mu)\ \mathrm{d}\mu$. Note that it is well defined for Re $s < -m$ (recall that $m < 0$), since g_2 is supported in $[1/2,\infty)$, and that it decays fast at infinity with respect to $|\mathrm{Im}s|$, for fixed Re s. It is then easy to check that

$$\int_{\mathrm{Re}s - i\infty}^{\mathrm{Re}s+i\infty} \mathrm{M}[g_2](r,s) \left(\int_{\Gamma_s} z^s M_N(z)\ \mathrm{d}z \right)\ \mathrm{d}s \in C^\infty(\mathbb{R}_r, \mathcal{L}(H^\kappa, H^{\kappa+N})),$$

so, by Mellin's inversion formula, i.e. $g_2(r,\mu) = (2i\pi)^{-1} \int_{\mathrm{Re}s=\mathrm{const}} \mathrm{M}[g_2](r,s)\mu^{-s}\ \mathrm{d}s$, and by setting

$$\mathcal{R}_N^{\theta,Y}(r) = \theta g(r,\Delta_h) - \Psi^* \left((\Psi_*\theta) g(p_h) + (\Psi_*\theta) \sum_{j=1}^N \sum_l (-1)^l d_{jl} \partial_\mu^l g(r,p_h)/l! \right) \Psi_*\ \tilde{\theta},$$

we get

$$\sup_{r>r_0} \left|\left| D_r^k \mathcal{R}_N^{\theta,Y}(r) \right|\right|_{H^\kappa \to H^{\kappa+N}} < \infty, \qquad \forall\ k.$$

The latter easily follows from the boundedness of the derivatives of g (or g_2) with respect to r. In order to prove (3.8), with N replaced by $N/8$ (which can be assumed to be an integer), we first remark that $\mathcal{R}_N^\theta$ is defined on generators of $L^2(I) \otimes L^2(Y)$ by

$$\mathcal{R}_N^\theta(\varphi_k \otimes \psi_k)(r,\omega) = \varphi_k(r) \left(\mathcal{R}_N^{\theta,Y}(r)\psi_k \right)(\omega)$$

with $\varphi_k \in L^2(I)$. We then note that, by writing $\psi_k = (\mu_k+i)^{-N/4}(\Delta_h+i)^{N/4}\psi_k$, we have, for $j,l \leq N/8$,

$$\left|\left| \Delta_h^j \mathcal{R}_N^\theta \Delta_y^l (\varphi_k \otimes \psi_k) \right|\right|_{L^2(I\times Y)} \leq C_N \langle \mu_k \rangle^{-N/4} ||\varphi_k||_{L^2(I)} \sup_{r>r_0} ||\mathcal{R}_N^{\theta,Y}(r)||_{H^{-3N/4}\to H^{N/4}}$$

and thus, if N is large enough so that $\sum_k \langle \mu_k \rangle^{-N/2} < \infty$, Parseval's formula yields

$$\left|\left| \Delta_h^j \mathcal{R}_N^\theta \Delta_h^l (\sum_k \varphi_k \otimes \psi_k) \right|\right|_{L^2(I\times Y)} \leq C_N \left(\sum_k ||\varphi_k||^2_{L^2(I)} \right)^{1/2}.$$

This proves (3.8). The proof of (3.9) is similar. □

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