Sharp low frequency resolvent estimates on asymptotically conical manifolds

Jean-Marc Bouclet & Julien Royer
Institut de Mathématiques de Toulouse
118 route de Narbonne
F-31062 Toulouse Cedex 9
jean-marc.bouclet@math.univ-toulouse.fr
julien.royer@math.univ-toulouse.fr

Abstract

On a class of asymptotically conical manifolds, we prove two types of low frequency estimates for the resolvent of the Laplace-Beltrami operator. The first result is a uniform $L^2 \to L^2$ bound for $\langle r \rangle^{-1}(-\Delta_G - z)^{-1}\langle r \rangle^{-1}$ when $\mathrm{Re}(z)$ is small, with the optimal weight $\langle r \rangle^{-1}$. The second one is about powers of the resolvent. For any integer N, we prove uniform $L^2 \to L^2$ bounds for $\langle \epsilon r \rangle^{-N}(-\epsilon^{-2}\Delta_G - Z)^{-N}\langle \epsilon r \rangle^{-N}$ when $\mathrm{Re}(Z)$ belongs to a compact subset of $(0, +\infty)$ and $0 < \epsilon \ll 1$. These results are obtained by proving similar estimates on a pure cone with a long range perturbation of the metric at infinity.

1 Introduction and main results

The long range scattering theory of the Laplace-Beltrami operator on asymptotically Euclidean or conical manifolds has been widely studied. It has reached a point where our global understanding of the spectrum, in particular the behaviour of the resolvent at low, medium and high frequencies, allows to extend to curved settings many results which are well known on $\mathbb{R}^n$. We have typically in mind global in time Strichartz estimates [34, 24, 28, 19, 37] or various instances of the local energy decay [2, 4, 35, 36, 6, 33, 8] which are important tools in nonlinear PDE arising in mathematical physics. We refer to the recent paper [33] which surveys resolvent estimates (or limiting absorption principle) and some of their applications in this geometric framework.

In this picture, the results on low frequency estimates are relatively recent, compared to the longer history of the high frequency regime, and some of them are not yet optimal. The main result of this paper is a low frequency bound for the resolvent of the Laplace-Beltrami operator with sharp weight. The interest is twofold. On one hand, we obtain the same type of sharp inequality as on $\mathbb{R}^n$ for a general class of manifolds which contains both $\mathbb{R}^n$ with an asymptotically flat metric and the class of scattering manifolds (see [25, 26]). On the other hand, in the spirit of the applications quoted above, our result can be used in the proof of global Strichartz estimates: it allows to handle in a fairly simple and intuitive fashion the phase space region which cannot be treated by semiclassical (or microlocal) techniques.

Let us describe more precisely our framework and our results.

In this paper we consider an asymptotically conical manifold $(\mathcal{M}, G)$, that is a connected Riemannian manifold isometric outside a compact subset to a product $(R_0, +\infty) \times \mathcal{S}$, with $(\mathcal{S}, h_0)$ a closed Riemannian manifold, equipped with a metric approaching the conical metric $dr^2 + r^2 h_0$

as $r \to \infty$. More precisely this means that for some compact, connected manifold with boundary $\mathcal{K} \Subset \mathcal{M}$ and some $R_0 > 0$, there is a diffeomorphism

$$\kappa : \mathcal{M} \setminus \mathcal{K} \ni m \to \big(r(m), \omega(m)\big) \in (R_0, +\infty) \times \mathcal{S}, \tag{1.1}$$

through which the metric reads

$$G = \kappa^* \big(a(r)dr^2 + 2rb(r)dr + r^2 h(r)\big), \tag{1.2}$$

with $a \to 1$, $b \to 0$ and $h \to h_0$ as $r \to \infty$ in the following sense: for each $r > R_0$, $a(r)$ is a function on $\mathcal{S}$, $b(r)$ is a 1-form on $\mathcal{S}$ and $h(r)$ is a Riemannian metric on $\mathcal{S}$, with $a(\cdot), b(\cdot)$ and $h(\cdot)$ all depending smoothly on r so that, for some $\rho > 0$,

$$||\partial_r^j(a(r)-1)||_{\Gamma^0(\mathcal{S})} + ||\partial_r^j b(r)||_{\Gamma^1(\mathcal{S})} + ||\partial_r^j(h(r)-h_0)||_{\Gamma^2(\mathcal{S})} \lesssim r^{-j-\rho}, \tag{1.3}$$

where, for $k = 0, 1, 2$, $|| \cdot ||_{\Gamma^k(\mathcal{S})}$ is any seminorm of the space of smooth sections of $(T^*\mathcal{S})^{\otimes^k}$. In usual terms, this means that G is a long range perturbation of $\kappa^*(dr^2 + r^2 h_0)$ near infinity. In (1.1) r is the first component of κ. It defines a coordinate on $\mathcal{M} \setminus \mathcal{K}$ taking its values in (R_0, ∞). We also assume that κ is an homeomorphism between $\overline{\mathcal{M} \setminus \mathcal{K}}$ and $[R_0, \infty) \times \mathcal{S}$. We may then assume without loss of generality that r is a globally defined smooth function which is proper[1], but which is a coordinate only near infinity. This allows us to define the weights $\langle r \rangle^\mu = (1 + r^2)^{\mu/2}$ globally on $\mathcal{M}$.

Our definition is more general than the one of scattering metrics [25, 26] and than the one used in [33, Definition 1.4] where h has a polyhomogeneous expansion at infinity. It also covers the usual case of long range perturbations of the Euclidean metric as considered in [3, 5, 6].

We will allow the possibility for $\mathcal{M}$ to have a boundary. We thus introduce $C_c^\infty(\mathcal{M})$, the set of smooth functions vanishing outside a compact set (these functions do not need to vanish on $\partial\mathcal{M}$), and $C_0^\infty(\mathcal{M}) = C_c^\infty(\mathcal{M} \setminus \partial\mathcal{M})$ the subset of those which also vanish near ∂M. We let $\hat{P}$ be the Friedrichs extension of $-\Delta_G$ on $C_0^\infty(\mathcal{M})$. It is self-adjoint on $L^2(\mathcal{M}) = L^2(\mathcal{M}, d\mathrm{vol}_G)$. If $\mathcal{M}$ has no boundary, it is the unique self-adjoint realization of $-\Delta_G$ and if $\partial\mathcal{M}$ is non empty it is the Dirichlet realization. The connectedness of $\mathcal{M}$ ensures that

$$0 \text{ is not an eigenvalue of } \hat{P}. \tag{1.4}$$

Our assumptions also imply that

$$\chi(\hat{P} + i)^{-1} \text{ is compact on } L^2(\mathcal{M}), \text{ for all } \chi \in C_c^\infty(\mathcal{M}). \tag{1.5}$$

We let $n = \dim(\mathcal{M})$ and assume everywhere that $n \geq 3$.

Our first main result is the following.

Theorem 1.1. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for all $z \in \mathbb{C} \setminus \mathbb{R}$ satisfying $|\mathrm{Re}(z)| < \varepsilon_0$,*

$$\big|\big|\langle r \rangle^{-1}(\hat{P} - z)^{-1}\langle r \rangle^{-1}\big|\big|_{L^2(\mathcal{M}) \to L^2(\mathcal{M})} \leq C.$$

In [15, 5, 3, 16, 6, 18, 33]), uniform estimates on $\langle r \rangle^{-s}(\hat{P} - z)^{-1}\langle r \rangle^{-s}$ (for $|\mathrm{Re}(z)|$ small) were proved for $s > 1$. The novelty of our result is that we use the weight $\langle r \rangle^{-1}$ which is sharp. We also cover more general manifolds than the ones considered in the aforementioned papers. This result is satisfactory for it answers the natural question of what the optimal weight is, but it also has useful applications which we describe below.

Our second main result is the following.

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[1]*i.e. $r^{-1}([r_1, r_2])$ is a compact subset of $\mathcal{M}$ for all $r_1 \leq r_2$*

Theorem 1.2. *Fix an integer $N \geq 1$ and a compact interval $[E_1, E_2] \subset (0, \infty)$. There exist $C > 0$ and $\epsilon_0 > 0$ such that*

$$\left|\left|\langle \epsilon r \rangle^{-N} \left(\epsilon^{-2}\hat{P} - Z\right)^{-N} \langle \epsilon r \rangle^{-N}\right|\right|_{L^2(\mathcal{M}) \to L^2(\mathcal{M})} \leq C, \tag{1.6}$$

for all $Z \in \mathbb{C} \setminus \mathbb{R}$ such that $\mathrm{Re}(Z) \in [E_1, E_2]$ and all $\epsilon \in (0, \epsilon_0)$.

These estimates are low frequency inequalities for they are equivalent to the spectrally localized versions

$$\left|\left|\langle \epsilon r \rangle^{-N} \left(\epsilon^{-2}\hat{P} - Z\right)^{-N} \phi(\epsilon^{-2}\hat{P}) \langle \epsilon r \rangle^{-N}\right|\right|_{L^2(\mathcal{M}) \to L^2(\mathcal{M})} \leq C$$

for any $\phi \in C_0^\infty$ which is equal to 1 near $[E_1, E_2]$, that is when $\hat{P}$ is spectrally localized near $[\epsilon^2 E_1, \epsilon^2 E_2]$. Let us remark that for the Laplacian on $\mathbb{R}^n$, Theorem 1.2 follows directly from the usual estimates on $\langle r \rangle^{-N}(-\Delta - Z)^{-N}\langle r \rangle^{-N}$ by a simple rescaling argument. Such a global rescaling argument is of course meaningless on a manifold, but Theorem 1.2 says that this scaling intuition remains correct.

We record a last result which is a byproduct of our analysis but which is also interesting on its own.

Theorem 1.3. *Fix $s \in (0, 1/2)$. There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for all $z \in \mathbb{C} \setminus \mathbb{R}$ satisfying $|\mathrm{Re}(z)| < \varepsilon_0$,*

$$\left|\left|\langle r \rangle^{-2-s}(\hat{P} - z)^{-2}\langle r \rangle^{-2-s}\right|\right|_{L^2(\mathcal{M}) \to L^2(\mathcal{M})} \leq C|\mathrm{Re}(z)|^{s-1}. \tag{1.7}$$

The estimate (1.7) is nearly sharp with respect to $|\mathrm{Re}(z)|^{s-1}$ in dimension 3 (one can take exactly $s = 1/2$ in the asymptotically Euclidean case [6], but this is not clear if $\mathcal{S} \neq \mathbb{S}^2$) but certainly not in higher dimensions (see [6] where we get better estimates in higher dimensions in the asymptotically Euclidean case). To get sharper estimates, one would need to use improved Hardy inequalities (*e.g.* improve Lemma 3.14 to be able to consider higher order derivatives in higher dimensions). We did not consider this technical question since the main focus of this paper is on Theorems 1.1 and 1.2 for which Theorem 1.3 will be essentially a tool.

We now discuss some motivations and applications of Theorems 1.1 and 1.2. The first application is on the global smoothing effect.

Corollary 1.4. *Assume that $\mathcal{M}$ has no boundary and no trapped geodesics. Then, there exists $C > 0$ such that*

$$\int_{\mathbb{R}} ||\langle r \rangle^{-1}(1 + \hat{P})^{1/4} e^{it\hat{P}} u_0||_{L^2}^2 dt \leq C||u_0||_{L^2}^2.$$

We state this result in the case of boundaryless manifolds only for simplicity. However, it extends to manifolds with boundary, under the non trapping condition for the generalized billard flow of Melrose-Sjöstrand (see [27] and [10] for related problems).

On manifolds, the local in time version of this corollary is classical (see *e.g.* [13]). We refer to [1, 22] for the global in time version in the flat case. Here we derive a global in time version with the sharp weight $\langle r \rangle^{-1}$. According to the standard approach, Corollary 1.4 follows from the resolvent estimates

$$\left|\left|\langle r \rangle^{-1}(\hat{P} - \lambda \pm i\varepsilon)^{-1}\langle r \rangle^{-1}\right|\right|_{L^2(\mathcal{M}) \to L^2(\mathcal{M})} \leq C\langle \lambda \rangle^{-1/2}, \qquad \lambda \in \mathbb{R}, \ \varepsilon > 0, \tag{1.8}$$

and the Kato theory of smooth operators [32]. The resolvent estimates (1.8) follow from [11] at high energy, using the non trapping condition, from Theorem 1.1 at low energy and, when λ belongs

to any compact subset of $(0, \infty)$, from the standard Mourre theory [29, 21, 14] combined with the absence of embedded eigenvalues for $\hat{P}$ (see [20]).

Another range of applications is given by Strichartz estimates which were actually the main original motivation of this paper. The related results will appear in a forthcoming article [7]. We only point out here that both the sharp weight $\langle r \rangle^{-1}$ of Theorem 1.1 (or Corollary 1.4) and the rescaled estimates of Theorem 1.2 are important inputs. The former is crucial to handle large compact (frequency dependent) subsets while the latter provide non trivial propagation estimates which can be used to construct long time low frequency parametrices of the Schrödinger group in (frequency dependent) neighborhoods of infinity.

We have already noticed that for the flat Laplacian on $\mathbb{R}^n$ Theorem 1.2 follows from resolvent estimates at energy one thanks to a rescaling argument. We now explain the proof of Theorem 1.1 for this model case. Let $R_0(z) = (-\Delta - z)^{-1}$ be the resolvent of the Laplacian on $\mathbb{R}^n$. The first observation is that for $\mathrm{Re}(z) \leq 0$, the estimate of Theorem 1.1 follows from the Hardy inequality. Indeed, one can write

$$|x|^{-1} R_0(z) |x|^{-1} = |x|^{-1} |D|^{-1} |D| R_0(z) |D| |D|^{-1} |x|^{-1}$$

where $|x|^{-1}|D|^{-1}$ and its adjoint $|D|^{-1}|x|^{-1}$ are bounded on L^2 thanks to the Hardy inequality (if $n \geq 3$) and where, by the spectral theorem,

$$\left| \left| |D| R_0(z) |D| \right| \right|_{L^2 \to L^2} = \left\| \frac{|D|^2}{|D|^2 - z} \right\|_{L^2 \to L^2} \leq 1, \qquad \mathrm{Re}(z) \leq 0.$$

Thus $|x|^{-1} R_0(z) |x|^{-1}$ is uniformly bounded on L^2 for $\mathrm{Re}(z) \leq 0$ and hence so is $\langle x \rangle^{-1} R_0(z) \langle x \rangle^{-1}$. If $\mathrm{Re}(z) > 0$, we use the generator of dilations $A = x \cdot D_x + \frac{n}{2i}$ to write

$$|x|^{-1} R_0(z) |x|^{-1} = |x|^{-1} (A + i) \left((A + i)^{-1} R_0(z) (A - i)^{-1} \right) (A - i) |x|^{-1}.$$

By rescaling the right-hand side, ie by writing it as

$$e^{itA} |x|^{-1} (A + i) \left((A + i)^{-1} R_0(1 + i\delta) (A - i)^{-1} \right) (A - i) |x|^{-1} e^{-itA},$$

with $\delta = \mathrm{Im}(z)/\mathrm{Re}(z)$ and $t = \frac{1}{2} \ln \mathrm{Re}(z)$, we see that

$$\left| \left| |x|^{-1} R_0(z) |x|^{-1} \right| \right|_{L^2 \to L^2} \leq \left| \left| |x|^{-1} (A + i) \right| \right|^2_{H^1 \to L^2} \sup_{\mathrm{Re}(\zeta) = 1} \left| \left| (A + i)^{-1} R_0(\zeta) (A - i)^{-1} \right| \right|_{H^{-1} \to H^1}.$$

The boundedness of the supremum follows from the Mourre theory (in the $H^{-1} \to H^1$ topology) based on a standard positive commutator estimate at energy 1. The finiteness of the squared norm follows from the fact that

$$|x|^{-1} (A + i) = \frac{x}{|x|} \cdot D_x + \frac{n}{2i|x|} + \frac{i}{|x|}$$

maps H^1 in L^2 thanks to the Hardy inequality. Thus $|x|^{-1} R_0(z) |x|^{-1}$ and hence $\langle x \rangle^{-1} R_0(z) \langle x \rangle^{-1}$ are uniformly bounded on L^2 for $\mathrm{Re}(z) > 0$.

We point out in passing that the Hardy inequality $\left| \left| |x|^{-1} u \right| \right|_{L^2} \lesssim \left| \left| |D| u \right| \right|_{L^2}$ and the Sobolev inequality $||u||_{L^{2^*}} \lesssim \left| \left| |D| u \right| \right|_{L^2}$ have the same scaling (here and below $2^* = \frac{2n}{n-2}$ and $2_* = \frac{2n}{n+2}$) so, in this sense, the uniform boundedness of $|x|^{-1} R_0(z) |x|^{-1}$ has the same scaling as the $L^{2_*} \to L^{2^*}$ inequality

$$\left| \left| R_0(z) \right| \right|_{L^{2_*} \to L^{2^*}} \lesssim 1, \qquad z \in \mathbb{C} \setminus [0, \infty).$$

This estimate for the flat Laplacian is a consequence of [23] and has recently been extended to non trapping scattering manifolds in [17]. Note that $\langle r \rangle^{-s} L^{2^*} \subset L^2$ and $\langle r \rangle^{-s} L^2 \subset L^{2^*}$ for any $s > 1$ so any $L^{2^*} \to L^{2^*}$ estimate on $R(z)$ yield $L^2 \to L^2$ estimates on $\langle r \rangle^{-s} R(z) \langle r \rangle^{-s}$ with $s > 1$. But it is not clear that $L^{2^*} \to L^{2^*}$ estimates on $R(z)$ imply $L^2 \to L^2$ estimates on $\langle r \rangle^{-1} R(z) \langle r \rangle^{-1}$ (nor the reverse). In particular, our results and those of [17] do not overlap.

To prove the sharpness of Theorem 1.1, we consider again the flat case. We observe that if we could replace the weight $\langle x \rangle^{-1}$ by $\langle x \rangle^{-s}$ for some $s < 1$, then by letting $z \to 0$, we would obtain the $L^2 \to L^2$ boundedness of $(1 + |x|)^{-s} \Delta^{-1} (1 + |x|)^{-s}$ (Δ^{-1} being understood as the Fourier multiplier by $-|\xi|^{-2}$) and then, by rescaling, we would have

$$\left\| (|x| + 1)^{-s} \Delta^{-1} (|x| + 1)^{-s} \right\|_{L^2 \to L^2} = \epsilon^{2-2s} \left\| (|x| + \epsilon^{-1})^{-s} \Delta^{-1} (|x| + \epsilon^{-1})^{-s} \right\|_{L^2 \to L^2} \lesssim \epsilon^{2-2s}$$

hence get $(1 + |x|)^{-s} \Delta^{-1} (1 + |x|)^{-s} = 0$ which is obviously wrong.

The proofs of Theorems 1.1 and 1.2 in the general case follow the ideas sketched above for the flat Laplacian combined with a perturbation argument. We will first show that the above strategy can be adapted to small enough perturbations (in a suitable topology) of the Laplace operator on the exact cone $\mathcal{M}_0 := ((0,\infty) \times \mathcal{S}, dr^2 + r^2 h_0)$ where one can use scaling arguments. Then, we will get the estimate on $\mathcal{M}$ by using a perturbation argument based on the fact that the resolvent of $\hat{P}$ can be suitably approximated by the resolvent of an operator on $\mathcal{M}_0$.

The paper is organized as follows. In Section 2, we prove Theorems 1.1, 1.2 and 1.3 by using Theorem 2.3 which postulates the existence of an operator $\hat{P}_T$ on $\mathcal{M}_0$ coinciding with $\hat{P}$ near infinity and satisfying suitable resolvent estimates. Theorem 2.3 follows basically from an appropriate choice of the radial coordinate (Proposition 2.1) and a version of the Mourre Theory on a cone. This theory is developed in Section 3 and then applied in Section 4 to prove Theorem 2.3. Proposition 2.1 is also proved in Section 4.

2   Proofs of the main results

The basic idea to prove Theorems 1.1, 1.2 and 1.3 is to extrapolate $\mathcal{M} \setminus \mathcal{K} \approx (R_0, \infty) \times \mathcal{S}$ into a pure cone $\mathcal{M}_0 = (0, \infty) \times \mathcal{S}$, where one can use a global scaling argument to reduce the proof to the case of estimates at frequency 1. The contribution of $\mathcal{K}$ is then treated by means of a compactness trick, using that 0 is not an eigenvalue of $\hat{P}$.

We record here the main steps of this analysis and then derive the proofs of Theorems 1.1, 1.2 and 1.3.

Proposition 2.1. *We may assume that the diffeomorphism κ in (1.1) is such that*

$$d\mathrm{vol}_G = \kappa^* \big(r^{n-1} dr d\mathrm{vol}_{h_0} \big), \qquad on \ \mathcal{M} \setminus \mathcal{K}. \tag{2.9}$$

More precisely, we can find a new $\tilde{\mathcal{K}} \Subset \mathcal{M}$, a new diffeomorphism $\tilde{\kappa}$ and a new proper function $\tilde{r} : \mathcal{M} \to [0, \infty)$, such that (1.1), (1.2), (1.3) hold with $(\tilde{\kappa}, \tilde{r})$, such that $\langle r \rangle / \langle \tilde{r} \rangle$ is bounded from above and below on $\mathcal{M}$ by positive constants, and such that $d\mathrm{vol}_G = \tilde{\kappa}^* \big(\tilde{r}^{n-1} d\tilde{r} d\mathrm{vol}_{h_0} \big)$ on $\mathcal{M} \setminus \tilde{\mathcal{K}}$.

Proof. See Section 4.

The interest of working with the density $r^{n-1} dr d\mathrm{vol}_{h_0}$ is that, on the cone $(0, \infty) \times \mathcal{S}$, the group $(e^{itA})_{t \in \mathbb{R}}$ of L^2 scalings (see (3.9)) is unitary on $L^2 \big((0, \infty) \times \mathcal{S}, r^{n-1} dr d\mathrm{vol}_{h_0} \big)$. This guarantees

that both A and a suitable extrapolation $\hat{P}_T$ of $\hat{P}$ on $(0, \infty) \times \mathcal{S}$ (see Theorem 2.3 below) are self-adjoint with respect to the same measure.

We next record useful results related to the Hardy inequality. We define the operator

$$\hat{P}^{-1/2} := f(\hat{P}), \qquad f(\lambda) = \mathbb{1}_{(0,+\infty)}(\lambda)\lambda^{-1/2},$$

by means of the spectral theorem (see *e.g.* [31, p. 263]). It is an unbounded self-adjoint operator and it is a routine to check that it maps its domain into the domain of $\hat{P}^{1/2}$. Moreover, we have

$$\hat{P}^{1/2}\hat{P}^{-1/2} = I, \qquad \text{on } \mathrm{Dom}(\hat{P}^{-1/2}), \tag{2.10}$$

which is a consequence of the spectral theorem and the property (1.4).

Proposition 2.2. *1. There is a constant C such that, for all $u \in \mathrm{Dom}\big(\hat{P}^{1/2}\big)$,*

$$||\langle r \rangle^{-1} u||_{L^2(\mathcal{M})} \le C ||\hat{P}^{1/2} u||_{L^2(\mathcal{M})}. \tag{2.11}$$

2. The operator $\langle r \rangle^{-1}$ maps $L^2(\mathcal{M})$ into $\mathrm{Dom}(\hat{P}^{-1/2})$ and $\hat{P}^{-1/2}\langle r \rangle^{-1}$ is bounded on $L^2(\mathcal{M})$.

3. For all real number $s > 1$, $\hat{P}^{-1/2}\langle r \rangle^{-s}$ is compact on $L^2(\mathcal{M})$.

Proof. 1. Since $\hat{P}$ is the Friedrichs extension of $-\Delta_G$, $C_0^\infty(\mathcal{M})$ is dense in $\mathrm{Dom}\big(\hat{P}^{1/2}\big)$ for the graph norm so we may assume that $u \in C_0^\infty(\mathcal{M})$. Fix $\chi = \chi(r)$ a smooth function which is equal to 1 near $\mathcal{K}$ and vanishes for $r \gg 1$. Then

$$||\langle r \rangle^{-1} u||_{L^2(\mathcal{M})} \le ||\chi u||_{L^2(\mathcal{M})} + ||\langle r \rangle^{-1}(1-\chi)u||_{L^2(\mathcal{M})}.$$

From the Poincaré inequality on a compact manifold with boundary containing $\mathrm{supp}(\chi)$, we get

$$||\chi u||_{L^2(\mathcal{M})} \lesssim ||\nabla_G(\chi u)||_{L^2(\mathcal{M})} \lesssim ||\nabla_G u||_{L^2(\mathcal{M})} + ||\chi'(r)u||_{L^2(\mathcal{M})}.$$

On the other hand, using the Hardy inequality on $(0,\infty) \times \mathcal{S}$ (see (3.19) in Section 3) we also have

$$||\langle r \rangle^{-1}(1-\chi)u||_{L^2(\mathcal{M})} \lesssim ||\partial_r((1-\chi)u)||_{L^2(\mathcal{M})} \lesssim ||\nabla_G u||_{L^2(\mathcal{M})} + ||\chi'(r)u||_{L^2(\mathcal{M})}.$$

To complete the proof it suffices to observe that, if $\chi'(r)$ is supported in $\{r_1 < r < r_2\}$ with $r_1 > R_0$,

$$||\chi'(r)u||_{L^2(\mathcal{M})} \lesssim ||u||_{L^2(r_1 < r < r_2)} \lesssim ||\nabla_G u||_{L^2(\mathcal{M})}. \tag{2.12}$$

Indeed, for all $r > R_0$ and all $\omega \in \mathcal{S}$, we have

$$|\kappa^* u(r,\omega)| = \left| \int_r^\infty \partial_s(\kappa^* u)(s,\omega)ds \right| \le \frac{r^{\frac{2-n}{2}}}{(n-2)^{1/2}} \left(\int_r^\infty |\partial_s(\kappa^* u)(s,\omega)|^2 s^{n-1}ds \right)^{1/2},$$

using the Cauchy-Schwartz inequality. Squaring and integrating over $[r_1, r_2] \times \mathcal{S}$ with respect to $r^{n-1}dr\mathrm{dvol}_{h_0}(\omega)$, we get (2.12). Since $||\nabla_G u||_{L^2(\mathcal{M})} = ||\hat{P}^{1/2}u||_{L^2(\mathcal{M})}$, (2.11) follows.

2. For all $u \in \mathrm{Dom}(\hat{P}^{-1/2})$ and $v \in L^2(\mathcal{M})$, we have

$$\left| (\langle r \rangle^{-1} v, \hat{P}^{-1/2}u) \right| = \left| (v, \langle r \rangle^{-1}\hat{P}^{-1/2}u) \right| \le C||v||_{L^2(\mathcal{M})}||u||_{L^2(\mathcal{M})},$$

using (2.10) and (2.11). This implies both that $\langle r \rangle^{-1}v$ belongs to $\mathrm{Dom}\big((\hat{P}^{-1/2})^*\big) = \mathrm{Dom}(\hat{P}^{-1/2})$ and that $\hat{P}^{-1/2}\langle r \rangle^{-1}$ is bounded.

3. Fix $\Phi \in C_0^\infty(\mathbb{R})$ which is equal to 1 near zero. Then, using the spectral theorem, we have the decomposition

$$\hat{P}^{-1/2}\langle r \rangle^{-1} = \Phi(\hat{P})\hat{P}^{-1/2}\langle r \rangle^{-s} + (1 - \Phi)(\hat{P})\hat{P}^{-1/2}\langle r \rangle^{-s}.$$

The second term in the right-hand side is compact since (1.5) implies that $f(\hat{P})g(r)$ is compact whenever f and g are continuous on $\mathbb{R}$ and vanish at infinity. We then rewrite the first term of the right-hand side as $\hat{P}^{-1/2}\langle r \rangle^{-1}\big(\langle r \rangle \Phi(\hat{P})\langle r \rangle^{-s}\big)$ so the result will follow from the compactness of $\langle r \rangle \Phi(\hat{P})\langle r \rangle^{-s}$ proved as follows. By the Helffer-Sjöstrand formula [12], we have

$$\langle r \rangle \Phi(\hat{P})\langle r \rangle^{-s} = \int_{\mathbb{C}} \overline{\partial}\widetilde{\Phi}(z)\langle r \rangle(\hat{P} - z)^{-1}\langle r \rangle^{-s}L(dz), \tag{2.13}$$

where $\widetilde{\Phi} \in C_c^\infty(\mathbb{C})$ is an almost analytic extension of Φ, i.e. $\overline{\partial}\widetilde{\Phi}(z) = \mathcal{O}(\mathrm{Im}(z)^\infty)$, and

$$\langle r \rangle(\hat{P} - z)^{-1}\langle r \rangle^{-s} = (\hat{P} - z)^{-1}\langle r \rangle^{-(s-1)} + (\hat{P} - z)^{-1}[\hat{P}, \langle r \rangle](\hat{P} - z)^{-1}\langle r \rangle^{-s}.$$

Using on one hand that $(\hat{P} - z)^{-1}[\hat{P}, \langle r \rangle]$ has a bounded closure on $L^2(\mathcal{M})$ with norm $\mathcal{O}(\mathrm{Im}(z)^{-1})$ on $\mathrm{supp}(\widetilde{\Phi})$, and on the other hand that, for $\mu = s$ or $s - 1$,

$$(\hat{P} - z)^{-1}\langle r \rangle^{-\mu} = (\hat{P} + i)(\hat{P} - z)^{-1}(\hat{P} + i)^{-1}\langle r \rangle^{-\mu},$$

we obtain easily the compactness of (2.13) from (1.5). This completes the proof. $\qquad\square$

Remark. In the sequel we shall use freely the fact that, for any real number M, operators of the form $\langle r \rangle^M(\hat{P} + 1)^{-1}\langle r \rangle^{-M}$ or $\langle r \rangle^M\Phi(\hat{P})\langle r \rangle^{-M}$, with $\Phi \in C_0^\infty(\mathbb{R})$, are bounded on $L^2(\mathcal{M})$ and even map $L^2(\mathcal{M})$ into $\mathrm{Dom}(\hat{P}^{1/2})$. This is basically well known and follows from the same type of standard argument as the ones used in the proof of the item 3 above.

In the next theorem, we summarize the main technical results of this paper, which deal with resolvent estimates of model operators on

$$\mathcal{M}_0 := (0, +\infty) \times \mathcal{S}, \tag{2.14}$$

equipped with the conical volume density $r^{n-1}dr\mathrm{dvol}_{h_0}$. We will set everywhere

$$L^2(\mathcal{M}_0) := L^2\big((0, +\infty) \times \mathcal{S}, r^{n-1}dr\mathrm{dvol}_{h_0}\big).$$

In the sequel, when nothing is specified, $||\cdot||$ will denote both norms $||\cdot||_{L^2(\mathcal{M}_0) \to L^2(\mathcal{M}_0)}$ and $||\cdot||_{L^2(\mathcal{M}) \to L^2(\mathcal{M})}$. This won't cause any ambiguity in practice but will simplify the notation.

We will also use the standard notation κ^* for the composition with κ and κ_* for the composition with κ^{-1}.

Theorem 2.3. *Fix $N \geq 1$. There exists a self-adjoint operator $\hat{P}_T$ on $L^2(\mathcal{M}_0)$ such that, for some $R \gg 1$, we have*

$$\kappa^*\hat{P}_T = \hat{P}\kappa^*, \qquad on\ (R, \infty) \times \mathcal{S},$$

and such that $\hat{P}_T$ satisfies the following resolvent estimates:

1. there exists C such that

$$\big|\big|r^{-1}\big(\hat{P}_T - z\big)^{-1}r^{-1}\big|\big|_{L^2(\mathcal{M}_0) \to L^2(\mathcal{M}_0)} \leq C, \tag{2.15}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ such that $|\mathrm{Re}(z)| \leq 1$.

2. *For all $[E_1, E_2] \Subset (0, +\infty)$ and all $0 \leq k \leq N-1$, there exists C_k such that*

$$\left|\left|(\epsilon r)^{-1}\langle \epsilon r\rangle^{-k}\left(\epsilon^{-2}\hat{P}_T - Z\right)^{-1-k}\langle \epsilon r\rangle^{-k}(\epsilon r)^{-1}\right|\right|_{L^2(\mathcal{M}_0)\to L^2(\mathcal{M}_0)} \leq C_k, \tag{2.16}$$

for all $Z \in \mathbb{C} \setminus \mathbb{R}$ such that $\mathrm{Re}(Z) \in [E_1, E_2]$ and all $\epsilon \in (0, 1]$.

3. *For all $s \in (0, 1/2)$, there exists C_s such that*

$$\left|\left|\langle r\rangle^{-2-s}(\hat{P}_T - z)^{-2}\langle r\rangle^{-2-s}\right|\right|_{L^2(\mathcal{M}_0)\to L^2(\mathcal{M}_0)} \leq C_s|\mathrm{Re}(z)|^{s-1}, \tag{2.17}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ such that $0 < |\mathrm{Re}(z)| \leq 1$.

Proof. See Section 4. We only mention here that the operator $\hat{P}_T$ will be constructed as the self-adjoint realization of an operator in divergence form given by (4.6). $\hfill\square$

Note that the first item of Theorem 2.3 means essentially that $\hat{P}$ and $\hat{P}_T$ coincide close to infinity, though they are not defined on the same manifold.

We then want to compare the resolvent of $\hat{P}$ with the resolvent of $\hat{P}_T$. Since they are not defined on the same space, we will use the following identification operators,

$$\begin{aligned} J_0 &= \chi_{\mathcal{M}\setminus\mathcal{K}} \circ \kappa^* \circ \varrho_{(R_0,\infty)\times\mathcal{S}} : L^2(\mathcal{M}_0) \to L^2(\mathcal{M}), \\ J &= \chi_{(R_0,\infty)\times\mathcal{S}} \circ \kappa_* \circ \varrho_{\mathcal{M}\setminus\mathcal{K}} : L^2(\mathcal{M}) \to L^2(\mathcal{M}_0) \end{aligned}$$

Everywhere ϱ_Ω stands for the restriction operator to Ω. It is straightforward to check that $J_0J : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is the multiplication operator

$$J_0J = \mathbb{1}_{\mathcal{M}\setminus\mathcal{K}}. \tag{2.18}$$

Proposition 2.4 (Generalized resolvent identity). *Let $\psi \in C^\infty(\mathcal{M})$ depend only on r, be supported on $\{r > R\}$ with R as in Theorem 2.3, and $\psi \equiv 1$ near infinity. Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\begin{aligned} (\hat{P} - z)^{-1}\psi J_0 &= \psi J_0(\hat{P}_T - z)^{-1} - (\hat{P} - z)^{-1}[\hat{P}, \psi]J_0(\hat{P}_T - z)^{-1}, \tag{2.19} \\ J\psi(\hat{P} - z)^{-1} &= (\hat{P}_T - z)^{-1}J\psi + (\hat{P}_T - z)^{-1}J[\hat{P}, \psi](\hat{P} - z)^{-1}. \tag{2.20} \end{aligned}$$

As a consequence, we also have

$$\begin{aligned} (\hat{P} - z)^{-1}\psi &= \psi J_0(\hat{P}_T - z)^{-1}J - (\hat{P} - z)^{-1}[\hat{P}, \psi]J_0(\hat{P}_T - z)^{-1}J, \tag{2.21} \\ \psi(\hat{P} - z)^{-1} &= J_0(\hat{P}_T - z)^{-1}J\psi + J_0(\hat{P}_T - z)^{-1}J[\hat{P}, \psi](\hat{P} - z)^{-1}. \tag{2.22} \end{aligned}$$

Proof. By the first item of Theorem 2.3 and the support property of ψ, we have $\psi\hat{P}J_0 = \psi J_0\hat{P}_T$ and therefore

$$(\hat{P} - z)\psi J_0(\hat{P}_T - z)^{-1} = [\hat{P}, \psi]J_0(\hat{P}_T - z)^{-1} + \psi J_0.$$

After composition to the left with $(\hat{P} - z)^{-1}$ we obtain (2.19). The proof of (2.20) is similar. The identities (2.21) and (2.22) follow from (2.19) and (2.20) respectively, combined with (2.18) and the fact that $\psi\mathbb{1}_{\mathcal{M}\setminus\mathcal{K}} = \psi$. $\hfill\square$

In the sequel, we fix a real number $M > \max(N, 3)$, N being the integer fixed in Theorem 2.3, and introduce the weighted resolvent

$$R_M(z) = \langle r\rangle^{-M}(\hat{P} - z)^{-1}\langle r\rangle^{-M}.$$

We will first obtain estimates on $R_M(z)$ and its derivatives. With those estimates at hand, it will be fairly easy to derive estimates with sharper weights. To prove estimates on $R_M(z)$ we will use the following proposition.

Proposition 2.5. *One can find $\varepsilon_0 > 0$ and bounded operators S_1, S_2, T_1, T_2 and $F_M(\cdot), S(\cdot), T(\cdot)$ depending holomorphically on $z \in \{|\mathrm{Re}(z)| < \varepsilon_0\} \cap \{\mathrm{Im}(z) \neq 0\}$ such that for all z in this set, we have*

$$R_M(z) = F_M(z) + S_1 R_M(z) T_1 + T_2 R_M(z) S_2 + T(z) R_M(z) S(z), \tag{2.23}$$

and

$$||S_1||\,||T_1|| + ||S_2||\,||T_2|| + ||S(z)||\,||T(z)|| \leq 3/4, \tag{2.24}$$

and such that, for all $s \in (0, 1/2)$ and all $1 \leq k \leq N$,

$$
\begin{aligned}
||F_M(z)|| &\leq C, & |\mathrm{Re}(z)| < \varepsilon_0, & \quad(2.25)\\
||\partial_z F_M(z)|| + ||\partial_z S(z)|| + ||\partial_z T(z)|| &\leq C_s |\mathrm{Re}(z)|^{s-1}, & 0 \neq |\mathrm{Re}(z)| < \varepsilon_0, & \quad(2.26)\\
||\partial_z^k F_M(z)|| + ||\partial_z^k S(z)|| + ||\partial_z^k T(z)|| &\leq C_k \mathrm{Re}(z)^{-k}, & 0 < \mathrm{Re}(z) < \varepsilon_0. & \quad(2.27)
\end{aligned}
$$

This proposition is based on Proposition 2.4. Before proving it, we need to establish several intermediate lemmas. Consider three functions ψ_1, ψ_2 and φ in $C^\infty(\mathcal{M})$ supported in $\{r > R\}$, depending only on r, equal to 1 near infinity and such that

$$\psi_2 \equiv 1 \ \text{ near supp}(\psi_1), \qquad \psi_1 \equiv 1 \ \text{ near supp}(\varphi).$$

By Proposition 2.4 and the easily verified fact that $J[\hat{P}, \psi_1]J_0 = [\hat{P}_T, \psi_1]$ (we identify ψ_1 and $\kappa_* \psi_1$ in the obvious fashion since ψ_1 depends only on r), we obtain

$$
\begin{aligned}
\varphi(\hat{P} - z)^{-1}\varphi = {}& \varphi J_0(\hat{P}_T - z)^{-1}J\varphi - \varphi J_0(\hat{P}_T - z)^{-1}[\hat{P}_T, \psi_1](\hat{P}_T - z)^{-1}J\varphi \\
& - \varphi J_0(\hat{P}_T - z)^{-1}J[\hat{P}, \psi_2](\hat{P} - z)^{-1}[\hat{P}, \psi_1]J_0(\hat{P}_T - z)^{-1}J\varphi. \quad(2.28)
\end{aligned}
$$

The interest of this formula is that $[\hat{P}, \psi_1]$ and $[\hat{P}, \psi_2]$ (as well as $[\hat{P}_T, \psi_1]$) have compactly supported coefficients. The smallness condition (2.24) will be a consequence of the following lemma.

Lemma 2.6. *Let $\Phi \in C_0^\infty(\mathbb{R})$ be equal to 1 near 0. For all $\delta > 0$, we can choose $\varepsilon > 0$ and $\nu > 0$ such that*

$$\left|\left|\langle r \rangle^M (1 - \varphi)\Phi(\hat{P}/\varepsilon)\right|\right| \leq \delta, \tag{2.29}$$

and, for all $z \in \mathbb{C} \setminus \mathbb{R}$ satisfying $|\mathrm{Re}(z)| < \nu$,

$$\left|\left|\langle r \rangle^{-M} J_0(\hat{P}_T - z)^{-1}J\varphi\Phi(\hat{P}/\varepsilon)\langle r \rangle^{-M}\right|\right| \leq \delta. \tag{2.30}$$

Proof. The first inequality is standard. We recall the proof for completeness. If ε is small enough, we have $\Phi(\lambda/\varepsilon) = \Phi(\lambda)\Phi(\lambda/\varepsilon)$ so that

$$\langle r \rangle^M (1 - \varphi)\Phi(\hat{P}/\varepsilon) = \left(\langle r \rangle^M (1 - \varphi)\Phi(\hat{P})\right)\Phi(\hat{P}/\varepsilon).$$

By (1.5) and the compact support of $1 - \varphi$, the first term in the right-hand side is a compact operator. On the other hand, the property (1.4) implies that $\Phi(\hat{P}/\varepsilon) \to 0$ in the strong sense as $\varepsilon \to 0$, hence in operator norm when composed with a compact operator. This yields (2.29). Let us prove (2.30). We set $y = \mathrm{Im}(z)$ and split the resolvent as

$$(\hat{P}_T - z)^{-1} = (\hat{P}_T - iy)^{-1} + \left((\hat{P}_T - z)^{-1} - (\hat{P}_T - iy)^{-1}\right).$$

We start with the contribution of the first term which, using Proposition 2.2, we write as

$$\langle r \rangle^{-M}(\hat{P}_T - iy)^{-1}J\varphi\Phi(\hat{P}/\varepsilon)\langle r \rangle^{-M} = \big(\langle r \rangle^{-M}(\hat{P}_T - iy)^{-1}J\varphi\hat{P}^{1/2}\big)\big(\Phi(\hat{P}/\varepsilon)\hat{P}^{-1/2}\langle r \rangle^{-M}\big).$$

By Proposition 2.2, $\hat{P}^{-1/2}\langle r \rangle^{-M}$ is compact, hence the second bracket in the right-hand side goes to zero in operator norm as $\varepsilon \to 0$. Therefore, it suffices to obtain a uniform estimate on the first term. To do so, we use the form of $\hat{P}_T$ which is (the Friedrichs extension of) the elliptic operator in divergence form (4.6). Using the form of $\hat{P}_T$, the Hardy inequality (3.19) on $\mathcal{M}_0$ and the spectral theorem, we have

$$
\begin{aligned}
\big|\big|\langle r \rangle^{-M}(\hat{P}_T - iy)^{-1}J\varphi\hat{P}^{1/2}\big|\big| &= \big|\big|\hat{P}^{1/2}\varphi J^*(\hat{P}_T + iy)^{-1}\langle r \rangle^{-M}\big|\big| \\
&\lesssim \big|\big|\nabla_G\varphi J^*(\hat{P}_T + iy)^{-1}\langle r \rangle^{-1}\big|\big| \\
&\lesssim \big|\big|\hat{P}_T^{1/2}(\hat{P}_T + iy)^{-1}\langle r \rangle^{-1}\big|\big| + \big|\big|\langle r \rangle^{-1}(\hat{P}_T + iy)^{-1}\langle r \rangle^{-1}\big|\big| \\
&\lesssim \big|\big|\hat{P}_T^{1/2}(\hat{P}_T + iy)^{-1}\hat{P}_T^{1/2}\big|\big| \lesssim 1.
\end{aligned}
$$

We can therefore fix ε such that

$$\big|\big|\langle r \rangle^{-M}(\hat{P}_T - iy)^{-1}J\varphi\Phi(\hat{P}/\varepsilon)\langle r \rangle^{-M}\big|\big| \leq \delta/2.$$

Then, by writing

$$(\hat{P}_T - z)^{-1} - (\hat{P}_T - iy)^{-1} = \int_0^{\mathrm{Re}(z)}(\hat{P}_T - x - iy)^{-2}dx$$

and using (2.17), say with $s = 1/4$, we obtain

$$\big|\big|\langle r \rangle^{-M}\left((\hat{P}_T - z)^{-1} - (\hat{P}_T - iy)^{-1}\right)J\langle r \rangle^{-M}\big(\langle r \rangle^{M}\varphi\Phi(\hat{P}/\varepsilon)\langle r \rangle^{-M}\big)\big|\big| \leq C_\varepsilon|\mathrm{Re}(z)|^{1/4}.$$

Here we also used that J preserves the decay in r. The above norm can therefore be made smaller than $\delta/2$ if $\mathrm{Re}(z)$ is small enough and the result follows. $\qquad\square$

Lemma 2.7. *Fix two integers $k, M \geq 0$. Then there exists $C > 0$ such that*

$$
\begin{aligned}
\big|\big|\partial_z^k\big(w(\hat{P}_T - z)^{-1}[\hat{P}_T, \psi_1](\hat{P}_T - z)^{-1}w\big)\big|\big| &\leq C\sum_{k_1=0}^{k}\big|\big|w(\hat{P}_T - z)^{-1-k_1}\langle r \rangle^{-M}\big|\big| + \\
C\sum_{k_1+k_2=k}&\big|\big|w(\hat{P}_T - z)^{-1-k_1}\langle r \rangle^{-M}\big|\big|\,\big|\big|\langle r \rangle^{-M}(\hat{P}_T - z)^{-1-k_2}w\big|\big|,
\end{aligned}
$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ such that $|\mathrm{Re}(z)| \leq 1$ and all weight w such that $\|w\|_{L^\infty} \leq 1$.

Proof. Fix $\Phi \in C_0^\infty(\mathbb{R})$ such that $\Phi \equiv 1$ near $[-1, 1]$ and write

$$[\hat{P}_T, \psi_1](\hat{P}_T - z)^{-1} = [\hat{P}_T, \psi_1]\Phi(\hat{P}_T)(\hat{P}_T - z)^{-1} + [\hat{P}_T, \psi_1](1 - \Phi)(\hat{P}_T)(\hat{P}_T - z)^{-1}.$$

The last term can be written $[\hat{P}_T, \psi_1](\hat{P}_T + i)^{-1}(1 - \Phi)(\hat{P}_T)(\hat{P}_T + i)(\hat{P}_T - z)^{-1}$, which is holomorphic and bounded (uniformly in z) in the strip $\{|\mathrm{Re}(z)| \leq 1\}$. Note also that, by the compact support of $[\hat{P}_T, \psi_1]$ the same property holds for $\langle r \rangle^{M}[\hat{P}_T, \psi_1](1 - \Phi)(\hat{P}_T)(\hat{P}_T - z)^{-1}$. On the other hand, $[\hat{P}_T, \psi_1]\Phi(\hat{P}_T)$ is bounded and, by a Helffer-Sjöstrand formula argument (see the remark after Proposition 2.2), $\langle r \rangle^{M}[\hat{P}_T, \psi_1]\Phi(\hat{P}_T)\langle r \rangle^{M}$ is bounded too. All this implies that

$$(\hat{P}_T - z)^{-1}[\hat{P}_T, \psi_1](\hat{P}_T - z)^{-1} = (\hat{P}_T - z)^{-1}\langle r \rangle^{-M}B_M\langle r \rangle^{-M}(\hat{P}_T - z)^{-1} + (\hat{P}_T - z)^{-1}\langle r \rangle^{-M}O_M(z),$$

for some bounded operators B_M and $O_M(z)$, $O_M(\cdot)$ being holomorphic and uniformly bounded in the strip $\{|\mathrm{Re}(z)| \leq 1\}$. The result then follows easily. $\qquad\square$

Lemma 2.8. *Fix $0 \le k \le N - 1$. Then*

$$\left| \left| \langle r \rangle^{-M} (\hat{P}_T - z)^{-1-k} \langle r \rangle^{-M} \right| \right| \lesssim \mathrm{Re}(z)^{-k}, \tag{2.31}$$

$$\left| \left| \langle r \rangle^{-M} \partial_z^k \big((\hat{P}_T - z)^{-1} [\hat{P}_T, \psi_1] (\hat{P}_T - z)^{-1} \big) \langle r \rangle^{-M} \right| \right| \lesssim \mathrm{Re}(z)^{-k}, \tag{2.32}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ such that $\mathrm{Re}(z) > 0$ is small enough

Proof. We rewrite $z = \epsilon^2 Z$ with $\mathrm{Re}(Z)$ in a compact subset of $(0, \infty)$ and $\epsilon \in (0, 1]$. Then, by using (2.16) and the fact that

$$(\epsilon r)^{-1} \langle \epsilon r \rangle^{-k} (\epsilon^{-2} \hat{P}_T - Z)^{-1-k} \langle \epsilon r \rangle^{-k} (\epsilon r)^{-1} = \epsilon^{2k} r^{-1} \langle \epsilon r \rangle^{-k} (\hat{P}_T - \epsilon^2 Z)^{-1-k} \langle \epsilon r \rangle^{-k} r^{-1},$$

whose left-hand side is bounded by (2.16), we get (2.31) once observed that

$$\langle r \rangle^{-M} = \big(\langle r \rangle^{-M} r \langle \epsilon r \rangle^{k} \big) r^{-1} \langle \epsilon r \rangle^{-k},$$

where the bracket is bounded uniformly in ϵ. The estimate (2.32) follows from Lemma 2.7 and (2.31). $\qquad \square$

Proof of Proposition 2.5. We use the spatial cutoffs φ, ψ_1, ψ_2 introduced after Proposition 2.5. We write

$$(\hat{P} - z)^{-1} = \varphi (\hat{P} - z)^{-1} \varphi + \varphi (\hat{P} - z)^{-1} (1 - \varphi) + (1 - \varphi)(\hat{P} - z)^{-1}, \tag{2.33}$$

where we recall that $1 - \varphi$ is compactly supported on $\mathcal{M}$. We next consider three functions $\Phi, \phi_1, \phi_2 \in C_0^\infty(\mathbb{R})$ equal to 1 near 0, which we will use as spectral cutoffs. We fix Φ arbitrarily but we will choose ϕ_1, ϕ_2 below. We use Φ to rewrite the second line of (2.28) as the sum of

$$-\varphi J_0 (\hat{P}_T - z)^{-1} J \left\{ [\hat{P}, \psi_2] \Phi(\hat{P}) (\hat{P} - z)^{-1} \Phi(\hat{P}) [\hat{P}, \psi_1] \right\} J_0 (\hat{P}_T - z)^{-1} J \varphi \tag{2.34}$$

and

$$-\varphi J_0 (\hat{P}_T - z)^{-1} J \left\{ [\hat{P}, \psi_2] (1 - \Phi^2)(\hat{P}) (\hat{P} - z)^{-1} [\hat{P}, \psi_1] \right\} J_0 (\hat{P}_T - z)^{-1} J \varphi. \tag{2.35}$$

The interest of this decomposition is one hand that $[\hat{P}, \psi_2] \Phi(\hat{P})$ and $\Phi(\hat{P})[\hat{P}, \psi_1]$ are bounded (*i.e.* have bounded closures) on $L^2(\mathcal{M})$ and on the other hand that the operator $\{ \cdots \}$ in (2.35) is bounded and holomorphic with respect to z for $|\mathrm{Re}(z)|$ small enough. Moreover, all these operators have a fast spatial decay in the sense that, for any fixed $M \ge 1$, there exist $B_1, B_2, B(z)$ bounded on $L^2(\mathcal{M})$ with $B(z)$ depending boundedly and holomorphically on z for $\mathrm{Re}(z)$ small, such that

$$[\hat{P}, \psi_2] \Phi(\hat{P}) = \langle r \rangle^{-M} B_2 \langle r \rangle^{-M}, \qquad \Phi(\hat{P})[\hat{P}, \psi_1] = \langle r \rangle^{-M} B_1 \langle r \rangle^{-M},$$

and

$$[\hat{P}, \psi_2] (1 - \Phi^2)(\hat{P}) (\hat{P} - z)^{-1} [\hat{P}, \psi_1] = \langle r \rangle^{-M} B(z) \langle r \rangle^{-M}.$$

This follows from the compact support of $[\hat{P}, \psi_1]$ and $[\hat{P}, \psi_2]$ and standard arguments (using for instance the Helffer-Sjöstrand formula). Setting

$$
\begin{aligned}
F(z) \;=\; & \varphi J_0 (\hat{P}_T - z)^{-1} J \varphi - \varphi J_0 (\hat{P}_T - z)^{-1} \big[\hat{P}_T, \psi_1 \big] (\hat{P}_T - z)^{-1} J \varphi \\
& - \varphi J_0 (\hat{P}_T - z)^{-1} J \langle r \rangle^{-M} B(z) \langle r \rangle^{-M} J_0 (\hat{P}_T - z)^{-1} J \varphi,
\end{aligned}
$$

11

it follows from (2.28) and (2.33) that

$$
\begin{aligned}
(\hat{P} - z)^{-1} &= F(z) + (1 - \varphi)(\hat{P} - z)^{-1} + \varphi(\hat{P} - z)^{-1}(1 - \varphi) \\
&\quad - \varphi J_0(\hat{P}_T - z)^{-1} J \langle r \rangle^{-M} B_2 \langle r \rangle^{-M} (\hat{P} - z)^{-1} \langle r \rangle^{-M} B_1 \langle r \rangle^{-M} J_0(\hat{P}_T - z)^{-1} J \varphi.
\end{aligned}
$$

We now use this expression to study a spectrally localized version of $(\hat{P} - z)^{-1}$. We consider $\phi_1(\hat{P})(\hat{P} - z)^{-1}\phi_2(\hat{P})$ where we let ϕ_1 and ϕ_2 take the form

$$
\phi_1(\lambda) = \Phi(\lambda/\varepsilon_1), \qquad \phi_2 = \Phi(\lambda/\varepsilon_2).
$$

By Lemma 2.6, we can choose first ε_1 small enough such that

$$
||\Phi(\hat{P}/\varepsilon_1)(1 - \varphi)\langle r \rangle^M|| \le 1/4.
$$

Once ε_1 is chosen, we can use again Lemma 2.6 to pick $\varepsilon_2 > 0$ and $\nu > 0$ such that

$$
\left|\left|\langle r \rangle^{-M}\Phi(\hat{P}/\varepsilon_1)\varphi\langle r \rangle^M\right|\right| \left|\left|\langle r \rangle^M(1 - \varphi)\Phi(\hat{P}/\varepsilon_2)\right|\right| \le 1/4,
$$

and

$$
\left|\left|\langle r \rangle^{-M}\Phi(\hat{P}/\varepsilon_1)\varphi J_0(\hat{P}_T - z)^{-1}J\langle r \rangle^{-M}B_2\right|\right| \left|\left|B_1\langle r \rangle^{-M}J_0(\hat{P}_T - z)^{-1}J\varphi\Phi(\hat{P}/\varepsilon_2)(\hat{P})\langle r \rangle^{-M}\right|\right| \le 1/4,
$$

for $|\mathrm{Re}(z)| < \nu$ (recall that the first factor in the left-hand side is bounded uniformly with respect to $|\mathrm{Re}(z)| \le 1$ by (2.15)). Summing up, by choosing

$$
\begin{array}{llll}
S_1 &= \langle r \rangle^{-M}\Phi(\hat{P}/\varepsilon_1)(1 - \varphi)\langle r \rangle^M, & T_1 &= I, \\
T_2 &= \langle r \rangle^{-M}\Phi(\hat{P}/\varepsilon_1)\varphi\langle r \rangle^M, & S_2 &= \langle r \rangle^M(1 - \varphi)\Phi(\hat{P}/\varepsilon_2)\langle r \rangle^{-M}
\end{array}
$$

and

$$
\begin{aligned}
T(z) &= \langle r \rangle^{-M}\Phi(\hat{P}/\varepsilon_1)\varphi J_0(\hat{P}_T - z)^{-1}J\langle r \rangle^{-M}B_2, \\
S(z) &= B_1\langle r \rangle^{-M}J_0(\hat{P}_T - z)^{-1}J\varphi\Phi(\hat{P}/\varepsilon_2)(\hat{P})\langle r \rangle^{-M}
\end{aligned}
$$

which satisfy (2.24), we have shown that

$$
\begin{aligned}
\langle r \rangle^{-M}(\hat{P} - z)^{-1}\phi_1(\hat{P})\phi_2(\hat{P})\langle r \rangle^{-M} &= S_1 R_M(z)T_1 + T_2 R_M(z)S_2 + T(z)R_M(z)S(z) \\
&\quad + \langle r \rangle^{-M}\phi_1(\hat{P})F(z)\phi_2(\hat{P})\langle r \rangle^{-M}.
\end{aligned}
$$

If we restrict z to $|\mathrm{Re}(z)| < \varepsilon_0$ with ε_0 small enough, then $(\hat{P} - z)^{-1}\phi_1(\hat{P})\phi_2(\hat{P}) - (\hat{P} - z)^{-1}$ becomes holomorphic and bounded in this strip. Therefore, upon adding a bounded holomorphic term to the right-hand side, we may replace the left-hand side by $\langle r \rangle^{-M}R_M(z)\langle r \rangle^{-M}$ and (2.23) holds with

$$
F_M(z) = \langle r \rangle^{-M}\left((1 - \phi_1\phi_2)(\hat{P})(\hat{P} - z)^{-1} + \phi_1(\hat{P})F(z)\phi_2(\hat{P})\right)\langle r \rangle^{-M}.
$$

The estimates (2.25) and (2.26) follow directly from (2.15), (2.17) and Lemma 2.7. The estimates (2.27) follow from Lemma 2.8. $\qquad\square$

To derive estimates with sharper weights, we will use the following proposition.

Proposition 2.9. *Let $(w_\epsilon)_{\epsilon \in (0,1]}$ be the family of weights*

$$w_\epsilon(r) = \langle r \rangle^{-\mu_1} \langle \epsilon r \rangle^{-\mu_2}$$

where $\mu_1, \mu_2 \geq 0$ are fixed real numbers. Fix $\varphi \in C^\infty(\mathcal{M})$ supported in $\{r > R\}$, equal to 1 near infinity, and let us define

$$R_{w_\epsilon}(z) = w_\epsilon(\hat{P} - z)^{-1} w_\epsilon, \qquad R_{w_\epsilon, \varphi}(z) = w_\epsilon \varphi(\hat{P} - z)^{-1} \varphi w_\epsilon.$$

Then, there exist

1. *families of bounded operators $(A_{\epsilon,j})_{\epsilon \in (0,1]}, (B_{\epsilon,j})_{\epsilon \in (0,1]}, (S_{\epsilon,j})_{\epsilon \in (0,1]}$ on $L^2(\mathcal{M})$, $j = 1, 2$, such that, for all $\epsilon \in (0, 1]$,*

$$||B_{\epsilon,1}|| \ ||S_{\epsilon,1}|| + ||B_{\epsilon,2}|| \ ||S_{\epsilon,2}|| \quad < \quad \frac{1}{2}, \tag{2.36}$$

$$||A_{\epsilon,1}|| + ||A_{\epsilon,1}|| \quad \leq \quad C, \tag{2.37}$$

2. *a real number $\varepsilon_0 > 0$ and a bounded holomorphic mapping $z \mapsto K(z) \in \mathcal{B}(L^2(\mathcal{M}))$ defined in a neighborhood of $\{|z| \leq \varepsilon_0\}$,*

such that, for all $z \in \mathbb{C} \setminus \mathbb{R}$ satisfying $|z| < \varepsilon_0$ and all $\epsilon \in (0, 1]$,

$$R_{w_\epsilon}(z) \quad = \quad w_\epsilon K(z) w_\epsilon + A_{\epsilon,2} R_{w_\epsilon, \varphi}(z) A_{\epsilon,1} + B_{\epsilon,1} R_{w_\epsilon}(z) S_{\epsilon,1} + S_{\epsilon,2} R_{w_\epsilon}(z) B_{\epsilon,2}. \tag{2.38}$$

In particular, for all $k \geq 0$, there exists C_k such that

$$\left| \left| \partial_z^k R_{w_\epsilon}(z) \right| \right| \leq C_k \left(1 + \left| \left| \partial_z^k R_{w_\epsilon, \varphi}(z) \right| \right| \right), \tag{2.39}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ such that $|z| < \varepsilon_0$ and all $\epsilon \in (0, 1]$.

The main interest of this proposition is (2.39) which allows to estimate the full resolvent by its cutoff near infinity. This will allow to use (2.28) in a convenient way.

Proof. The proof is similar, and simpler, than that of Proposition 2.5. Let us set $R(z) = (\hat{P} - z)^{-1}$. For functions $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R})$ equal to 1 near zero and to be chosen later, we have

$$R(z)\phi_1(\hat{P}) = R(z)\varphi\phi_1(\hat{P}) + R(z)(1 - \varphi)\phi_1(\hat{P}),$$

where

$$R(z)\varphi\phi_1(\hat{P}) = (1 - \phi_2)(\hat{P})R(z)\varphi\phi_1(\hat{P}) + \phi_2(\hat{P})R(z)\varphi\phi_1(\hat{P}).$$

The second term in the right-hand side of the last formula reads

$$\phi_2(\hat{P})(1 - \varphi)R(z)\varphi\phi_1(\hat{P}) + \phi_2(\hat{P})\varphi R(z)\varphi\phi_1(\hat{P}).$$

Setting for simplicity $W_\epsilon = w_\epsilon^{-1}$, and writing $R(z) = R(z)\phi_1(\hat{P}) + R(z)(1 - \phi_1)(\hat{P})$ whose second term will be holomorphic with respect to z in a vertical strip around 0, we find that (2.38) holds with

$$\begin{aligned}
B_{\epsilon,1} &= I, & S_{\epsilon,1} &= W_\epsilon(1 - \varphi)\phi_1(\hat{P})w_\epsilon, \\
B_{\epsilon,2} &= W_\epsilon \varphi \phi_1(\hat{P})w_\epsilon, & S_{\epsilon,2} &= w_\epsilon \phi_2(\hat{P})(1 - \varphi)W_\epsilon, \\
A_{\epsilon,1} &= W_\epsilon \phi_1(\hat{P})w_\epsilon, & A_{\epsilon,2} &= w_\epsilon \phi_2(\hat{P})W_\epsilon,
\end{aligned}$$

13

and
$$K(z) = R(z)(1 - \phi_1)(\hat{P}) + (1 - \phi_2)(\hat{P})R(z)\varphi\phi_1(\hat{P}).$$

We choose ϕ_1, ϕ_2 successively as follows. By using the uniform boundedness of $W_\epsilon\langle r\rangle^{-\mu_1-\mu_2}$ and of w_ϵ, we can choose first $\phi_1(\lambda) = \Phi(\lambda/\varepsilon_1)$ such that, by using (2.29) with $M = \mu_1 + \mu_2$, we have $||B_{\epsilon,1}|| \, ||S_{\epsilon,1}|| < 1/4$ (recall that $1 - \varphi$ is a compactly support spatial cutoff). Then $B_{\epsilon,2}$, which depends on ϕ_1, is bounded on $L^2(\mathcal{M})$, uniformly in ϵ, by routine arguments. We choose then $\phi_2(\lambda) = \Phi(\lambda/\varepsilon_2)$ such that the norm of $S_{\epsilon,2}$ is small enough to guarantee that $||B_{\epsilon,2}|| ||S_{\epsilon,2}|| < 1/4$. We therefore get (2.36). The operators $A_{\epsilon,1}$ and $A_{\epsilon,2}$ are uniformly bounded on $L^2(\mathcal{M})$ similarly to $B_{\epsilon,2}$, which yields (2.37). We then fix ε_0 small enough such that both ϕ_1 and ϕ_2 are equal to 1 near $[-\varepsilon_0, \varepsilon_0]$. This implies that $K(z)$ is holomorphic and bounded near the strip $\{|\text{Re}(z)| \leq \varepsilon_0\}$. Finally, (2.39) is an easy consequence of (2.36), (2.37) and (2.38) and the fact that $||\partial_z^k K(z)||$ is bounded on $\{|\text{Re}(z)| \leq \varepsilon_0\}$. $\qquad\square$

We now apply the results of this section to prove Theorems 1.1, 1.2 and 1.3. The three proofs follow the same strategy, but we present them separately for pedagogical reasons: we consider first the simplest case which is Theorem 1.1 and then explain how to modify the arguments for the other two.

Proof of Theorem 1.1. We prove first a uniform estimate in z, but with a rough weight. By Proposition 2.5, we have

$$\left|\left|\langle r\rangle^{-M}(\hat{P} - z)^{-1}\langle r\rangle^{-M}\right|\right| \leq C + \frac{3}{4}\left|\left|\langle r\rangle^{-M}(\hat{P} - z)^{-1}\langle r\rangle^{-M}\right|\right|,$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ such that $|\text{Re}(z)| < \varepsilon_0$. This implies that

$$\left|\left|\langle r\rangle^{-M}(\hat{P} - z)^{-1}\langle r\rangle^{-M}\right|\right| \leq 4C. \tag{2.40}$$

We now use (2.28) to replace $\langle r\rangle^{-M}$ by the optimal weight $\langle r\rangle^{-1}$ as follows. We start by observing that (2.40) and the compact support of $[\hat{P}, \psi_1]$ and $[\hat{P}, \psi_2]$ in (2.28) imply that

$$\left|\left|\langle r\rangle[\hat{P}, \psi_2](\hat{P} - z)^{-1}[\hat{P}, \psi_1]\langle r\rangle\right|\right| \leq C'. \tag{2.41}$$

The weight $\langle r\rangle$ could be replaced by any power of $\langle r\rangle$ but (2.41) will be sufficient. Here the unboundedness of the operators $[\hat{P}, \psi_1]$ and $[\hat{P}, \psi_2]$ can easily be overcome for instance by writing

$$[\hat{P}, \psi_2](\hat{P} - z)^{-1}[\hat{P}, \psi_1] = [\hat{P}, \psi_2]\Phi(\hat{P})(\hat{P} - z)^{-1}\Phi(\hat{P})[\hat{P}, \psi_1] + [\hat{P}, \psi_2](\hat{P} - z)^{-1}(1 - \Phi^2(\hat{P}))[\hat{P}, \psi_1],$$

with $\Phi \in C_0^\infty(\mathbb{R})$, $\Phi \equiv 1$ near $[-\varepsilon_0, \varepsilon_0]$. Both $\langle r\rangle[\hat{P}, \psi_2]\Phi(\hat{P})\langle r\rangle^M$ and $\langle r\rangle^M \Phi(\hat{P})[\hat{P}, \psi_1]\langle r\rangle$ are bounded on $L^2(\mathcal{M})$ and the second term is holomorphic and bounded near the strip $\{|\text{Re}(z)| \leq \varepsilon_0\}$, so (2.41) follows clearly from (2.40). Then, by composing with $\langle r\rangle^{-1}$ to the left and to the right of both sides of (2.28), we obtain

$$\begin{aligned}
\left|\left|\langle r\rangle^{-1}\varphi(\hat{P} - z)^{-1}\varphi\langle r\rangle^{-1}\right|\right| \lesssim \ & \left|\left|\langle r\rangle^{-1}(\hat{P}_T - z)^{-1}\langle r\rangle^{-1}\right|\right| \\
& + \left|\left|\langle r\rangle^{-1}(\hat{P}_T - z)^{-1}[\hat{P}_T, \psi_1](\hat{P}_T - z)^{-1}\langle r\rangle^{-1}\right|\right| \\
& + \left|\left|\langle r\rangle^{-1}(\hat{P}_T - z)^{-1}\langle r\rangle^{-1}\right|\right|^2,
\end{aligned}$$

where the last term is the contribution of the last term of (2.28) combined with (2.41). The second term in the right-hand side can be estimated thanks to Lemma 2.7 with $w = \langle r\rangle^{-1}$ and $k = 0$ so, using (2.15), we conclude that $\left|\left|\langle r\rangle^{-1}\varphi(\hat{P} - z)^{-1}\varphi\langle r\rangle^{-1}\right|\right|$ is uniformly bounded for $|\text{Re}(z)|$ small.

We can then remove the cutoff φ by using (2.39) (with $\mu_2 = 0$ and $\mu_1 = 1$) and this completes the proof. $\qquad\square$

Since its proof is less technical, we prove Theorem 1.3 before Theorem 1.2.

Proof of Theorem 1.3. By differentiating (2.23) with respect to z, we obtain

$$\left|\left|\partial_z R_M(z)\right|\right| \le \frac{3}{4}\left|\left|\partial_z R_M(z)\right|\right| + \left|\left|\partial_z F_M(z)\right|\right| + C\left|\left|\partial_z T_M(z)\right|\right| + C\left|\left|\partial_z S_M(z)\right|\right|,$$

using (2.40) to control the contribution of the non differentiated $R_M(z)$ in the right-hand side. Using (2.26), this implies that

$$\left|\left|\langle r\rangle^{-M}(\hat{P} - z)^{-2}\langle r\rangle^{-M}\right|\right| = \left|\left|\partial_z R_M(z)\right|\right| \lesssim |\mathrm{Re}(z)|^{s-1},$$

when $|\mathrm{Re}(z)| > 0$ is small enough. It remains to replace the weight $\langle r\rangle^{-M}$ by $\langle r\rangle^{-2-s}$. The first observation is that, for the very same reason we got (2.41), the above estimate implies that

$$\left|\left|\langle r\rangle^M[\hat{P}, \psi_2](\hat{P} - z)^{-2}[\hat{P}, \psi_1]\langle r\rangle^M\right|\right| \lesssim |\mathrm{Re}(z)|^{s-1}. \tag{2.42}$$

Using (2.15), (2.17) and Lemma 2.7 with $w = \langle r\rangle^{-2-s}$, we also have

$$\left|\left|\partial_z\left(\langle r\rangle^{-2-s}(\hat{P}_T - z)^{-1}[\hat{P}_T, \psi_1](\hat{P}_T - z)^{-1}\langle r\rangle^{-2-s}\right)\right|\right| \lesssim |\mathrm{Re}(z)|^{s-1}. \tag{2.43}$$

Therefore, it follows from (2.28) that

$$\left|\left|\langle r\rangle^{-2-s}\varphi(\hat{P} - z)^{-2}\varphi\langle r\rangle^{-2-s}\right|\right| \lesssim |\mathrm{Re}(z)|^{s-1},$$

using again (2.17) for the first term as well as (2.40) and (2.42) to handle the contribution of the last term. We can then remove the factors φ by using (2.39) with $\mu_2 = 0$ and $\mu_1 = 2 + s$. The result follows. $\qquad\square$

Proof of Theorem 1.2. Using Proposition 2.5, it is not hard to check by (finite) induction on k that, for $0 \le k \le N$,

$$||\partial_z^k R_M(z)|| \lesssim |\mathrm{Re}(z)|^{-k}, \qquad 0 < \mathrm{Re}(z) < \varepsilon_0. \tag{2.44}$$

Using that

$$(\epsilon^{-2}\hat{P} - Z)^{-1-k} = \frac{\epsilon^2}{k!}\partial_Z^k(\hat{P} - \epsilon^2 Z)^{-1} = \epsilon^{2(k+1)}\frac{\partial_z^k}{k!}(\hat{P} - z)^{-1}\bigg|_{z=\epsilon^2 Z}$$

we obtain from (2.44) the a priori estimates

$$||\langle r\rangle^{-M}(\epsilon^{-2}\hat{P} - Z)^{-1-k}\langle r\rangle^{-M}|| \lesssim \epsilon^{2(k+1)}|\mathrm{Re}(\epsilon^2 Z)|^{-k} \lesssim \epsilon^2. \tag{2.45}$$

On the other hand, it is not hard to check that (1.6) is equivalent to

$$||\langle \epsilon r\rangle^{-1-k}\partial_Z^k(\hat{P} - \epsilon^2 Z)^{-1}\langle \epsilon r\rangle^{-1-k}|| \lesssim \epsilon^{-2}, \tag{2.46}$$

with $k = N - 1$. We will show that (2.46) holds for all k between 0 and $N - 1$. By (2.45), we already know that

$$||\langle r\rangle^{-M}\partial_Z^k(\hat{P} - \epsilon^2 Z)^{-1}\langle r\rangle^{-M}|| \lesssim 1, \tag{2.47}$$

15

which are better estimates, but with the much stronger ϵ independent weight $\langle r \rangle^{-M}$. We shall combine (2.47) with a priori estimates on $(\epsilon^{-2}\hat{P}_T - Z)^{-1}$ to derive (1.6). Our first observation is that, by (2.16), we have slightly more precise estimates for $\hat{P}_T$, namely

$$||\langle \epsilon r \rangle^{-k-1} \partial_Z^k (\hat{P}_T - \epsilon^2 Z)^{-1} \langle \epsilon r \rangle^{-k-1}|| \lesssim \epsilon^{-2}, \tag{2.48}$$
$$||\langle \epsilon r \rangle^{-k-1} \partial_Z^k (\hat{P}_T - \epsilon^2 Z)^{-1} \langle \epsilon r \rangle^{-k} \langle r \rangle^{-1}|| \lesssim \epsilon^{-1}, \tag{2.49}$$

This follows from (2.16) since we are allowed to use one power of the homogeneous weight r^{-1} at most on each side. For instance, (2.49) is obtained by

$$\begin{aligned}
||\langle \epsilon r \rangle^{-k-1} \partial_Z^k (\hat{P}_T - \epsilon^2 Z)^{-1} \langle \epsilon r \rangle^{-k} \langle r \rangle^{-1}|| &\lesssim \epsilon^{2k} ||\langle \epsilon r \rangle^{-k-1} (\hat{P}_T - \epsilon^2 Z)^{-1-k} \langle \epsilon r \rangle^{-k} \langle r \rangle^{-1}|| \\
&\lesssim \epsilon^{-2} ||\langle \epsilon r \rangle^{-k-1} (\epsilon^{-2}\hat{P}_T - Z)^{-1-k} \langle \epsilon r \rangle^{-k} \langle r \rangle^{-1}|| \\
&\lesssim \epsilon^{-1} ||\langle \epsilon r \rangle^{-k-1} (\epsilon^{-2}\hat{P}_T - Z)^{-1-k} \langle \epsilon r \rangle^{-k} (\epsilon r)^{-1}|| \\
&\lesssim \epsilon^{-1},
\end{aligned}$$

where we pass from the second to the third line by replacing $\langle r \rangle^{-1}$ by r^{-1}, and get the final estimate by using (2.16). Using (2.48), (2.49) and Lemma 2.7, it follows that

$$\left|\left| \partial_Z^k \left(\langle \epsilon r \rangle^{-k-1} (\hat{P}_T - \epsilon^2 Z)^{-1} [\hat{P}_T, \psi] (\hat{P}_T - \epsilon^2 Z)^{-1} \langle \epsilon r \rangle^{-k-1} \right) \right|\right| \lesssim \epsilon^{-2}. \tag{2.50}$$

Also, using (2.47) and (2.49), the contribution of the last term of (2.28) is

$$\left|\left| \partial_Z^k \left(\langle \epsilon r \rangle^{-k-1} J_0 (\hat{P}_T - \epsilon^2 Z)^{-1} J[\hat{P}, \psi_2] (\hat{P} - \epsilon^2 Z)^{-1} [\hat{P}, \psi_1] J_0 (\hat{P}_T - \epsilon^2 Z)^{-1} J \langle \epsilon r \rangle^{-k-1} \right) \right|\right| \lesssim \epsilon^{-2}.$$

Therefore, this last estimate together with (2.48), (2.50) and (2.28) imply that for all k between 0 and $N-1$,

$$\left|\left| \partial_Z^k \left(\langle \epsilon r \rangle^{-k-1} \varphi (\hat{P} - \epsilon^2 Z)^{-1} \varphi \langle \epsilon r \rangle^{-k-1} \right) \right|\right| \lesssim \epsilon^{-2}.$$

We can then drop the cutoff φ by using (2.39). We have thus proved (2.46) which, in the special case $k = N - 1$, yields (1.6). $\qquad\square$

3 Mourre theory on a cone

In this section, we develop a Mourre theory for elliptic operators in divergence form on an exact cone, which will be crucial to prove the estimates (2.15), (2.16) and (2.17).

3.1 Operators in divergence form

We start by introducing a class of tensors which will be convenient to handle operators in divergence form on the cone $\mathcal{M}_0 = (0, +\infty) \times \mathcal{S}$ introduced in (2.14). Given a Riemannian metric g on $\mathcal{M}_0$, we denote the associated co-metric (*i.e.* the inner product on the fibers of $T^*\mathcal{M}_0$) by g^*. Then there exists a unique tensor $T^g \in C^\infty\big(\mathcal{M}_0, \mathrm{Hom}(T^*\mathcal{M}_0, T\mathcal{M}_0)\big)$, *i.e.* a section of the vector bundle $\mathrm{Hom}(T^*\mathcal{M}_0, T\mathcal{M}_0) \approx (T^*\mathcal{M}_0)^* \otimes T\mathcal{M}_0$, such that for all 1-forms ξ, η on $\mathcal{M}_0$,

$$\eta \cdot T^g \xi = g^*(\xi, \eta), \tag{3.1}$$

where $\cdot$ is the intrinsinc duality between a 1-form and a vector. In usual terms, T^g raises indices. It is automatically symmetric in the sense that we have

$$\xi \cdot T^g \eta = \eta \cdot T^g \xi, \tag{3.2}$$

16

for all ξ, η, and it allows to write the Laplace-Beltrami operator Δ_g as

$$\Delta_g u = \mathrm{div}_g\big(T^g du\big),$$

for all smooth functions u.

Using the isomorphisms $T\mathcal{M}_0 \approx T\mathbb{R}^+ \times T\mathcal{S}$ and $T^*\mathcal{M}_0 \approx T^*\mathbb{R}^+ \times T^*\mathcal{S}$, we can write any tensor $T \in C^\infty\big(\mathcal{M}_0, \mathrm{Hom}(T^*\mathcal{M}_0, T\mathcal{M}_0)\big)$ in matrix form as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

with $T_{11} \in C^\infty(\mathbb{R}^+ \times \mathcal{S})$, $T_{22} \in C^\infty\big(\mathbb{R}^+, C^\infty\big(\mathcal{S}, \mathrm{Hom}(T^*\mathcal{S}, T\mathcal{S})\big)\big)$ and

$$T_{12} \in C^\infty\big(\mathbb{R}^+, C^\infty\big(\mathcal{S}, \mathrm{Hom}(T^*\mathcal{S}, \mathbb{R})\big)\big), \qquad T_{21} \in C^\infty\big(\mathbb{R}^+, C^\infty\big(\mathcal{S}, \mathrm{Hom}(\mathbb{R}, T\mathcal{S})\big)\big).$$

For all $\omega \in \mathcal{S}$, any element of $u_\omega \in \mathrm{Hom}(\mathbb{R}, T_\omega \mathcal{S})$ has an intrinsic adjoint (or transpose) denoted by $u_\omega^\dagger \in \mathrm{Hom}(T_\omega^* \mathcal{S}, \mathbb{R})$ and defined by

$$u_\omega^\dagger(\xi) = \xi \cdot u_\omega, \qquad \xi \in T_\omega^* \mathcal{S}.$$

If $u \in C^\infty\big(\mathcal{S}, \mathrm{Hom}(\mathbb{R}, T\mathcal{S})\big)$ is a section, we define the section $u^\dagger \in C^\infty\big(\mathcal{S}, \mathrm{Hom}(T^*\mathcal{S}, \mathbb{R}))\big)$ in the obvious way $(u^\dagger(\omega) = u(\omega)^\dagger)$. It is then easy to check the following characterization of symmetric tensors, in the sense of (3.2),

$$T \text{ is symmetric} \iff \text{for each } r \in \mathbb{R}^+, \ \ T_{12}(r) = T_{21}(r)^\dagger \ \text{ and } T_{22}(r) \text{ is symmetric}$$

where, for each r, the tensors in the right-hand side belong respectively to $C^\infty\big(\mathcal{S}, \mathrm{Hom}(T^*\mathcal{S}, \mathbb{R})\big)$ and $C^\infty\big(\mathcal{S}, \mathrm{Hom}(T^*\mathcal{S}, T\mathcal{S})\big)$.

If we consider the conical metric

$$g_0 = dr^2 + r^2 h_0, \tag{3.3}$$

on $\mathcal{M}_0$ and use the above formalism, we then have

$$T^{g_0} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} I_{T\mathcal{S}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T^{h_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} I_{T^*\mathcal{S}} \end{pmatrix}.$$

We now introduce a class of perturbations of this tensor. For any $V \in C^\infty\big(\mathcal{S}, \mathrm{Hom}(\mathbb{R}, T\mathcal{S})\big)$, which can be identified with a vector field on $\mathcal{S}$, we set

$$\|V\|_{L^\infty(\mathcal{S})} := \sup_{\omega \in \mathcal{S}} |V_\omega|_{h_0, \omega}, \tag{3.4}$$

$|\cdot|_{h_0, \omega}$ denoting the norm on $T_\omega \mathcal{S}$ associated to h_0. If $W \in C^\infty\big(\mathcal{S}, \mathrm{Hom}(T^*\mathcal{S}, T\mathcal{S})\big)$, we set

$$\|W_\omega\|_{h_0} = \sup \frac{|\eta \cdot W_\omega(\xi)|}{|\xi|_{h_{0,\omega}^*} |\eta|_{h_{0,\omega}^*}}, \tag{3.5}$$

the sup being taken over all $\xi, \eta \in T_\omega^* \mathcal{S} \setminus \{0\}$, and $|\cdot|_{h_{0,\omega}^*}$ being the norm on $T_\omega^* \mathcal{S}$. Then, we set

$$\|W\|_{L^\infty(\mathcal{S})} := \sup_{\omega \in \mathcal{S}} \|W_\omega\|_{h_0}.$$

Definition 3.1 (admissible perturbation). *Fix an integer $N \geq 0$. A symmetric tensor $K \in C^\infty\big(\mathcal{M}_0, \mathrm{Hom}(T^*\mathcal{M}_0, T\mathcal{M}_0)\big)$ of the form*

$$K = \begin{pmatrix} K_{11}(r) & K_{21}(r)^\dagger \\ K_{21}(r) & K_{22}(r) \end{pmatrix}$$

is an N-admissible perturbation if for all $k \leq N + 1$,

$$|||K|||_k := \sup_{r>0} \big||(r\partial_r)^k K_{11}(r)\big||_{L^\infty(\mathcal{S})} + \big||(r\partial_r)^k K_{21}(r)\big||_{L^\infty(\mathcal{S})} + \big||(r\partial_r)^k K_{22}(r)\big||_{L^\infty(\mathcal{S})} < \infty.$$

We note that although we assume K to be smooth, we only require a control on the derivatives with respect to r. The interest of this class is to behave nicely under rescaling in r. For future purposes we record here the notation,

$$K^t(r) := K(e^t r), \tag{3.6}$$

defined for $t \in \mathbb{R}$ and any admissible perturbation K.

We now consider differential operators associated to such perturbations. In the sequel, we will denote the Riemannian measure associated to g_0 by,

$$d\mu = r^{n-1} dr d\mathrm{vol}_{h_0}.$$

For a given N-admissible perturbation K, we can consider the sesquilinear form

$$Q_K(u,v) = \int_{\mathbb{R}^+ \times \mathcal{S}} \begin{pmatrix} \partial_r \overline{u} \\ d_{\mathcal{S}} \overline{u}/r \end{pmatrix} \cdot \left\{ \begin{pmatrix} 1 & 0 \\ 0 & T^{h_0} \end{pmatrix} + \begin{pmatrix} K_{11} & K_{21}^\dagger \\ K_{21} & K_{22} \end{pmatrix} \right\} \begin{pmatrix} \partial_r v \\ d_{\mathcal{S}} v/r \end{pmatrix} d\mu, \tag{3.7}$$

first for $u, v \in C_0^\infty(\mathcal{M}_0)$. For simplicity, everywhere in the sequel we set

$$C_0^\infty = C_0^\infty(\mathcal{M}_0).$$

If we let $(.,.)_{L^2}$ be the inner product of $L^2(\mathcal{M}_0, d\mu)$, we see by integration by parts that

$$Q_K(u,v) = (P_K u, v)_{L^2},$$

with

$$P_K v = -\Delta_{g_0} v - \mathrm{div}_{g_0}\big(K^{\mathrm{sc}} dv\big), \tag{3.8}$$

where

$$K^{\mathrm{sc}} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} I_{T\mathcal{S}} \end{pmatrix} \begin{pmatrix} K_{11}(r) & K_{21}(r)^\dagger \\ K_{21}(r) & K_{22}(r) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} I_{T^*\mathcal{S}} \end{pmatrix}.$$

Notice that, by the symmetry assumption on K, the operator P_K is symmetric with respect to $d\mu$. All this will allow to define closed realizations of the differential operators P_K by means of sesquilinear forms (see Subsection 3.3). Before doing so, we need to introduce the relevant Sobolev norms, as well as useful intermediate results in the next subsection.

3.2 Sobolev spaces and dilations

Let us consider the operators

$$D_r = i^{-1}\partial_r, \qquad |D_\mathcal{S}| = (-\Delta_\mathcal{S})^{1/2}$$

that preserve C_0^∞. We let e^{itA} be the unitary group defined on L^2 by

$$\left(e^{itA}u\right)(r,\omega) = e^{tn/2}u(e^t r, \omega), \tag{3.9}$$

whose generator is the differential operator $A = \frac{n}{2i} - ir\partial_r$. Note that e^{itA} preserves C_0^∞.

Proposition 3.2. *On C_0^∞, the following identities hold*

$$e^{-itA}D_r e^{itA} \;=\; e^t D_r, \tag{3.10}$$

$$e^{-itA}r^s e^{itA} \;=\; e^{-ts}r^s, \qquad s \in \mathbb{R}. \tag{3.11}$$

If K is an N-admissible perturbation and K_t is given by (3.6), then

$$Q_K(e^{-itA}u, v) \;=\; e^{-2t}Q_{K^t}(u, e^{itA}v), \tag{3.12}$$

for all $u, v \in C_0^\infty$. In particular, if $N \geq 1$,

$$i\left(Q_K(u, Av) - Q_K(Au, v)\right) = \left(u, i[P_K, A]v\right)_{L^2} = 2Q_{K_1}(u, v) \tag{3.13}$$

with

$$K_1 = \left(1 - \frac{r\partial_r}{2}\right)K.$$

Proof. The formulas (3.10), (3.11), (3.12) are routine and (3.13) follows from

$$2Q_K(u, v) - \frac{d}{dt}Q_{K^t}(u, v)_{|t=0} = i\left(Q_K(u, Av) - Q_K(Au, v)\right),$$

by differentiating (3.12) in t and evaluating it at $t = 0$. $\qquad\square$

We next define the norm

$$||u||_{H_0^1} = \left(||u||_{L^2}^2 + ||D_r u||_{L^2}^2 + \left|\left|r^{-1}|D_\mathcal{S}|u\right|\right|_{L^2}^2\right)^{1/2},$$

first on C_0^∞ and then on $H_0^1 := H_0^1(\mathcal{M}_0)$ defined as

$$H_0^1 = \text{closure of } C_0^\infty \text{ for the norm } || \cdot ||_{H_0^1}. \tag{3.14}$$

The operators D_r and $r^{-1}|D_\mathcal{S}|$ have unique continuous extensions as linear maps from H_0^1 to L^2. It is also convenient to introduce the homogeneous Sobolev norm

$$||u||_{\dot{H}_0^1} := \left(||D_r u||_{L^2}^2 + \left|\left|r^{-1}|D_\mathcal{S}|u\right|\right|_{L^2}^2\right)^{1/2}.$$

We note that we will use this norm only for functions $u \in H_0^1$ so we do not need to define the space $\dot{H}_0^1$. We finally set

$$H^{-1} = \text{topological dual space to } H_0^1.$$

We denote the antilinear duality between $u \in H_0^1$ and $E \in H^{-1}$ by (E, u), with the convention that it is linear in u and conjugate linear in E. In other words, if $\langle ., . \rangle$ is the bilinear pairing between H_0^1 and its dual, we have set

$$(E, u) := \overline{\langle E, \bar{u} \rangle}. \tag{3.15}$$

To make this definition more symmetric, we also set

$$(u, E) := \overline{(E, \bar{u})}, \tag{3.16}$$

for all $E \in H^{-1}$ and $u \in H_0^1$. We have the following useful and elementary result which we record at least for notational purpose.

Proposition 3.3. *For all $f \in L^2$, there exists a unique $E_f \in H^{-1}$ such that, for all $u \in H_0^1$*

$$(f, u)_{L^2} = (E_f, u).$$

The map $f \mapsto E_f$ is linear, continuous and injective, thus realizes an embedding from L^2 into H^{-1}. We denote it by $\bar{I}$. Moreover, L^2 (i.e. $\bar{I} L^2$) is dense in H^{-1}.

We omit the proof which is standard. The interest of this proposition is to be able to consider L^2 as a (dense) subspace of H^{-1}. We shall use this convenient identification everywhere in the sequel. For instance, if E belongs to L^2, (3.15) and (3.16) correspond to L^2 inner products.

We next summarize several useful properties on H_0^1 and H^{-1} related to the group (3.9). We will be in particular interested in the properties of the resolvent of A, as an operator on L^2, when restricted to H_0^1. We recall that, if $\alpha > 0$ is a real parameter and $\zeta \in \mathbb{C} \setminus \mathbb{R}$, we have

$$(\alpha A - \zeta)^{-1} = \frac{1}{i} \int_0^{\pm\infty} e^{-it\zeta} e^{it\alpha A} dt, \qquad \mp \mathrm{Im}(\zeta) > 0. \tag{3.17}$$

Proposition 3.4.   *1. H_0^1 is stable by multiplication by smooth functions of r which are bounded together with their derivatives.*

2. *H_0^1 is stable by e^{itA}, e^{itA} is strongly continuous on H_0^1 and*

$$\|e^{itA} u\|_{H_0^1} \lesssim (1 + e^t) \|u\|_{H_0^1}, \qquad t \in \mathbb{R}, \ u \in H_0^1.$$

Furthermore, if $0 < \alpha < |\mathrm{Im}(\zeta)|$, H_0^1 is stable by $(\alpha A - \zeta)^{-1}$.

3. *The group e^{itA} extends from L^2 to an $H^{-1} \to H^{-1}$ strongly continuous group. Its adjoint is e^{-itA} (acting on H_0^1). Furthermore, if $0 < \alpha < |\mathrm{Im}(\zeta)|$, $(\alpha A - \zeta)^{-1}$ extends from L^2 to a bounded $H^{-1} \to H^{-1}$ operator, whose adjoint (acting on H_0^1) is $(\alpha A - \bar{\zeta})^{-1}$.*

4. *Fix $0 < \alpha < |\mathrm{Im}(\zeta)|$. For all $t \in \mathbb{R}$,*

$$e^{itA}(\alpha A - \zeta)^{-1} = (\alpha A - \zeta)^{-1} + \frac{i}{\alpha} \int_0^t e^{isA} \left(I + \zeta (\alpha A - \zeta)^{-1} \right) ds$$

as an equality between operators on H_0^1 (resp. H^{-1}). Here the integral converges in the strong sense. In particular $e^{itA}(\alpha A - \zeta)^{-1}$ is strongly differentiable with respect to t on H_0^1 and H^{-1}.

5. *Fix $\alpha \in (0,1)$ and an integer $N \geq 1$. We have the interpolation estimate*

$$||A(\alpha A + i)^{-N}u||_{H_0^1} \leq C||(\alpha A + i)^{-N}u||_{H_0^1}^{1-\frac{1}{N}}||u||_{H_0^1}^{\frac{1}{N}},$$

for all $u \in H_0^1$.

Proof. The item 1 is straightforward by density of C_0^∞. The estimate of the item 2 holds true on C_0^∞ by (3.10) and (3.11) (with $s = -1$) hence on H_0^1 by density. Checking the strong continuity is a routine. The boundedness of $(\alpha A - \zeta)^{-1}$ on H_0^1 is then a consequence of (3.17). The item 3 follows from Proposition 3.3, the item 2 of the present proposition and the formula (3.17) combined with routine duality arguments. The identity of the item 4 holds clearly on L^2 since $e^{itA}(\alpha A - \zeta)^{-1}$ is strongly C^1 in t (note that $i\alpha^{-1}e^{isA}\big(I + \zeta(\alpha A - \zeta)^{-1}\big) = e^{isA}iA(\alpha A - \zeta)^{-1}$). That the integral converges in the strong sense on H_0^1 (resp. H^{-1}) follows from the item 2 (resp. item 3). To prove the item 5, we recall first that

$$||\langle A \rangle (\alpha A + i)^{-N}f||_{L^2} \leq C||(\alpha A + i)^{-N}f||_{L^2}^{1-\frac{1}{N}}||f||_{L^2}^{\frac{1}{N}}, \tag{3.18}$$

using the spectral theorem and the Hadamard three lines theorem. In particular, we have

$$
\begin{aligned}
||A(\alpha A + i)^{-N}u||_{L^2} &\leq C||(\alpha A + i)^{-N}u||_{L^2}^{1-\frac{1}{N}}||u||_{L^2}^{\frac{1}{N}} \\
&\leq C||(\alpha A + i)^{-N}u||_{H_0^1}^{1-\frac{1}{N}}||u||_{H_0^1}^{\frac{1}{N}}.
\end{aligned}
$$

It remains to estimate $\big||LA(\alpha A + i)^{-N}u\big||_{L^2}$, when $L = D_r$ or $r^{-1}|D_{\mathcal{S}}|$. We observe that, for $k = 0, 1$,

$$LA^k(\alpha A + i)^{-N} = (A^k - ki)\big(\alpha A + i(1-\alpha)\big)^{-N}L,$$

which follows easily from (3.10), (3.11) and (3.17) (see also (3.20) below). Thus, using (3.18),

$$
\begin{aligned}
\big||LA(\alpha A + i)^{-N}u\big||_{L^2} &\leq C||\big(\alpha A + i(1-\alpha)\big)^{-N}Lu||_{L^2}^{1-\frac{1}{N}}||Lu||_{L^2}^{\frac{1}{N}} \\
&\leq C\big||L(\alpha A + i)^{-N}u\big||_{L^2}^{1-\frac{1}{N}}||Lu||_{L^2}^{\frac{1}{N}} \\
&\leq C||\big(\alpha A + i\big)^{-N}u||_{H_0^1}^{1-\frac{1}{N}}||u||_{H_0^1}^{\frac{1}{N}}.
\end{aligned}
$$

The result follows. $\qquad \square$

We next record the basic Hardy inequality. Recall that we assume $n \geq 3$.

Proposition 3.5. *For all $u \in C_0^\infty$, we have*

$$||r^{-1}u||_{L^2} \leq \frac{2}{n-2}||\partial_r u||_{L^2}. \tag{3.19}$$

As a consequence, the multiplication by r^{-s}, $s \in [0,1]$, is bounded from H_0^1 to L^2. Furthermore,

$$(\alpha A - \zeta)^{-1}r^{-s}u = r^{-s}(\alpha A - \zeta + i\alpha s)^{-1}u, \qquad (\alpha A - \zeta)^{-1}\partial_r u = \partial_r(\alpha A - \zeta + i\alpha)^{-1}u, \tag{3.20}$$

for all $s \in [0,1]$, $u \in H_0^1$, $\alpha > 0$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$ such that $|\mathrm{Im}(\zeta)| > \alpha$.

Proof. The inequality (3.19) is a direct consequence of the same one dimensional inequality (on $L^2(\mathbb{R}^+, r^{n-1}dr)$) which is standard. The boundedness of r^{-1} on H_0^1, hence of r^{-s} by interpolation, is then straighforward. The identities in (3.20) follow easily from (3.10), (3.11) and (3.17). $\qquad\square$

We will need later the following proposition.

Proposition 3.6. *The space $\langle r\rangle^{-1}H_0^1$ is contained in $\mathrm{Dom}(A)$. Furthermore, for any symbol σ of order -1 (i.e. $|\sigma^{(k)}(r)| \lesssim \langle r\rangle^{-1-k}$), there exists $C > 0$ such that*

$$\|A(\sigma(r)v)\|_{L^2} \leq C\|v\|_{H_0^1},$$

for all $v \in H_0^1$.

Proof. Let $u = \sigma(r)v$ with $v \in H_0^1$. Let $v_k \in C_0^\infty$ be a sequence approaching v in H_0^1. Then, for all $w \in \mathrm{Dom}(A)$, we have

$$(Aw, u)_{L^2} = \lim_k (Aw, \sigma(r)v_k)_{L^2} = \lim_{k\to\infty}(w, A(\sigma(r)v_k))_{L^2} = (w, Bv)_{L^2},$$

where B is the (closure to H_0^1 of the) differential operator

$$r\sigma(r)D_r + \frac{n}{2i}\sigma(r) + \frac{1}{i}r\sigma'(r),$$

which is bounded on H_0^1. In particular

$$\|Au\|_{L^2} = \|Bv\|_{L^2} \leq C\big(\|D_r v\|_{L^2} + \|v\|_{L^2}\big) \leq C\|v\|_{H_0^1},$$

and this completes the proof. $\qquad\square$

We finally record simple weighted estimates. When W is a function, we set

$$\|W\nabla_{g_0}u\|_{L^2}^2 = \|W\partial_r u\|_{L^2}^2 + \big\|Wr^{-1}|D_\mathcal{S}|u\big\|_{L^2}^2.$$

Proposition 3.7. *There exists $C > 0$ such that for all $u,v \in C_0^\infty$, all non vanishing smooth functions $W : (0, +\infty) \to \mathbb{C}$ of r, and all admissible perturbations K*

$$\big|\big(u, \mathrm{div}_{g_0}(K^{\mathrm{sc}}dv)\big)\big| \leq C|||K|||_0 \|W(r)\nabla_{g_0}u\|_{L^2}\|W(r)^{-1}\nabla_{g_0}v\|_{L^2}. \tag{3.21}$$

Proof. We start by writing

$$\big(u, \mathrm{div}_{g_0}(K^{\mathrm{sc}}dv)\big) = \int_{\mathbb{R}^+\times\mathcal{S}} \begin{pmatrix} W\partial_r\overline{u} \\ Wd_\mathcal{S}\overline{u}/r \end{pmatrix} \cdot \begin{pmatrix} K_{11} & K_{21}^\dagger \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} W^{-1}\partial_r v \\ W^{-1}d_\mathcal{S}v/r \end{pmatrix} d\mu,$$

since the multiplication by W commutes with K. Then, using (3.4), (3.5) and the fact that, if $\xi \in T_\omega^*\mathcal{S}, V \in T_\omega\mathcal{S}$, $|\xi \cdot V| \leq |\xi|_{h_{0,\omega}^*}|V|_{h_{0,\omega}}$, one sees that $\big|\big(u, \mathrm{div}_{g_0}(K^{\mathrm{sc}}dv)\big)\big|$ is not greater than

$$|||K|||_0 \int_{(0,+\infty)\times\mathcal{S}} \big(|W\partial_r\overline{u}| + |Wr^{-1}d_\mathcal{S}\overline{u}|_{h_{0,\omega}^*}\big)\big(|W^{-1}\partial_r v| + |W^{-1}r^{-1}d_\mathcal{S}v|_{h_{0,\omega}^*}\big)d\mu.$$

By using the Cauchy-Schwarz inequality combined with the fact that

$$\int_\mathcal{S} |W^{-1}r^{-1}d_\mathcal{S}v|_{h_{0,\omega}^*}^2 d\mathrm{vol}_{h_0} = r^{-2}W^{-2}\int_\mathcal{S} |d_\mathcal{S}v|_{h_{0,\omega}^*}^2 d\mathrm{vol}_{h_0} = r^{-2}W^{-2}\big(v, -\Delta_\mathcal{S}v\big)_{L^2(\mathcal{S})}$$

$$= \big\|W^{-1}(r)r^{-1}|D_\mathcal{S}|v(r)\big\|_{L^2(\mathcal{S})}^2$$

the conclusion follows easily. $\qquad\square$

3.3 Jensen-Mourre-Perry estimates

In this subsection, we define closed realizations of operators of the form P_K (see (3.8)) and prove resolvent estimates thereon. Although we follow closely the Jensen-Mourre-Perry techniques [21], the proofs of resolvent estimates will be self contained. We have to review the proof for we will need to control most estimates with respect to the perturbation K and also since we need to prove $H^{-1} \to H_0^1$ estimates.

Starting from (3.7), we observe first that the sesquilinear form Q_K satisfies

$$|Q_K(u,v)| \leq C||u||_{H_0^1}||v||_{H_0^1},$$

for all $u, v \in C_0^\infty$ hence has a unique continuous extension to $H_0^1 \times H_0^1$. Everywhere in the sequel, we denote this extension by $\bar{Q}_K$.

Proposition 3.8. *Let K be a N-admissible perturbation tensor.*

- *The operator $P_K : C_0^\infty \to C_0^\infty$ has a unique linear continuous extension $\bar{P}_K : H_0^1 \to H^{-1}$ and this extension satisfies*

$$(\bar{P}_K u, v) = \bar{Q}_K(u,v), \qquad u, v \in H_0^1.$$

Furthermore, there exists C independent of K such that

$$\left(1 - C|||K|||_0\right)||u||_{\dot{H}_0^1}^2 \leq (\bar{P}_K u, u) \leq \left(1 + C|||K|||_0\right)||u||_{\dot{H}_0^1}^2, \tag{3.22}$$

for all $u \in H_0^1$.

- *If in addition $||K||_{L^\infty}$ is small enough, then one defines a self-adjoint operator $\hat{P}_K$ on L^2 by*

  1. $\mathrm{Dom}(\hat{P}_K) = \{u \in H_0^1 \mid \text{there exists } C_u > 0 \; |\bar{Q}_K(u,v)| \leq C_u||v||_{L^2}, \text{ for all } v \in H_0^1\}$.

  2. If $u \in \mathrm{Dom}(\hat{P}_K)$, $\hat{P}_K u$ is defined as the unique element of L^2 such that

  $$(\hat{P}_K u, v)_{L^2} = \bar{Q}_K(u,v), \qquad v \in H_0^1.$$

  3. The operator $\hat{P}_K$ is nonnegative and has the property that

  $$\mathrm{Dom}(\hat{P}_K^{1/2}) = H_0^1, \qquad ||u||_{H_0^1}/2 \leq ||(1 + \hat{P}_K)^{1/2}u||_{L^2} \leq 2||u||_{H_0^1}.$$

The proof is standard hence is omitted. We simply note that (3.22) follows straightforwardly by density from (3.8) and Proposition 3.7 (with $W \equiv 1$), once noticed that $(\bar{P}_0 u, u) = ||u||_{\dot{H}_0^1}^2$.

We now study the resolvent of $\hat{P}_K$. Let us introduce the notation

$$\Sigma(\varepsilon_0, \delta_0) = \{(\delta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \mid 0 < |\delta| \leq \delta_0, \; |\varepsilon| \leq \varepsilon_0, \; \varepsilon\delta \geq 0\}.$$

We also set

$$P_K(\varepsilon) = P_K + \sum_{k=1}^{N} \frac{\varepsilon^k}{k!}\mathrm{ad}_A^k P_K, \tag{3.23}$$

where, as usual, $\mathrm{ad}_A P = [P, A] = PA - AP$ and $\mathrm{ad}_A^{k+1} P = [\mathrm{ad}_A^k P, A]$. This definition makes sense as an equality between operators on C_0^∞. Its interest is the following easily verified property,

$$[A, P_K(\varepsilon)] = -\frac{\partial}{\partial \varepsilon} P_K(\varepsilon) - \frac{\varepsilon^N}{N!} \mathrm{ad}_A^{N+1} P_K. \tag{3.24}$$

Using (3.13), we have the formula

$$\mathrm{ad}_A^k P_K = \frac{2^k}{i^k} P_{K_k}, \qquad K_k := \left(1 - \frac{r\partial_r}{2}\right)^k K. \tag{3.25}$$

We can thus rewrite (3.23) as

$$P_K(\varepsilon) = \left(P_K + \sum_{2 \leq 2j \leq N} (-1)^j \frac{(2\varepsilon)^{2j}}{(2j)!} P_{K_{2j}}\right) - 2i\varepsilon \left(P_{K_1} + \sum_{3 \leq 2j+1 \leq N} (-1)^j \frac{(2\varepsilon)^{2j}}{(2j+1)!} P_{K_{2j+1}}\right) \tag{3.26}$$

where both brackets are symmetric on C_0^∞ with respect to $d\mu$. The operator $P_K(\varepsilon)$ can be extended to an $H_0^1 \to H^{-1}$ operator by Proposition 3.8 (item 1), ie

$$\bar{P}_K(\varepsilon) = \bar{P}_K + \sum_{k=1}^N \frac{(-2i\varepsilon)^k}{k!} \bar{P}_{K_k}. \tag{3.27}$$

Similarly, the identity (3.26) can accordingly be extended as an equality between $H_0^1 \to H^{-1}$ operators.

Proposition 3.9. *Fix $N \geq 1$ and $C > 0$. There exist $\varrho > 0$ and $\delta_0, \varepsilon_0 > 0$ such that, for all N-admissible perturbations satisfying*

$$|||K|||_0 + |||K|||_1 \leq \varrho, \qquad |||K|||_2 + \cdots + |||K|||_N \leq C$$

and all $(\delta, \varepsilon) \in \Sigma(\varepsilon_0, \delta_0)$, we have the following results.

1. *The operator*
$$\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I} : H_0^1 \to H^{-1}$$
is a bounded isomorphism (see Proposition 3.3 for $\bar{I}$). Its inverse
$$R(\varepsilon, \delta) := \left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)^{-1}$$
satisfies
$$\|R(\varepsilon, \delta)\|_{H^{-1} \to H_0^1} \leq \min\left(\frac{10}{|\varepsilon|}, \frac{8}{|\delta|}\right). \tag{3.28}$$

2. *When $\varepsilon = 0$ and $f \in L^2$,*
$$\left(\bar{P}_K(0) - (1 + i\delta)\bar{I}\right)^{-1} \bar{I} f = (\hat{P}_K - 1 - i\delta)^{-1} f.$$

3. *For all $(\varepsilon, \delta) \in \Sigma(\varepsilon_0, \delta_0)$,*
$$R(\varepsilon, \delta)^* = R(-\varepsilon, -\delta).$$

24

Proof. By symmetry of K, hence of all K_k, $(\bar{P}_{K_k}u, u)$ is real for all $u \in H_0^1$. Using that $|||K|||_0$ is small enough, it follows from (3.22) and (3.26) that

$$\left(\frac{3}{4} - \gamma\varepsilon^2\right)||u||_{\dot{H}_0^1}^2 - ||u||_{L^2}^2 \leq \mathrm{Re}\left(\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right) \leq \left(\frac{5}{4} + \gamma\varepsilon^2\right)||u||_{\dot{H}_0^1}^2 - ||u||_{L^2}^2,$$

for some constant $\gamma > 0$ independent of K as long as $|||K|||_0 + \cdots + |||K|||_N$ remains bounded. Similarly, if $|||K_1|||_0$ is small enough, we have

$$\mathrm{sgn}(\varepsilon)\mathrm{Im}\left(\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right) \geq |\varepsilon|(1 - \gamma\varepsilon^2)||u||_{\dot{H}_0^1}^2 + |\delta|\left|\left|u\right|\right|_{L^2}^2$$

since the first term of $\bar{P}_K(\varepsilon)$ contributing in the imaginary part is $2\varepsilon(\bar{P}_{K_1}u, u)$ and $\mathrm{sgn}(\varepsilon)\delta = |\delta|$ by definition of $\Sigma(\varepsilon_0, \delta_0)$. If $|\varepsilon| \leq \varepsilon_0$ is small enough, we obtain

$$\mathrm{Re}\left(\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right) \leq \frac{3}{2}||u||_{\dot{H}_0^1}^2 - ||u||_{L^2}^2, \tag{3.29}$$

$$\mathrm{Re}\left(\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right) \geq \frac{1}{2}||u||_{\dot{H}_0^1}^2 - ||u||_{L^2}^2, \tag{3.30}$$

$$\mathrm{sgn}(\varepsilon)\mathrm{Im}\left(\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right) \geq \frac{|\varepsilon|}{2}||u||_{\dot{H}_0^1}^2 + |\delta|||u||_{L^2}^2. \tag{3.31}$$

We wish to get lower bounds in term of the H_0^1 norm. Let $\theta = \theta(\varepsilon, \delta)$ be such that $\cos(\theta) = -|\varepsilon|/4$ and $\mathrm{sgn}(\varepsilon)\sin(\theta) = (1 - \varepsilon^2/16)^{1/2}$. We then have the coercivity estimate

$$\mathrm{Re}\left(e^{i\theta}\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right) \geq \frac{|\varepsilon|}{10}||u||_{H_0^1}^2, \tag{3.32}$$

where we now have the H_0^1 norm in the right-hand side. Indeed, using that the duality $(.,.)$ is antilinear in the first factor, the left-hand side of (3.32) reads

$$\cos\theta\,\mathrm{Re}\left(\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right) + \sin\theta\,\mathrm{Im}\left(\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right).$$

Multiplying (3.29) by $\cos\theta$ (which is negative) and (3.31) by $\mathrm{sgn}(\varepsilon)\sin(\theta)$ allows to bound from below this expression by

$$\frac{|\varepsilon|}{4}||u||_{L^2}^2 + |\varepsilon|\left(\frac{(1 - \varepsilon^2/16)^{1/2}}{2} - \frac{3}{8}\right)||u||_{\dot{H}_0^1}^2 \geq \frac{|\varepsilon|}{10}||u||_{H_0^1}^2$$

if ε is small enough. It follows from (3.32) that the operator $\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}$ is injective and has a closed range (this would also follow from (3.31)). Using the usual Lax-Milgram argument, the estimate (3.32) implies that $e^{i\theta}\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)$ and hence $\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}$ are isomorphisms between H_0^1 and H^{-1}, which proves the existence of $R(\varepsilon, \delta)$. To complete the proof of the first item, it remains to prove (3.28). The bound $10/|\varepsilon|$ follows from (3.32). We prove the bound $8/|\delta|$ in a similar fashion as follows. We choose $\beta \in \mathbb{R}$ such that $\mathrm{sgn}(\varepsilon)\sin(\beta) = (1 - \delta^2/4)^{1/2}$ and $\cos(\beta) = |\delta|/2$. Then, it is not hard to see as above that using (3.30) and (3.31) we have

$$\mathrm{Re}\left(e^{i\beta}\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)u, u\right) \geq \frac{|\delta|}{8}||u||_{H_0^1}^2,$$

provided that δ is small enough. This last estimate and (3.32) imply (3.28).

To prove the second item, we observe that $\left(\bar{P}_K(0) - (1 + i\delta)\bar{I}\right)^{-1}\bar{I}f$ is the unique $u \in H_0^1$ such that

$$\bar{Q}_K(u, v) - \left((1 + i\delta)u, v\right)_{L^2} = (f, v)_{L^2},$$

25

for all $v \in H_0^1$. This implies precisely that u belongs to $\mathrm{Dom}(\hat{P}_K)$ and that

$$\left(\hat{P}_K u - (1 + i\delta)u, v\right)_{L^2} = (f, v)_{L^2},$$

which shows that $u = (\hat{P}_K - 1 - i\delta)^{-1} f$.

To prove the third item, we use the definition (3.16) to write

$$\begin{aligned}
(f, R(-\varepsilon, -\delta)g) &= \left((\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I})R(\varepsilon, \delta)f, R(-\varepsilon, -\delta)g\right) \\
&= \left(R(\varepsilon, \delta)f, (\bar{P}_K(-\varepsilon) - (1 - i\delta)\bar{I})R(-\varepsilon, -\delta)g\right) \\
&= (R(\varepsilon, \delta)f, g),
\end{aligned}$$

since, by the definition of $\bar{P}_K(\pm\varepsilon)$, it is clear that $(\bar{P}_K(\varepsilon)u, v) = (u, \bar{P}_K(-\varepsilon)v)$ for all $u, v \in H_0^1$ (see for instance (3.26)). The result follows. $\qquad\square$

We next recall a classical lemma (see [21]) on differential inequalities.

Lemma 3.10. *Let $C > 0$, $\varepsilon_0 > 0$, $\gamma > 0$ and $0 \le \beta < 1$ be fixed constants. Then there exists $C' > 0$ such that, for all differentiable maps $F : (0, \varepsilon_0) \to \mathcal{L}(H^{-1}, H_0^1)$ satisfying*

$$\left\|\frac{d}{d\varepsilon}F(\varepsilon)\right\|_{H^{-1} \to H_0^1} \le C(\|F(\varepsilon)\|_{H^{-1} \to H_0^1} + 1)\varepsilon^{-\beta}, \tag{3.33}$$

$$\|F(\varepsilon)\|_{H^{-1} \to H_0^1} \le C\varepsilon^{-\gamma}, \tag{3.34}$$

for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\|F(\varepsilon)\|_{H^{-1} \to H_0^1} \le C',$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. Consider the sequence $(\gamma_k)_{k \in \mathbb{N}}$ defined by

$$\gamma_0 = \gamma, \qquad \gamma_{k+1} = \begin{cases} \gamma_k + \beta - 1 & \text{if } \gamma_k + \beta > 1, \\ 0 & \text{if } \gamma_k + \beta \le 1. \end{cases}$$

It is easy to check that $\gamma_k = 0$ for all k large enough. The lemma then follows from the observation that for all $k \ge 0$ there exists $C_k > 0$ such that, for all F satisfying (3.33) and (3.34),

$$\|F(\varepsilon)\|_{H^{-1} \to H_0^1} \le C_k \varepsilon^{-\gamma_k}, \qquad \varepsilon \in (0, \varepsilon_0),$$

which is obtained by an elementary induction. $\qquad\square$

In the following proposition K is a fixed $N + 1$-admissible perturbation. For simplicity, when $k \ge 1$ is an integer, we will use $R(\varepsilon, \delta)^k$ as the obvious short hand for $(R(\varepsilon, \delta)\bar{I})^{k-1}R(\varepsilon, \delta)$.

Proposition 3.11. *For all $(\varepsilon, \delta) \in \Sigma(\varepsilon_0, \delta_0)$, the function $t \mapsto e^{itA}R(\varepsilon, \delta)e^{-itA}$ can be weakly differentiated at $t = 0$ and*

$$\frac{1}{i}\frac{d}{dt}\left(e^{itA}R(\varepsilon, \delta)e^{-itA}\right)\Big|_{t=0} = -\frac{\partial}{\partial\varepsilon}R(\varepsilon, \delta) + R(\varepsilon, \delta)\left((-2i)^{N+1}\frac{\varepsilon^N}{N!}\bar{P}_{K_{N+1}}\right)R(\varepsilon, \delta). \tag{3.35}$$

Similarly, $\frac{1}{i}\frac{d}{dt}\left(e^{itA}R(\varepsilon, \delta)^N e^{-itA}\right)\big|_{t=0}$, defined in the weak sense, reads

$$-\sum_{k=0}^{N-1} R(\varepsilon, \delta)^k \left(\frac{\partial}{\partial\varepsilon}R(\varepsilon, \delta) - R(\varepsilon, \delta)\left((-2i)^{N+1}\frac{\varepsilon^N}{N!}\bar{P}_{K_{N+1}}\right)R(\varepsilon, \delta)\right) R(\varepsilon, \delta)^{N-1-k}.$$

Recall that the action of e^{itA} on H_0^1 and H^{-1} respectively is described in Proposition 3.4. We also point out that the derivative $\partial R(\varepsilon, \delta)/\partial \varepsilon$ is well defined, in the $\mathcal{L}(H^{-1}, H_0^1)$ topology, since $R(\varepsilon, \delta)$ is the inverse of $\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}$ which depends polynomially on ε in $\mathcal{L}(H_0^1, H^{-1})$ (see (3.27)).

Proof. Since e^{itA} is an isomorphism on H_0^1 and e^{-itA} an isomorphism on H^{-1}, $e^{itA}R(\varepsilon, \delta)e^{-itA}$ is the inverse of

$$e^{itA}\left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I}\right)e^{-itA} = e^{-2t}\bar{P}_{K^t}(\varepsilon) - (1 + i\delta)\bar{I},$$

where the equality with the right-hand side follows from (3.12) and the first item of Proposition 3.8. We claim that this operator can be weakly differentiated in t. Indeed, if $u, v \in H_0^1$, we have

$$\left(e^{-2t}\bar{P}_{K^t}(\varepsilon)u, v\right) = \left(\bar{P}_K(\varepsilon)u, v\right) - 2\int_0^t \left(\bar{P}_{K_1^s}(\varepsilon)u, v\right)ds. \tag{3.36}$$

This is easily seen first with $u, v \in C_0^\infty$ by using (3.12), and then on H_0^1 by density. By writing $(\bar{P}_{K_1^s}(\varepsilon)u, v) = \bar{Q}_{K_1^s(\varepsilon)}(u, v)$, we see that this quantity depends continously on s, by the strong continuity on L^2 of $s \mapsto (r\partial_r)^k K(e^s r)$, for $k \leq N + 1$. This implies on one hand that $(e^{-2t}\widetilde{P}_{K^t}(\varepsilon)u, v)$ can be differentiated at $t = 0$ and on the other hand that

$$\left\|e^{-2t}\bar{P}_{K^t}(\varepsilon) - \bar{P}_K(\varepsilon)\right\|_{H_0^1 \to H^{-1}} \leq C|t|$$

hence that

$$\left\|\left(e^{-2t}\bar{P}_{K^t}(\varepsilon) - (1 + i\delta)\bar{I}\right)^{-1} - R(\varepsilon, \delta)\right\|_{H^{-1} \to H_0^1} \leq C_{\varepsilon, \delta}|t|. \tag{3.37}$$

Using the resolvent identity, this shows that

$$\left(e^{-2t}\bar{P}_{K^t}(\varepsilon) - (1 + i\delta)\bar{I}\right)^{-1} - R(\varepsilon, \delta) = -R(\varepsilon, \delta)\left(e^{-2t}\bar{P}_{K^t}(\varepsilon) - \bar{P}_K(\varepsilon)\right)R(\varepsilon, \delta) + O(t^2),$$

where the $O(t^2)$ holds in operator norm. This justifies the weak differentiability and the fact that

$$\frac{d}{dt}\left(e^{itA}R(\varepsilon, \delta)e^{-itA}\right)\Big|_{t=0} = -R(\varepsilon, \delta)\frac{d}{dt}\left(e^{itA}\bar{P}_K(\varepsilon)e^{-itA}\right)_{|t=0}R(\varepsilon, \delta).$$

To compute the derivative, we use on one hand (3.36) to see that

$$\frac{d}{dt}\left(e^{itA}\bar{P}_K(\varepsilon)e^{-itA}\right)_{|t=0} = -2\bar{P}_{K_1}(\varepsilon).$$

On the other hand, using (3.27), we can compute $\partial \bar{P}_K(\varepsilon)/\partial \varepsilon$ directly and check that

$$-2\bar{P}_{K_1}(\varepsilon) = -i\frac{\partial}{\partial \varepsilon}\bar{P}_K(\varepsilon) - i(-2i)^{N+1}\frac{\varepsilon^N}{N!}\bar{P}_{K_{N+1}},$$

from which (3.35) follows. In a similar fashion, we have

$$e^{itA}R(\varepsilon, \delta)^N e^{-itA} = R(\varepsilon, \delta)^N + \sum_{k=0}^{N-1} R(\varepsilon, \delta)^k \left(e^{itA}R(\varepsilon, \delta)^N e^{-itA} - R(\varepsilon, \delta)\right)R(\varepsilon, \delta)^{N-1-k} + O(t^2).$$

This allows to justify the weak differentiability and obtain the second assertion. $\quad\square$

The next lemma is a convenient version of the standard quadratic estimates of [29].

Lemma 3.12. *Let $B : H^{-1} \to H^{-1}$ be a bounded linear operator. Then for all $(\varepsilon, \delta) \in \Sigma(\varepsilon_0, \delta_0)$ and all K as in Proposition 3.9,*

$$\left|\left| R(\varepsilon, \delta) B \right|\right|_{H^{-1} \to H_0^1} \leq \frac{4}{|\varepsilon|^{1/2}} \left|\left| B^* R(\varepsilon, \delta) B \right|\right|_{H^{-1} \to H_0^1}^{1/2}.$$

Proof. Using (3.32), we have

$$\operatorname{Re}\left(e^{i\theta} \left(\bar{P}_K(\varepsilon) - (1 + i\delta)\bar{I} \right) R(\varepsilon, \delta) B f, R(\varepsilon, \delta) B f \right) \geq \frac{|\varepsilon|}{10} \| R(\varepsilon, \delta) B f \|_{H_0^1}^2,$$

for all $f \in H^{-1}$. The left-hand side reads

$$\operatorname{Re}\left(e^{i\theta} B f, R(\varepsilon, \delta) B f \right) = \operatorname{Re}\left(e^{i\theta} f, B^* R(\varepsilon, \delta) B f \right).$$

By bounding $1/10$ from below by $1/16$, this implies that

$$\| f \|_{H^{-1}}^2 \| B^* R(\varepsilon, \delta) B \|_{H^{-1} \to H_0^1} \geq \frac{|\varepsilon|}{16} \| R(\varepsilon, \delta) B f \|_{H_0^1}^2,$$

from which the result follows. $\qquad \square$

Proposition 3.13. *Fix $N \geq 1$, $M \geq 1$ and $0 < \alpha < 1$. There exist $C > 0$ large enough and $\varrho, \varepsilon_0, \delta_0 > 0$ small enough such that, for all $(\varepsilon, \delta) \in \Sigma(\varepsilon_0, \delta_0)$ and all $N + 1$-admissible perturbation K satisfying*

$$|||K|||_0 + |||K|||_1 < \varrho, \qquad |||K|||_2 + \cdots + |||K|||_{N+1} \leq M,$$

we have

$$\left|\left| (\alpha A + i)^{-N} R(\varepsilon, \delta)^N (\alpha A - i)^{-N} \right|\right|_{H^{-1} \to H_0^1} \leq C.$$

Proof. Let us set first $F^1(\varepsilon) := (\alpha A - i)^{-1} R(\varepsilon, \delta)(\alpha A + i)^{-1}$. By Lemma 3.12 and the item 3 of Proposition 3.4 with $\zeta = -i$, we have

$$\| R(\varepsilon, \delta)(\alpha A + i)^{-1} \|_{H^{-1} \to H_0^1} \leq \frac{4}{|\varepsilon|^{1/2}} \| F^1(\varepsilon) \|_{H^{-1} \to H_0^1}^{1/2}. \tag{3.38}$$

By taking the adjoint and using the third items of Propositions 3.4 and 3.9, the same estimate holds for $\| (\alpha A - i)^{-1} R(\varepsilon, \delta) \|_{H^{-1} \to H_0^1}$. On the other hand, by using the item 4 of Proposition 3.4 and Proposition 3.11, we obtain

$$\begin{aligned}
\frac{d}{d\varepsilon} F^1(\varepsilon) &= iA(\alpha A - i)^{-1} R(\varepsilon, \delta)(\alpha A + i)^{-1} - i(\alpha A - i)^{-1} R(\varepsilon, \delta) A(\alpha A + i)^{-1} + \\
&\quad (\alpha A - i)^{-1} R(\varepsilon, \delta) \left((-2i)^{N+1} \frac{\varepsilon^N}{N!} \bar{P}_{K_{N+1}} \right) R(\varepsilon, \delta)(\alpha A + i)^{-1},
\end{aligned}$$

as an equality between $H^{-1} \to H_0^1$ operators. Therefore, using (3.38) and the bound (3.28) to handle the last term above, we get

$$\left|\left| \frac{d}{d\varepsilon} F^1(\varepsilon) \right|\right|_{H^{-1} \to H_0^1} \leq C_\alpha |\varepsilon|^{-1/2} \| F^1(\varepsilon) \|_{H^{-1} \to H_0^1}^{1/2}.$$

By (3.28) and Lemma 3.10, we obtain that $||F^1(\varepsilon)||_{H^{-1}\to H^1_0} \leq C$. In particular, the right-hand side of (3.38) is at most of order $|\varepsilon|^{-1/2}$. If we now set $F^N(\varepsilon) := (\alpha A - i)^{-N}R(\varepsilon,\delta)^N(\alpha A + i)^{-N}$, we obtain similarly (using now the second part of Proposition 3.11) that

$$\left|\left|\frac{d}{d\varepsilon}F^N(\varepsilon)\right|\right|_{H^{-1}\to H^1_0} \lesssim_\alpha \left|\left|A(\alpha A + i)^{-N}R(\varepsilon,\delta)^N(\alpha A - i)^{-N}\right|\right|_{H^{-1}\to H^1_0} +$$

$$\left|\left|(\alpha A + i)^{-N}R(\varepsilon,\delta)^N(\alpha A - i)^{-N}A\right|\right|_{H^{-1}\to H^1_0} +$$

$$|\varepsilon|\left|\left|(\alpha A + i)^{-N}R(\varepsilon,\delta)\right|\right|_{H^{-1}\to H^1_0}\left|\left|R(\varepsilon,\delta)(\alpha A - i)^{-N}\right|\right|_{H^{-1}\to H^1_0}.$$

The last line is the contribution of

$$\sum_{k=0}^{N-1}(\alpha A + i)^{-N}R(\varepsilon,\delta)\left\{R(\varepsilon,\delta)^k\varepsilon^N\bar{P}_{K_{N+1}}R(\varepsilon,\delta)^{N-1-k}\right\}R(\varepsilon,\delta)(\alpha A - i)^{-N},$$

where each bracket $\{\cdots\}$ in the middle has a $H^{-1}\to H^{-1}$ norm of order ε by (3.28). This is then bounded since the right-hand side of (3.38) is at most of order $|\varepsilon|^{-1/2}$. Then, estimating $||R(\varepsilon,\delta)^N(\alpha A - i)^{-N}||_{H^{-1}\to H^1_0}$ by

$$||R(\varepsilon,\delta)||_{H^{-1}\to H^1_0}^{N-1}||R(\varepsilon,\delta)(\alpha A - i)^{-1}||_{H^{-1}\to H^1_0}||(\alpha A - i)^{1-N}||_{H^{-1}\to H^{-1}} \lesssim |\varepsilon|^{-(N-1)-\frac{1}{2}}$$

which follows from (3.28) and (3.38), and using a similar estimate for $(\alpha A + i)^{-N}R(\varepsilon,\delta)^N$ together with the interpolation estimate of Proposition 3.4, we thus obtain

$$\left|\left|\frac{d}{d\varepsilon}F^N(\varepsilon)\right|\right|_{H^{-1}\to H^1_0} \lesssim_\alpha 1 + ||F^N(\varepsilon)||_{H^{-1}\to H^1_0}^{1-\frac{1}{N}}\left(|\varepsilon|^{\frac{1}{2}-N}\right)^{\frac{1}{N}}$$

so the conclusion follows again from Lemma 3.10. $\qquad\square$

We end up this section with two technical results which will be useful when we will ultimately replace the powers of $(\alpha A \pm i)^{-1}$ by powers of $\langle r\rangle^{-1}$. More precisely, to prove Theorem 2.3, we will combine the estimates of Proposition 3.13 with refinements of Hardy type inequalities, as they appear for instance in [35], and which we now consider.

Lemma 3.14. *Let* $0 \leq s < \frac{n-2}{2}$. *Then, there exists* $C_s > 1$ *such that for all* $u \in C_0^\infty$ *and all* $\delta \geq 0$,

$$C_s^{-1}||\partial_r\left((r+\delta)^{-s}u\right)||_{L^2} \leq ||(r+\delta)^{-s}\partial_r u||_{L^2} \leq C_s||\partial_r\left((r+\delta)^{-s}u\right)||_{L^2}$$

and

$$||(r+\delta)^{-s-1}u||_{L^2} \leq C_s||(r+\delta)^{-s}\partial_r u||_{L^2}.$$

This last estimate is a Hardy inequality which is very close to [35, Lemma 3.2]. Here the additional information is the equivalence of the norms of $(r+\delta)^{-s}\partial_r u$ and $\partial_r((r+\delta)^{-s}u)$ which we will need below.

Proof. By decomposing u along an orthonormal basis of $L^2(\mathcal{S}, d\mathrm{vol}_{h_0})$, it suffices to prove the result for functions $v \in C_0^\infty(\mathbb{R}^+) \subset L^2(\mathbb{R}^+, r^{n-1}dr)$. Using the straighforward computation

$$\partial_r((r+\delta)^{-s}v) - (r+\delta)^{-s}\partial_r v = -s(r+\delta)^{-s-1}v,$$

and then using (3.19), we obtain

$$\left| ||\partial_r((r+\delta)^{-s}v)||_{L^2(r^{n-1}dr)} - ||(r+\delta)^{-s}\partial_r v||_{L^2(r^{n-1}dr)} \right| \le \frac{2s}{n-2}||\partial_r((r+\delta)^{-s}v)||_{L^2(r^{n-1}dr)}.$$

Since $2s/(n-2) < 1$, we obtain the equivalence of the norms. Using (3.19), these norms control $||(r+\delta)^{-s-1}v||_{L^2(r^{n-1}dr)}$ and the result follows. $\qquad\square$

The next proposition is a generalization of an estimate which can be found in [35, Proposition 4.1]. We have to modify it to allow additional weights depending on A. In passing, we also get the full range of exponents $s \in (0, (n-2)/2)$.

Proposition 3.15. *Fix $0 < s < (n-2)/2$ and constants $M, N \in \mathbb{N}$. Then there exist $C > 0$ large enough and $\varrho > 0$ small enough such that, for all α small enough, all $\delta \in (0,1)$, all N-admissible perturbations K such that*

$$|||K|||_0 \le \varrho \qquad and \qquad |||K|||_N \le M,$$

we have

$$\left||(r+\delta)^{-s}\nabla_{g_0}u\right||_{L^2} \le C\left||(\alpha A + i)^{-N}\hat{P}_K(\alpha A + i)^N u\right||_{L^2}^{1/2}\left||(r+\delta)^{-2s}u\right||_{L^2}^{1/2}, \tag{3.39}$$

for all $u \in (\alpha A + i)^{-N}\mathrm{Dom}(\hat{P}_K)$.

Before giving the detailed proof, we recall first the nice basic idea on which it rests when $N = 0$ and $\hat{P}_K = -\Delta_{g_0}$, *i.e.* when $K = 0$. We compute $\mathrm{Re}\,(-\Delta_{g_0}u, (r+\delta)^{-2s}u)_{L^2}$ or more precisely $\mathrm{Re}\,\bar{Q}_0(u, (r+\delta)^{-2s}u)$. The condition $\delta > 0$ guarantees that $(r+\delta)^{-2s}$ is bounded on H_0^1 by the item 1 of Proposition 3.4. By density of C_0^∞ in H_0^1, on which $\bar{Q}_0$ is continuous, we may assume that $u \in C_0^\infty$. Then, by integrating by part one finds

$$\begin{aligned}
\mathrm{Re}\,\bar{Q}_0(u, (r+\delta)^{-2s}u) &= s\int_{\mathcal{M}_0} r^{n-2}(r+\delta)^{-2s-2}\big((n-2-2s)r + (n-1)\delta\big)|u|^2 dr dvol_{h_0} \\
&\quad + ||(r+\delta)^{-s}D_r u||_{L^2}^2 + \left||(r+\delta)^{-s}r^{-1}|D_{\mathcal{S}}|u\right||_{L^2}^2 \\
&\ge ||(r+\delta)^{-s}\nabla_{g_0}u||_{L^2}^2, \tag{3.40}
\end{aligned}$$

since $(n-2-2s)r + (n-1)\delta \ge 0$ (here we may go up to $s = (n-2)/2$). If u belongs to $\mathrm{Dom}(\hat{P}_0)$, then $\bar{Q}_0(u, (r+\delta)^{-2s}u) = (\hat{P}_0 u, (r+\delta)^{-2s}u)_{L^2}$ and, using the Cauchy-Schwarz inequality, this yields the result with $C = 1$ when $K = 0$ and $N = 0$. The general case will follow from this model by a perturbation argument.

Proof of Proposition 3.15. We start by computing the commutator $\big[(\alpha A - i)^N, \hat{P}_K\big]$ in the form sense: for all $\psi, \varphi \in C_0^\infty(\mathcal{M}_0)$, we have

$$Q_K\big((\alpha A + i)^N\psi, \varphi\big) = Q_K\big(\psi, (\alpha A - i)^N\varphi\big) + \sum_{k=1}^N C_N^k(2i\alpha)^k Q_{K_k}\big(\psi, (\alpha A - i)^{N-k}\varphi\big), \tag{3.41}$$

where K_k is as in (3.25). This follows from (3.13) and a simple induction on N. Then, up to considering the closures of the quadratic forms, the identity (3.41) remains true if one only assumes that $\psi \in H_0^1$ and $\varphi \in H_0^1$ satisfy $(\alpha A + i)^N\psi \in H_0^1$ and $(\alpha A - i)^N\varphi \in H_0^1$ (this can be

proved as in Lemma 3.1 of [6][2]). Consequently, by choosing $\psi = u$ and $\varphi = (\alpha A - i)^{-N}(r+\delta)^{-2s}u$, we find that

$$\mathrm{Re}\left((\alpha A + i)^{-N}\hat{P}_K(\alpha A + i)^N u, (r+\delta)^{-2s}u\right)_{L^2} = \mathrm{Re}\,\bar{Q}_K\left((\alpha A + i)^N u, (\alpha A - i)^{-N}(r+\delta)^{-2s}u\right)$$

can be written as

$$\mathrm{Re}\left(\bar{Q}_K\left(u, (r+\delta)^{-2s}u\right) + \sum_{k=1}^N C_N^k(2i\alpha)^k\bar{Q}_{K_k}\left(u, (\alpha A - i)^{-k}(r+\delta)^{-2s}u\right)\right). \tag{3.42}$$

Our goal is to bound this expression from below similarly to (3.40). To study the contribution of the first term of (3.42), we use Lemma 3.7 and the expression of $P_K - (-\Delta_{g_0})$ given by (3.8) to obtain

$$\left|\mathrm{Re}\,\bar{Q}_0(u, (r+\delta)^{-2s}u)_{L^2} - \mathrm{Re}\,\bar{Q}_K(u, (r+\delta)^{-2s}u)\right| \lesssim$$
$$|||K|||_0||(r+\delta)^{-s}\nabla_{g_0}u||_{L^2}\left(||(r+\delta)^{-s}\nabla_{g_0}u||_{L^2} + ||(r+\delta)^{-s-1}u||_{L^2}\right).$$

By using Lemma 3.14 (which imposes $s < (n-2)/2$) and (3.40), we get

$$\mathrm{Re}\,\bar{Q}_K(u, (r+\delta)^{-2s}u)_{L^2} \geq \left(1 - C|||K|||_0\right)||(r+\delta)^{-s}\nabla_{g_0}u||_{L^2}^2. \tag{3.43}$$

The result will then follow if we prove that, for each term in the sum of (3.42),

$$\left|Q_{K_k}\left(u, (\alpha A - i)^{-k}(r+\delta)^{-2s}u\right)\right| \leq C(1 + |||K_k|||_0)||(r+\delta)^{-s}\nabla_{g_0}u||_{L^2}^2. \tag{3.44}$$

To justify (3.44), we start by observing that for all $v, w \in C_0^\infty$, we have

$$|Q_{K_k}(v, (r+\delta)^{-2s}w)| \leq C(1 + |||K_k|||_0)||(r+\delta)^{-s}\nabla_{g_0}v||_{L^2}||(r+\delta)^s\nabla_{g_0}(r+\delta)^{-2s}v||_{L^2}.$$

This follows easily from of (3.21) (strictly speaking (3.21) only yields the contribution of K_k but the one of Δ_{g_0} is similar). Using that

$$||(r+\delta)^s\nabla_{g_0}(r+\delta)^{-2s}v||_{L^2} \lesssim ||(r+\delta)^{-s}\nabla_{g_0}v||_{L^2} + ||(r+\delta)^{-s-1}v||_{L^2},$$

and Lemma 3.14, and then a density argument to replace v and w by any H_0^1 functions, we get

$$|Q_{K_k}(u, (\alpha A - i)^{-k}(r+\delta)^{-2s}u)| \lesssim (1 + |||K_k|||_0)||(r+\delta)^{-s}\nabla_{g_0}u||_{L^2}||(r+\delta)^{-s}\nabla_{g_0}\tilde{u}||_{L^2},$$

with

$$\tilde{u} = (r+\delta)^{2s}(\alpha A - i)^{-k}(r+\delta)^{-2s}u.$$

The proof will then be complete if we show that

$$||(r+\delta)^{-s}\nabla_{g_0}\tilde{u}||_{L^2} \leq C||(r+\delta)^{-s}\nabla_{g_0}u||_{L^2}. \tag{3.45}$$

This is obtained as follows. Given $\sigma \in \mathbb{R}$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$, if α is small enough we have

$$||(r+\delta)^\sigma(\alpha A - \zeta)^{-k}(r+\delta)^{-\sigma}||_{L^2 \to L^2} \leq C. \tag{3.46}$$

---

[2]for convenience, we recall that the idea is to set $\psi = (\alpha A + i)^{-N}v$ and to use on one hand that $(\alpha A + i)^{-N} = \lim_{\tau \to \infty} \frac{1}{i(N-1)!}\int_0^\tau (-it)^{N-1}e^{-t}e^{it\alpha A}dt$, where the integral preserves C_0^∞, and on the other hand that one can approximate $v \in H_0^1$ by some C_0^∞ function

This comes from the first identity of (3.20), since $(r + \delta)^{|\sigma|} \approx r^{|\sigma|} + \delta^{|\sigma|}$ (*i.e.* their quotient is bounded from above and below). Therefore, using (3.20) and (3.46)

$$
\begin{aligned}
||(r+\delta)^{-s}\partial_r \tilde{u}||_{L^2} &\lesssim ||(r+\delta)^s \partial_r (\alpha A - i)^{-k}(r+\delta)^{-2s}u||_{L^2} + ||(r+\delta)^{s-1}(\alpha A - i)^{-k}(r+\delta)^{-2s}u||_{L^2} \\
&\lesssim ||(r+\delta)^s(\alpha A - i + i\alpha)^{-k}\partial_r((r+\delta)^{-2s}u)||_{L^2} + ||(r+\delta)^{-s-1}u||_{L^2} \\
&\lesssim ||(r+\delta)^{-s}\partial_r u||_{L^2} + ||(r+\delta)^{-s-1}u||_{L^2} \\
&\lesssim ||(r+\delta)^{-s}\partial_r u||_{L^2}
\end{aligned}
$$

the last inequality following from Lemma 3.14. The estimate

$$
\left|\left|(r+\delta)^{-s}r^{-1}|D_{\mathcal{S}}|\tilde{u}\right|\right|_{L^2} \lesssim \left|\left|(r+\delta)^{-s}r^{-1}|D_{\mathcal{S}}|u\right|\right|_{L^2}
$$

is obtained in the very same way, using additionally that $|D_{\mathcal{S}}|$ commutes with functions of r and A. This yields (3.45) and completes the proof. $\qquad\square$

4 Proofs of Proposition 2.1 and Theorem 2.3

4.1 Proof of Proposition 2.1

Using (1.2), we recall that $g = a(r)dr^2 + 2rb(r)dr + r^2h(r)$ is a metric on $(R_0, \infty) \times \mathcal{S}$ such that $G = \kappa^* g$. This allows to recast the problem on a question on a half line times $\mathcal{S}$: it suffices to show that one can find $\widetilde{R}_0 > 0$ and a diffeomorphism

$$
\Xi : (\widetilde{R}_0, \infty) \times \mathcal{S} \to U \subset (R_0, \infty) \times \mathcal{S}
$$

of the form

$$
\Xi(\tilde{r}, \sigma) = \big(\bar{r}(\tilde{r}, \sigma), \sigma\big)
$$

such that

1. for some symbol $\xi \in S^{-\rho}$ on $(\widetilde{R}_0, \infty) \times \mathcal{S}$ (*i.e.* for all integer k $\partial_{\tilde{r}}^k \xi(\tilde{r}, .) = \mathcal{O}(\tilde{r}^{-\rho-k})$ in $C^\infty(\mathcal{S}, \mathbb{R})$)

$$
\bar{r}(\tilde{r}, \sigma) = \tilde{r}\big(1 + \xi(\tilde{r}, \sigma)\big),
$$

2. U contains $(R_0', +\infty) \times \mathcal{S}$ for some $R_0' > R_0$ large enough,

3. at each point $(\tilde{r}, \sigma) \in (\widetilde{R}_0, \infty) \times \mathcal{S}$, we have

$$
d\mathrm{vol}_{\Xi^* g} = \tilde{r}^{n-1} d\tilde{r} d\mathrm{vol}_{h_0}. \tag{4.1}
$$

To build $\bar{r}$ we check which conditions must be fulfilled. We first note that, at any point $(r, \sigma) \in (R_0, \infty) \times \mathcal{S}$,

$$
d\mathrm{vol}_g = F(r, \sigma)r^{n-1} dr d\mathrm{vol}_{h_0},
$$

with $F - 1 \in S^{-\rho}$. Therefore, the condition (4.1) reads

$$
\tilde{r}^{n-1} = \frac{\partial \bar{r}}{\partial \tilde{r}} \bar{r}^{n-1} F(\bar{r}, \sigma). \tag{4.2}
$$

If we assume that Ξ exists, the inverse diffeomorphism is of the form $(\underline{r}(r, \sigma), \sigma)$. By evaluating (4.2) at $\tilde{r} = \underline{r}(r, \sigma)$, we get

$$
\underline{r}^{n-1} \frac{1}{\frac{\partial \bar{r}}{\partial \tilde{r}}(\underline{r})} = r^{n-1} F(r, \sigma),
$$

32

that is

$$\underline{r}(r,\sigma)^{n-1}\frac{\partial \underline{r}}{\partial r}(r,\sigma) = r^{n-1}F(r,\sigma),$$

or equivalently

$$\frac{\partial \underline{r}^n}{\partial r}(r,\sigma) = nr^{n-1}F(r,\sigma), \tag{4.3}$$

which can be solved: we write F as $F(r,\sigma) = 1 + \delta(r,\sigma)$ with $\delta \in S^{-\rho}$ then, by following (4.3), we define for some $R_1 > R_0$

$$\underline{r}(r,\sigma) = \left(n\int_{R_1}^r (1 + \delta(t,\sigma))t^{n-1}dt\right)^{\frac{1}{n}} = r\left(1 - \frac{R_1^n}{r^n} + r^{-n}\int_{R_1}^r \delta(t,q)t^{n-1}dt\right)^{\frac{1}{n}}, \tag{4.4}$$

for $r > R_1$ and $\sigma \in \mathcal{S}$. Since $n \geq 2$ and by assuming $\rho \leq 1$, it is not hard to check that $r \mapsto \int_{R_1}^r \delta(t,\sigma)t^{n-1}dt$ belongs to $S^{n-\rho}$ hence that the last bracket in (4.4) is of the form $1 + S^{-\rho}$. It follows easily that, for $R_2 \gg 1$ and for all $\sigma \in \mathcal{S}$, $r \mapsto \underline{r}(r,\sigma)$ is a diffeomorphism from (R_2,∞) to $(\underline{r}(R_2,\sigma),\infty)$ hence that

$$(r,\sigma) \mapsto (\underline{r}(r,\sigma),\sigma)$$

is a diffeomorphism from $(R_2,\infty) \times \mathcal{S}$ to an open subset containing $\left(\sup_{\mathcal{S}} \underline{r}(R_2,.),\infty\right) \times \mathcal{S}$ which contains $(\tilde{R}_0,\infty) \times \mathcal{S}$ for some $\tilde{R}_0 \gg 1$. The inverse diffeomorphism provides a diffeomorphism of the form $\Xi : (\tilde{r},\sigma) \mapsto (\bar{r}(\tilde{r},\sigma),\sigma)$ which, by construction, satisfies (4.2) and hence (4.1). Using that $\underline{r}(r,\sigma) = r(1 + S^{-\rho})$ and by differentiating $\underline{r}(\bar{r}(\tilde{r},\sigma),\sigma) = \tilde{r}$, a routine analysis shows that $\bar{r}(\tilde{r},\sigma)$ is of the form $\tilde{r}(1 + S^{-\rho})$, which yields both item 1 and item 2. $\qquad\square$

4.2 Proof of Theorem 2.3

Let us recall that the goal of this theorem is to construct an operator $\hat{P}_T$ on $(0,\infty) \times \mathcal{S}$ such that on one hand $\hat{P}$ and $\hat{P}_T$ coincide near infinity and on the other hand $\hat{P}_T$ satisfy appropriate resolvent estimates ((2.15), (2.16) and (2.17)).

Construction of P_T. By Proposition 2.1, we assume that the metrics g_0 introduced in (3.3) and $g(= a(r)dr^2 + 2rb(r)dr + r^2 h(r)$, see (1.2)) satisfy $\operatorname{div}_{g_0} = \operatorname{div}_g$ near infinity. More precisely, for any vector field V on $(R,\infty) \times \mathcal{S}$, which we split as $V = (V_1, V^{\mathcal{S}})$ using the isomorphism $T((R,\infty) \times \mathcal{S}) \approx T(R,\infty) \times T\mathcal{S}$, we have

$$\operatorname{div}_g(V) = \operatorname{div}_{g_0}(V) = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}V_1\right) + \operatorname{div}_{h_0}(V^{\mathcal{S}}).$$

We then recall that $-\Delta_g$ (that is $\kappa^* \hat{P} \kappa_*$ near infinity) is for $r \gg 1$ of the form

$$-\Delta_g u = -\Delta_{g_0} u - \operatorname{div}_{g_0}\left(K_G^{\mathrm{sc}}u\right),$$

for some tensor K_G such that, by (1.3),

$$\|\partial_r^k K_G\|_{L^\infty} \leq C_k r^{-\rho-k}, \qquad k \geq 0. \tag{4.5}$$

(See also after (3.8) for the notation sc.) We introduce $\varphi_0 \in C^\infty(\mathbb{R})$ such that $\varphi_0 \equiv 1$ on $[1,\infty)$ and $\operatorname{supp}(\varphi_0) \subset [1/2,\infty)$. Then, we define

$$T = \varphi_0(r/R)K_G,$$

which is N-admissible for all N by (4.5), and set

$$P_T u := -\Delta_{g_0} u - \mathrm{div}_{g_0}\big(T^{\mathrm{sc}} u\big). \tag{4.6}$$

We let $\hat{P}_T$ be the associated self-adjoint realization defined according to Proposition 3.8. So defined, $\hat{P}_T$ satisfies the item 1 of Theorem 2.3. Furthermore, by (4.5), we have $|||T|||_0 \lesssim R^{-\rho}$ and $|||T|||_1 \lesssim R^{-\rho}$ (see Definition 3.1 for the norms $||| \cdot |||_k$), hence these norms are as small as we wish by choosing R large enough. This allows us to use the results of Propositions 3.13 and 3.15 to prove the estimates (2.15), (2.16) and (2.17) as follows.

Proof of (2.15) Let $z = \lambda + i\delta$ with $\delta \in \mathbb{R} \setminus \{0\}$. Assume first that $\lambda > 0$. Setting $\delta' = \delta/\lambda$, it is straightforward that

$$
\begin{aligned}
r^{-1}(\hat{P}_T - z)^{-1}r^{-1} &= (\lambda^{1/2}r)^{-1}(\lambda^{-1}\hat{P}_T - 1 - i\delta')^{-1}(\lambda^{1/2}r)^{-1} \\
&= e^{itA}r^{-1}(\hat{P}_{T^t} - 1 - i\delta')^{-1}r^{-1}e^{-itA},
\end{aligned}
\tag{4.7}
$$

by choosing $t = \ln(\lambda^{1/2})$ (see (3.9) for e^{itA}). Notice that $|||T|||_k = |||T^t|||_k$ for all k. We next write

$$
\begin{aligned}
r^{-1} &= r^{-1}(\alpha A + i)(\alpha A + i)^{-1} \\
&= B(\alpha A + i)^{-1},
\end{aligned}
\tag{4.8}
$$

where $B := \alpha D_r + (\alpha n/2i + i)r^{-1}$ is bounded from H_0^1 to L^2 by the Hardy inequality (3.19). This implies that

$$
\begin{aligned}
||r^{-1}(\hat{P}_T - z)^{-1}r^{-1}||_{L^2 \to L^2} &= ||B(\alpha A + i)^{-1}(\hat{P}_{T^t} - 1 - i\delta')^{-1}(\alpha A - i)^{-1}B^*||_{L^2 \to L^2}, \\
&\leq ||B||^2_{H_0^1 \to L^2}||(\alpha A + i)^{-1}(\hat{P}_{T^t} - 1 - i\delta')^{-1}(\alpha A - i)^{-1}||_{H^{-1} \to H^1}
\end{aligned}
$$

so the result follows from Proposition 3.13. When $\lambda \leq 0$, the proof is even simpler and does not use Proposition 3.13. It suffices to use the Hardy inequality (3.19) to see that

$$||r^{-1}(\hat{P}_K - z)^{-1}r^{-1}||_{L^2 \to L^2} \leq C||\hat{P}_K^{1/2}(\hat{P}_K - z)^{-1}\hat{P}_K^{1/2}||_{L^2 \to L^2},$$

whose right-hand side is bounded uniformly in z by the spectral theorem. $\qquad\square$

To prove (2.16) and (2.17), we still need two technical lemmas.

Lemma 4.1. *Fix $s \geq 0$ and $0 < \alpha < 1$. Then there exist $\varrho > 0$ and $C > 0$ such that*

$$||\langle r \rangle^{-s-2}(\hat{P}_K + 1)^{-1}u||_{L^2(\mathcal{M}_0)} \leq C||\langle r \rangle^{-s-1}(\alpha A + i)^{-1}u||_{L^2(\mathcal{M}_0)},$$

for all $u \in L^2$ and all K such that $|||K|||_0 \leq \varrho$.

Proof. Without loss of generality we can replace $\langle r \rangle$ by $\langle \delta r \rangle$, with $\delta > 0$ small enough to be fixed below. We next remark that $\bar{P}_0 + \bar{I}$ is an isomorphism from H_0^1 to H^{-1} (recall that $\bar{P}_0$ is the $H_0^1 \to H^{-1}$ closure of $-\Delta_{g_0}$) since $((\bar{P}_0 + \bar{I})u, v) = (u, v)_{H_0^1}$. Therefore, by Proposition 3.7, $\bar{P}_K + \bar{I}$ is also such an isomorphism (with norm in a fixed neighborhood of 1) if $|||K|||_0$ is small enough. If $\nu \geq 0$ is a fixed real number, the operator

$$P_{K,\delta,\nu} = \langle \delta r \rangle^{-\nu} P_K \langle \delta r \rangle^{\nu}, \tag{4.9}$$

defined first on C_0^∞, has a bounded closure $\bar{P}_{K,\delta,\nu}$ to H_0^1 such that

$$||\bar{P}_{K,\delta,\nu} - \bar{P}_K||_{H_0^1 \to H^{-1}} \leq C(1 + |||K|||_0)\delta,$$

34

where the constant is independent of ν as long as ν belongs to a bounded set. This is easily seen from (3.7) and (3.8), the factor δ coming from commutations between $\langle \delta r \rangle^{\pm \nu}$ and ∂_r. For δ small enough, $\bar{P}_{K,\delta,\nu} + \bar{I}$ is also an isomorphism between H_0^1 and H^{-1} and by construction (plus a routine verification which we omit), we obtain

$$\langle \delta r \rangle^{-\nu} (\bar{P}_K + \bar{I})^{-1} = (\bar{P}_{K,\delta,\nu} + \bar{I})^{-1} \langle \delta r \rangle^{-\nu}, \tag{4.10}$$

as operators from H^{-1} to H_0^1. Here $\langle \delta r \rangle^{-\nu}$ acts on H^{-1} in the distributions sense[3]. By composition with $\bar{I}$ and the item 2 of Proposition 3.9, we get

$$\langle \delta r \rangle^{-\nu} (\hat{P}_K + 1)^{-1} = (\bar{P}_{K,\delta,\nu} + \bar{I})^{-1} \bar{I} \langle \delta r \rangle^{-\nu}. \tag{4.11}$$

Using this identity with $\nu = s + 2$, we can thus write

$$\langle \delta r \rangle^{-s-2} (\hat{P}_K + 1)^{-1} = (\bar{P}_{K,\delta,\nu} + \bar{I})^{-1} \bar{I} \langle \delta r \rangle^{-1} (\alpha A + i)(\alpha A + i)^{-1} \langle \delta r \rangle^{-s-1},$$

where, using that $\langle \delta r \rangle^{-1} (\alpha A + i)$ maps L^2 in H^{-1}, we thus obtain

$$|| \langle \delta r \rangle^{-s-2} (\hat{P}_K + 1)^{-1} u ||_{L^2} \leq C || (\alpha A + i)^{-1} \langle \delta r \rangle^{-s-1} u ||_{L^2}.$$

To swap the positions of $(\alpha A + i)^{-1}$ and $\langle \delta r \rangle^{-s-1}$, we write

$$(\alpha A + i)^{-1} \langle \delta r \rangle^{-s-1} = (\alpha A + i)^{-1} \langle \delta r \rangle^{-s-1} (\alpha A + i)(\alpha A + i)^{-1},$$

and observe that $(\alpha A + i)^{-1} \langle \delta r \rangle^{-s-1} (\alpha A + i) = B(\alpha, s, \delta) \langle \delta r \rangle^{-s-1}$ for some bounded (and explicitly computable) operator $B(\alpha, s, \delta)$. The result follows. $\qquad \square$

Lemma 4.2. *Fix $M, N \geq 0$. There exist $\alpha_0 > 0$, $\varrho > 0$ and $C > 0$ such that, for all integer $0 \leq k \leq N - 1$,*

$$|| r^{-1} \langle r \rangle^{-k} (\hat{P}_K + 1)^{-1-N} u ||_{L^2(\mathcal{M}_0)} \leq C || (\alpha A + i)^{-1-k} u ||_{L^2(\mathcal{M}_0)},$$

for all $u \in L^2$, all $0 < \alpha < \alpha_0$ and all K such that $|||K|||_0 \leq \varrho$, $|||K|||_N \leq M$.

We state this result for general admissible perturbations K but we will apply it with $K = T^t$, using that $||T^t||_k = ||T||_k$ for all integer k and all $t \in \mathbb{R}$.

Proof. We will use (4.9) and (4.11) from the proof of Lemma 4.1. By iteration of (4.11), we obtain on one hand

$$
\begin{aligned}
\langle \delta r \rangle^{-k} (\hat{P}_K + 1)^{-k-1} &= (\bar{P}_{K,\delta,k} + \bar{I})^{-1} \bar{I} \langle \delta r \rangle^{-k} (\hat{P}_K + 1)^{-k} \\
&= (\bar{P}_{K,\delta,k} + \bar{I})^{-1} \bar{I} \langle \delta r \rangle^{-1} (\bar{P}_{K,\delta,k-1} + 1)^{-1} \bar{I} \langle \delta r \rangle^{-(k-1)} (\hat{P}_K + 1)^{-(k-2)} \\
&= (\bar{P}_{K,\delta,k} + \bar{I})^{-1} \bar{I} \prod_{\nu=k-1}^{0} \langle \delta r \rangle^{-1} (\bar{P}_{K,\delta,\nu} + 1)^{-1} \bar{I}, \tag{4.12}
\end{aligned}
$$

where the product is the composition from the left to the right decreasingly in ν. On the other hand, for any $\nu \geq 0$ and an integer $j \geq 0$, we can consider the operator

$$P_{K,\delta,\nu}^{j;\alpha} := (\alpha A + i)^{-j} P_{K,\delta,\nu} (\alpha A + i)^j, \tag{4.13}$$

[3]*i.e.* $(\langle \delta r \rangle^{-\nu} E, u) = (E, \langle \delta r \rangle^{-\nu} u)$ for all $E \in H^{-1}$ and $u \in H_0^1$

on C_0^∞. This can be written as the sum of $P_{K,\delta,\nu}$ and a linear combination of nonnegative powers of $(\alpha A + i)^{-1}$ composed with commutators of $P_{K,\delta,\nu}$ and αA. It follows that

$$\left\|(P_{K,\delta,\nu}^{j,\alpha} - P_{K,\delta,\nu})u\right\|_{H^{-1}} \leq C\alpha\|u\|_{H_0^1}, \qquad u \in C_0^\infty, 0 < \alpha \ll 1.$$

This implies that $P_{K,\delta,\nu}^{j,\alpha}$ has a closure $\bar{P}_{K,\delta,\nu}^{j,\alpha}$ to H_0^1. If α is small enough, $\bar{P}_{K,\delta,\nu}^{j,\alpha} + \bar{I}$ is an isomorphism between H_0^1 and H^{-1} since $\bar{P}_{K,\delta,\nu}$ is for δ small enough (cf the proof of Lemma 4.1). Moreover, (4.13) implies

$$(\alpha A + i)^{-j}\left(\bar{P}_{K,\delta,\nu} + \bar{I}\right)^{-1} = \left(\bar{P}_{K,\delta,\nu}^{j,\alpha} + \bar{I}\right)^{-1}(\alpha A + i)^{-j}. \qquad (4.14)$$

This is formally obvious but requires an argument since we cannot obviously compose both sides of (4.13) with $(\alpha A + i)^{-j}$ for this does not preserve C_0^∞ in general. To justify this formula we use the Lemma 3.1 of [6] as in the proof of Proposition 3.15, namely that for any $v \in H_0^1$ we can find a sequence $(u_m)_{m \in \mathbb{N}}$ of C_0^∞ such that

$$(\alpha A + i)^j u_m \to v \qquad \text{and} \qquad u_m \to (\alpha A + i)^{-j}v$$

in H_0^1. Then (4.13) yields $(\alpha A + i)^{-j}\bar{P}_{K,\delta,\nu}v = \bar{P}_{K,\delta,\nu}^{j,\alpha}(\alpha A + i)^{-j}v$ which then implies (4.14). The interest of (4.12) and (4.14) is the following one. After multiplication by r^{-1}, we write in the last line of (4.12)

$$\begin{aligned}
r^{-1}\left(\bar{P}_{K,\delta,k} + \bar{I}\right)^{-1}\bar{I} &= r^{-1}(\alpha A + i)(\alpha A + i)^{-1}\left(\bar{P}_{K,\delta,k} + \bar{I}\right)^{-1}\bar{I} \\
&= r^{-1}(\alpha A + i)\left(\bar{P}_{K,\delta,k}^{1,\alpha} + \bar{I}\right)^{-1}\bar{I}(\alpha A + i)^{-1}.
\end{aligned}$$

The operator $(\alpha A + i)^{-1}$ in the right-hand side, which falls on the operator $\langle \delta r \rangle^{-1}\left(\bar{P}_{K,\delta,k-1} + 1\right)^{-1}$, is then rewritten as $(\alpha A + i)(\alpha A + i)^{-2}$ so that we can use (4.14) with $j = 2$ and $\nu = k - 1$. By iteration, we see that $r^{-1}\langle \delta r \rangle^{-k}(\hat{P}_K + 1)^{-k-1}$ reads

$$r^{-1}(\alpha A + i)\left(\bar{P}_{K,\delta,k}^{1,\alpha} + \bar{I}\right)^{-1}\bar{I}\left(\prod_{l=k-1}^{0} B(k,l)(\alpha A + i)\langle \delta r \rangle^{-1}\left(\bar{P}_{K,\delta,l}^{k+1-l,\alpha} + 1\right)^{-1}\bar{I}\right)(\alpha A + i)^{-k-1},$$

with $B(k,l)$ such that $(\alpha A + i)^{-(k+1-l)}\langle \delta r \rangle^{-1}(\alpha A + i)^{k+1-l} = B(k,l)\langle \delta r \rangle^{-1}$. Each $B(k,l)$ is clearly bounded on L^2. By using Lemma 3.6 and the fact that $r^{-1}(\alpha A + i)$ is bounded on H_0^1, we conclude that

$$\begin{aligned}
\|r^{-1}\langle \delta r \rangle^{-k}(\hat{P}_K + 1)^{-N-1}u\|_{L^2} &\leq C\|(\alpha A + i)^{-k-1}(\hat{P}_K + 1)^{-(N-k)}u\|_{L^2} \\
&\leq C\|(\alpha A + i)^{-k-1}u\|_{L^2}
\end{aligned}$$

where, to get the second line, we used (4.14) with $j = k + 1$ and $\nu = 0$ (in this case we have $(\hat{P}_K + 1)^{-1} = (\bar{P}_{K,0,0} + \bar{I})^{-1}\bar{I}$). This completes the proof. $\qquad\square$

Proof of (2.16) We start with a general remark. By iterating the resolvent identity for any self-adjoint operator $H \geq 0$, we have

$$(H - \zeta)^{-1} = \sum_{j=0}^{2N}(1 + \zeta)^j(H + 1)^{-1-j} + (\zeta + 1)^{2N+1}(H + 1)^{-N}(H - \zeta)^{-1}(H + 1)^{-N}, \quad (4.15)$$

for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$. By differentiating k times in ζ, we see that if ζ belongs to a bounded subset of $\mathbb{C} \setminus \mathbb{R}$, there exists $C > 0$ such that, for all bounded operator W with operator norm at most 1,

$$||W(H-\zeta)^{-1-k}W|| \leq C\left(1 + \sum_{l\leq k}||W(H+1)^{-N}(H-\zeta)^{-1-l}(H+1)^{-N}||\right). \qquad (4.16)$$

Here $||\cdot||$ is the operator on the Hilbert space where H is defined. From now on, we consider $\hat{P}_T$ and $||\cdot||$ is the operator norm on $L^2(\mathcal{M}_0)$. We let $\epsilon^2 Z = \lambda(1 + i\delta')$, with $\delta' \in \mathbb{R} \setminus \{0\}$ (note that $\lambda \sim \epsilon^2$). Then

$$(\epsilon^{-2}\hat{P}_T - Z)^{-1-k} = \text{Re}(Z)^{-1-k}(\lambda^{-1}\hat{P}_T - 1 - i\delta')^{-1-k}. \qquad (4.17)$$

Similarly to (4.7), we consider the family of rescaled operators $\hat{P}_{T^t} = e^{itA}(\lambda^{-1}\hat{P}_T)e^{-itA}$. We observe that

$$||(\epsilon r)^{-1}\langle\epsilon r\rangle^{-k}(\epsilon^{-2}\hat{P}_T - Z)^{-1-k}\langle\epsilon r\rangle^{-k}(\epsilon r)^{-1}|| \lesssim ||r^{-1}\langle r\rangle^{-k}(\hat{P}_{T^t} - 1 - i\delta')^{-1-k}\langle r\rangle^{-k}r^{-1}||.$$

Indeed the right-hand side equals $||(\lambda^{1/2}r)^{-1}\langle\lambda^{1/2}r\rangle^{-k}(\lambda^{-1}\hat{P}_T - 1 - i\delta')^{-1-k}\langle\lambda^{1/2}r\rangle^{-k}(\lambda^{1/2}r)^{-1}||$ by rescaling, and this quantity is bounded from above and below by the left-hand side, using (4.17) and the fact that λ/ϵ^2 belongs to a compact see of $(0, \infty)$. Then, by using (4.16) and Lemma 4.2, we have

$$||r^{-1}\langle r\rangle^{-k}(\hat{P}_{T^t} - 1 - i\delta')^{-1-k}\langle r\rangle^{-k}r^{-1}|| \lesssim 1 + \sum_{l\leq k}||(\alpha A + i)^{-1-l}(\hat{P}_{T^t} - 1 - i\delta')^{-1-l}(\alpha A - i)^{-1-l}||$$

so the result follows from Proposition 3.13. $\qquad \square$

Proof of (2.17) By using (4.16) with $z = \zeta$, we obtain (as long as z is bounded)

$$||\langle r\rangle^{-2-s}(\hat{P}_T - z)^{-2}\langle r\rangle^{-2-s}|| \lesssim 1 + \sum_{k=1}^{2}||\langle r\rangle^{-2-s}(\hat{P}_T + 1)^{-1}(\hat{P}_T - z)^{-k}(\hat{P}_T + 1)^{-1}\langle r\rangle^{-2-s}||.$$

The term corresponding to $k = 1$ is clearly bounded for $|\text{Re}(z)| \leq 1$ and $0 < |\text{Im}(z)| \leq 1$ by using (2.15) and the fact that $(\hat{P}_T + 1)^{-1}$ preserves the decay $\langle r\rangle^{-2-s}$ (see (4.10)). Therefore, it suffices to consider the term corresponding to $k = 2$. By Lemma 4.1, this term is controlled by

$$||(r+1)^{-1-s}(\alpha A + i)^{-1}(\hat{P}_T - z)^{-2}(\alpha A - i)^{-1}(r+1)^{-1-s}||_{L^2 \to L^2}.$$

We assume first that $\text{Re}(z) =: \lambda$ is positive. By using the same rescaling as in (4.7), the above norm reads

$$\text{Re}(z)^{s-1}||(r+\lambda^{1/2})^{-1-s}(\alpha A + i)^{-1}(\hat{P}_{T^t} - 1 - i\delta')^{-2}(\alpha A - i)^{-1}(r+\lambda^{1/2})^{-1-s}||_{L^2 \to L^2}. \qquad (4.18)$$

Therefore, it suffices to show that the norm in (4.18) is bounded uniformly in δ' and λ (recall that $t = \ln(\lambda^{1/2})$). Using (4.8), we write

$$(r+\lambda^{1/2})^{-1-s}(\alpha A + i)^{-1} = (r+\lambda^{1/2})^{-s}\left(\frac{\alpha r}{r+\lambda^{1/2}}D_r + \frac{\alpha n/2i + i}{r+\lambda^{1/2}}\right)(\alpha A + i)^{-2}$$

$$=: (r+\lambda^{1/2})^{-s}B_\lambda(\alpha A + i)^{-2}$$

so that the norm in (4.18) equals precisely

$$\left|\left|(r+\lambda^{1/2})^{-s}B_\lambda(\alpha A+i)^{-2}(\hat{P}_{T^t}-1-i\delta')^{-2}(\alpha A-i)^{-2}B_\lambda^*(r+\lambda^{1/2})^{-s}\right|\right|_{L^2\to L^2}.$$

By Lemma 3.14 and Proposition 3.15, this norm is bounded by a constant (independent of λ and δ') times the product of the following powers of norms

$$\left|\left|(\alpha A+i)^{-2}\hat{P}_{T^t}^2(\hat{P}_{T^t}-1-i\delta')^{-2}(\alpha A-i)^{-2}\right|\right|_{L^2\to L^2}^{1/4}$$

$$\left|\left|(r+\lambda^{1/2})^{-2s}(\alpha A+i)^{-2}(\hat{P}_{T^t}-1\pm i\delta')^{-2}\hat{P}_{T^t}(\alpha A-i)^{-2}\right|\right|_{L^2\to L^2}^{1/4}$$

$$\left|\left|(r+\lambda^{1/2})^{-2s}(\alpha A+i)^{-2}(\hat{P}_{T^t}-1-i\delta')^{-2}(\alpha A-i)^{-2}(r+\lambda^{1/2})^{-2s}\right|\right|_{L^2\to L^2}^{1/4}.$$

Since $s < 1/2$, the Hardy inequality allows to drop the weight $(r+\lambda^{1/2})^{-2s}$ in the second and third lines up to the replacement of the $L^2 \to L^2$ norm by the $H^{-1} \to H_0^1$ norm. The uniform boundedness of these norms then follows from Proposition 3.13. The proof in the case $\mathrm{Re}(z) < 0$ is similar; one only has to replace $(\hat{P}_{T^t}-1-i\delta')$ by $(\hat{P}_{T^t}+1-i\delta')$ so that we do not need to use Proposition 3.13. This completes the proof. □

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