On global Strichartz estimates for non trapping metrics

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Abstract

We prove global Strichartz estimates (with spectral cutoff on the low frequencies) for non trapping metric perturbations of the Schrödinger equation, posed on the Euclidean space.

1 Introduction

Consider the Laplace Beltrami operator on $\mathbb{R}^d$, $d \geq 2$, associated to a Riemannian metric $G = (G_{jk})$,

$$\Delta_G = \det(G(x))^{-1/2} \frac{\partial}{\partial x_j} G^{jk}(x) \det(G(x))^{1/2} \frac{\partial}{\partial x_k},$$

using Einstein's summation convention and $(G^{jk}(x)) = (G_{jk}(x))^{-1}$. We suppose that the metric G is smooth (C^∞). Consider the Schrödinger equation

$$(i\partial_t + \Delta_G)u = 0, \quad u|_{t=0} = u_0 \in L^2(\mathbb{R}^d)\,. \tag{1.1}$$

Let us denote by P the self-adjoint realization of $-\Delta_G$ on $L^2(\mathbb{R}^d)$. The solutions of (1.1) are given by the unitary group e^{-itP} via the functional calculus of self-adjoint operators. The solutions of (1.1) satisfy

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}\,. \tag{1.2}$$

It follows from the explicit representation of the fundamental solution of $e^{it\Delta}$ that in the case $P = -\Delta = -\sum_j \partial_j^2$ (ie $G = \mathrm{Id}$) one has

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R}^d)} \leq C|t|^{-d/2}\|u_0\|_{L^1(\mathbb{R}^d)}, \tag{1.3}$$

which shows that, if in addition $u_0 \in L^1(\mathbb{R}^d)$, then the solution of (1.1) satisfies, for $p > 2$,

$$\lim_{|t|\to\infty} \|e^{it\Delta}u_0\|_{L^p(\mathbb{R}^d)} = 0\,. \tag{1.4}$$

Therefore $e^{it\Delta}$ enjoys a remarkable *dispersive property*, if we accept to replace $L^2(\mathbb{R}^d)$ by other phase spaces like $L^p(\mathbb{R}^d)$, $p > 2$.

This paper fits in the line of research studying the possible extensions of the dispersive properties of $e^{it\Delta}$ to e^{-itP}. A famous way to display the dispersive properties of e^{-itP} is via the classical local energy decay estimates, under a non trapping condition. Let us recall the local energy decay estimates. First, we assume that Δ_G is a long range perturbation of Δ, namely

$$\exists\, \nu > 0,\, \exists\, R_0 \geq 0,\ \forall\, \alpha \in \mathbb{N}^d,\, \exists\, C > 0,\, \forall\, |x| \geq R_0,\ \ |\partial^\alpha \left(G_{jk}(x) - \delta_{jk}\right)| \leq C\langle x\rangle^{-\nu-|\alpha|}, \tag{1.5}$$

where $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$, δ_{jk} is the Kronecker symbol and $\nu > 0$ is a real number (when $\nu > 1$, we deal with a short range perturbation of $-\Delta$). Let us remark that since G is a smooth metric (1.5) holds for $R_0 = 0$. The important point we wish to stress in assumption (1.5) is that G is close to Id near infinity only. Next, we make a global assumption. Namely

$$G \ \text{ is non trapping}, \tag{1.6}$$

which means that $|\exp_x^G(tv)| \to \infty$ as $|t| \to \infty$ for all $x \in \mathbb{R}^d$ and $v \in T_x\mathbb{R}^d \setminus 0$. We shall assume (1.6) throughout this paper. It is well known that, under our assumptions, the spectrum of P is $\mathrm{spec}(P) = [0, +\infty)$ and contains no singular continuous component (see [20]). It is also expected that the pure point spectrum $\mathrm{spec}_{\mathrm{pp}}(P)$, which is the closure of the set of eigenvalues of P, is empty. The latter is true in the short range case [8], without the non trapping assumption. In the long range case with the non trapping condition, it is also well known that, for some $E_0 > 0$ large enough, $\mathrm{spec}_{\mathrm{pp}}(P) \cap [E_0, +\infty)$ is empty (by the virial Theorem of [20] with the conjugate operator constructed in [24]). In other words,

$$\mathrm{spec}(P) \cap [E_0, +\infty) = \mathrm{spec}_{\mathrm{ac}}(P) \cap [E_0, +\infty), \tag{1.7}$$

for *all* $E_0 > 0$ if $\nu > 1$ and, at least, for some $E_0 > 0$ if $\nu > 0$ and G is non trapping. Let us choose $f_{\mathrm{ac}} \in C^\infty(\mathbb{R})$ such that

$$\mathrm{supp}(f_{\mathrm{ac}}) \subset [E_0, +\infty) \qquad \text{and} \qquad f_{\mathrm{ac}}(E) = 1 \ \text{ for } E \gg 1 \ \ (\text{for instance } E \geq 2E_0), \tag{1.8}$$

with $E_0 > 0$ such that (1.7) holds. The local energy decay reads :

$$\forall\, s > s' \geq 0,\ \exists\, C > 0\ :\ \forall\, t \in \mathbb{R},\ \|\langle x\rangle^{-s} e^{-itP} f_{\mathrm{ac}}(P) \langle x\rangle^{-s}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq C\langle t\rangle^{-s'}, \tag{1.9}$$

and is a consequence of the semi-classical local energy decay (2.7). In particular, if in (1.1) u_0 is such that $\langle x\rangle^s u_0 \in L^2(\mathbb{R}^d)$ then for every compact set $K \subset \mathbb{R}^d$,

$$\lim_{|t|\to\infty} \|e^{-itP} f_{\mathrm{ac}}(P) u_0\|_{L^2(K)} = 0\,. \tag{1.10}$$

Let us observe that assertion (1.10) is weaker than

$$\lim_{|t|\to\infty} \|e^{-itP} f_{\mathrm{ac}}(P) u_0\|_{L^p(\mathbb{R}^d)} = 0 \tag{1.11}$$

for some $p > 2$ since we may bound the norm in $L^2(K)$ by the norm in $L^p(K)$. Moreover, (1.11) displays a *global in space dispersive property*. On the other hand, we should also observe that (1.10) can be replaced by a quantitative bound for the convergence rate. Let us also remark that in (1.10), we can not have $K = \mathbb{R}^d$ because of the conservation law (1.2).

The goal of this paper is to show that in the cases where we have the semi-classical local energy decay (2.7), we can have the global in space dispersive property (1.11). In fact, we are going to prove that we have the following global in time Strichartz estimates.

Theorem 1.1. *Let us fix a Schrödinger admissible pair (p,q), ie such that*

$$\frac{2}{p}+\frac{d}{q}=\frac{d}{2}, \qquad (p,q)\neq(2,\infty). \tag{1.12}$$

Then under the assumptions (1.5), (1.6) and (1.8), there exists $C>0$ such that for all $u_0\in L^2(\mathbb{R}^d)$,

$$\left\|e^{-itP}f_{\mathrm{ac}}(P)u_0\right\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))}\leq C\|u_0\|_{L^2(\mathbb{R}^d)}\,. \tag{1.13}$$

Thanks to the L^2 nature of the right hand-side of (1.13), we may replace $f_{\mathrm{ac}}(P)$ by the characteristic function of an interval $[\alpha,\infty)$, $\alpha>E_0$. However, the problem of treating the long time behavior under the evolution e^{-itP} of the low frequencies, namely considering $e^{-itP}\chi_{[0,\alpha]}(P)u_0$, remains a challenging issue both in the context of the local energy decay or the global in time Strichartz estimates.

We expect that our proof of Theorem 1.1 could be applied to prove the same estimates for self-adjoint operators of the form $-\Delta_G+A(x)\cdot\nabla+V$ with long range A and V. The proof would be essentially the same, up to some technical modifications (like considering h dependent phases in the Isozaki-Kitada parametrix) which could however be an obstacle to the clarity of the exposition. This is the reason why we consider metric perturbations.

In principle, the method of proof of Theorem 1.1 would also give global Strichartz estimates with spectrally cutoff data for metric perturbations of the wave equation, posed on the Euclidean space. Indeed, all constructions we use have natural analogues in the context of the wave equation.

Let us observe that the result of Theorem 1.1 implies that if $u_0\in H^2(\mathbb{R}^d)$ then we have (1.11) at least for $2<p\leq\frac{2(d+2)}{d}$. Indeed using the equation solved by $u=e^{-itP}f_{\mathrm{ac}}(P)u_0$, we obtain that u_t enjoys global integrability properties similar to u and thus the function

$$t\longmapsto\|u(t,\cdot)\|^{\frac{2(d+2)}{d}}_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}$$

is integrable together with its derivative. This implies (1.11) for $p=\frac{2(d+2)}{d}$. The case $2<p\leq\frac{2(d+2)}{d}$ then can be treated by interpolation with the conservation law (1.2).

Let us remark that, by applying the Sobolev embedding to the low frequency part of $e^{-itP}u_0$, Theorem 1.1 imply all previously known *local in time* Strichartz estimates for variable (smooth) coefficients Schrödinger operators of [29, 13, 22, 4]. Let us mention the recent paper [30], where global in time Strichartz estimates for Schrödinger operators are studied. In [30], no low frequency cut-off is needed, but the assumptions on the metric are in the whole space in contrast with the situation considered here (recall that (1.5) is an assumption at infinity). Therefore the assertion of Theorem 1.1 and the result of [30] do not overlap.

Let us recall that in the case of *compactly supported perturbations* of $-\Delta$, we can obtain the global in time Strichartz estimates, without the low frequency cut-off, by the method of [29] and the resolvent estimate of the appendix of [5] (see also [6, 26]).

We also mention the recent papers [10, 32] and references therein which study non compactly supported first order perturbations of $-\Delta$. However, we don't see how the perturbative approaches of these papers could be applied to second order perturbations.

We end this introduction by giving a rough explanation of the method to prove our result. Thanks to previous works, the main issue is to control $e^{-itP}f_{\mathrm{ac}}(P)u_0$ outside a large ball. By some microlocalizations and the well-known duality $TT^\star$ argument, the main point is to prove that $\chi_+e^{-itP}f_{\mathrm{ac}}(P)\chi_+$ acts from L^1 to L^∞ with a norm bounded by $C|t|^{-d/2}$, where χ_+ localizes in a domain of the phase space included in the exterior of a large ball, in a fixed semi-classical frequency region, and in positions x of the physical space avoiding the opposite of the corresponding frequency ξ ($\cos(x,\xi) \neq -1$). Using the Isozaki-Kitada parametrix, we split $\chi_+ e^{-itP} f_{\rm ac}(P)\chi_+$ into a sum of 4 terms. The first term is represented by an oscillatory integral very similar to the one involved in the definition of $e^{it\Delta}$ and thus enjoys the dispersive bound (1.3) (an analysis already performed in [4], see also Section 4 below). The second one can be controlled fairly directly by the local energy decay. The third one has essentially the structure

$$\chi_+ \int_0^t e^{-i(t-\tau)P} f_{\rm ac}(P)\langle x\rangle^{-N} e^{i\tau\Delta}\chi_+ d\tau, \quad N \gg 1 \tag{1.14}$$

and the 4th one behaves essentially like

$$\chi_+ \int_0^t e^{-i(t-\tau)P} f_{\rm ac}(P)\tilde{\chi}_- e^{i\tau\Delta}\chi_+ d\tau, \tag{1.15}$$

where $\tilde{\chi}_-$ localizes in a zone such that, in contrast with χ_+, the localization is in positions of the physical space x such that the corresponding frequency ξ is essentially opposite ($\cos(x,\xi) \sim -1$). Basically the (outgoing) Isozaki-Kitada parametrix provides an approximation of $e^{-itP} f_{\rm ac}(P)\chi_+$ for $t \geq 0$. Therefore by duality, *a second use of the Isozaki-Kitada parametrix* and the local energy decay, we obtain a control on $\chi_+ e^{-i(t-\tau)P} f_{\rm ac}(P)\langle x\rangle^{-N}$, $t-\tau \leq 0$. This estimate combined with essentially free dispersion estimates provides a bound for (1.14) as far as $t \leq 0$. Amazingly enough, the estimate for $\chi_+ e^{-itP} f_{\rm ac}(P)\langle x\rangle^{-N}$, $t \leq 0$, we have just described and a use of the (incoming) Isozaki-Kitada parametrix, provides a propagation estimate for $\chi_+ e^{-i(t-\tau)P} f_{\rm ac}(P)\tilde{\chi}_-$, $t-\tau \leq 0$. Next, once again by free dispersion estimates, we obtain a control on (1.15) for $t \leq 0$. Finally, by the duality trick of [4], we deduce a control on $\chi_+ e^{-itP} f_{\rm ac}(P)\chi_+$ for positive times too.

We emphasize that the propagation estimates involved in this analysis are essentially well known ([21, 17, 15, 19] in the non semiclassical setting, [33] for semiclassical Schrödinger operators). They were however introduced for L^2 purposes, in contrast with the $L^1 \to L^\infty$ bounds considered here. For that reason and since they are rather straightforward consequences of the local energy decay and the Isozaki-Kitada parametrix, we prove them in Section 4, in the semiclassical case for metrics.

The rest of this paper is organized as follows. In the next section, we fix the pseudo-differential framework, we state the functional calculus for P in this framework, we recall the estimates for the resolvent of P (on the real axis) and its derivatives. Then we recall the classical consequences of these estimates, namely the local energy decay and the local (in space) smoothing effect. In Section 3, we perform the well-known reduction to a fixed frequency and the exterior of a large ball. In Section 4, we describe the Isozaki-Kitada parametrix in a form suitable to our purposes. We then derive the propagation estimates needed for the proof of our result. Finally, in Section 5, we complete the proof of our global Strichartz estimate.

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2 Functional calculus and propagation estimates

In this section, we record some well known results used in scattering theory.

We consider the symbol class $S_{\rm scat}(\mu, m)$, with $\mu, m \in \mathbb{R}$, which is the space of smooth functions on $\mathbb{R}^{2d}$ satisfying

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha\beta}\langle x\rangle^{\mu-|\alpha|}\langle\xi\rangle^{m-|\beta|}.$$

It is a Fréchet space for the semi-norms given by the best constants $C_{\alpha\beta}$. We will also need $S_{\rm scat}(\mu,-\infty) := \cap_{m\in\mathbb{R}} S_{\rm scat}(\mu,m)$. To any symbol $a \in S_{\rm scat}(\mu,m)$ and $h \in (0,1]$, we can associate the operator $Op_h(a)$ defined by

$$Op_h(a)u(x) = (2\pi h)^{-d} \iint e^{ih^{-1}(x-y)\cdot\xi} a\,(x,\xi)\, u(y) dy d\xi\,.$$

We recall that, if $a \in S_{\rm scat}(\mu_1,m_1)$ and $b \in S_{\rm scat}(\mu_2,m_2)$, then for all $N \geq 0$

$$Op_h(a)Op_h(b) = \sum_{j\leq N-1} h^j Op_h((a\#b)_j) + h^N Op_h(r_N(h)),$$

with

$$(a\#b)_j = \sum_{|\alpha|=j} (\partial_\xi^\alpha a)(D_x^\alpha b)/\alpha! \in S_{\rm scat}(\mu_1+\mu_2-j, m_1+m_2-j), \tag{2.1}$$

$$(r_N(h))_{h\in(0,1]} \ \text{ bounded in } \ S_{\rm scat}(\mu_1+\mu_2-N, m_1+m_2-N). \tag{2.2}$$

The latter is completely standard and follows from the symbolic calculus associated to the Hörmander metric $dx^2/(1+|x|^2) + d\xi^2/(1+|\xi|^2)$ [14, Sec. 18.5] and [23] (see also [1, App. A.1] for an elementary proof). A similar result holds for the adjoint $Op_h(a)^*$.

We then have

$$h^2 P = Op_h(p_2) + hOp_h(p_1) \qquad \text{with} \ \ p_{2-j} \in S_{\rm scat}(-j, 2-j), \quad j=1,2, \tag{2.3}$$

where p_{2-j} are homogeneous polynomial of degree $2-j$ with respect to ξ. Of course p_2 is the principal symbol in the usual sense, namely

$$p_2(x,\xi) = \sum_{j,k=1}^d G^{jk}(x)\xi_j\xi_k.$$

Note also that actually $p_1 \in S_{\rm scat}(-\nu-1, 1)$.

Proposition 2.1. *For all $\phi \in C_0^\infty(\mathbb{R})$, there exists a sequence of symbols $a_{\phi,j} \in S_{\rm scat}(-j,-\infty)$ such that, for all $N \geq 0$,*

$$\phi(h^2P) = \sum_{j=0}^{N-1} h^j Op_h(a_{\phi,j}) + h^N R_{\phi,N}^P(h),$$

with $a_{\phi,0} = \phi \circ p_2$, $\operatorname{supp}(a_{\phi,j}) \subset \operatorname{supp}(a_{\phi,0})$ and, for all $q \in [1,\infty]$,

$$\left\| \langle x\rangle^{N/2} R_{\phi,N}^P(h) \langle x\rangle^{N/2} \right\|_{L^q(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \lesssim 1, \qquad h \in (0,1],$$

and all $s \geq 0$,

$$\left\| \langle x\rangle^{N/2} R_{\phi,N}^P(h) \langle x\rangle^{N/2} \right\|_{H^{-s}(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \lesssim h^{-2s}, \qquad h \in (0,1].$$

The proof follows the lines of [4, Proposition 2.5], exploiting (2.2) and the fact that $\langle x\rangle^s(h^2P - z)\langle x\rangle^{-s}$ is bounded on $L^2(\mathbb{R}^d)$, for all $s \in \mathbb{R}$, with norm controlled by a power of $\langle z\rangle/|\mathrm{Im}(z)|$.

Here, the important point is that the terms of the expansion and the remainder term decay faster and faster with respect to x. This will be convenient to use the propagation estimates which we now recall.

Setting $R(z,h) = (h^2P - z)^{-1}$, it is well known that the boundary values $R(\lambda \pm i0, h)$ exist in weighted spaces, if $h^{-2}\lambda \geq E_0$, hence for all $h \in (0,1]$ and $\lambda \geq E_0$. They are smooth with respect to λ and $\partial_\lambda^k R(\lambda \pm i0, h) = k! R^{k+1}(\lambda \pm i0, h)$. This follows from [17]. Furthermore, for all $k \geq 0$ and $s > k + 1/2$, the following estimates hold locally uniformly with respect to λ,

$$\big\| \langle x \rangle^{-s} R^{k+1}(\lambda \pm i0, h) \langle x \rangle^{-s} \big\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \lesssim h^{-1-k}, \qquad h \in (0,1], \ \ \lambda \geq E_0 h^2. \tag{2.4}$$

When $h = 1$, which is the framework of [17], these estimates do not rely on the non trapping assumption but the control as $h \to 0$ requires the assumption (1.6) and the proof of (2.4) is given for instance in [24, 25], using an idea of [11].

Choose now $\phi \in C_0^\infty((0,+\infty))$. Then, as long as,

$$\operatorname{supp}([\lambda \mapsto \phi(h^2\lambda)]) \subset [E_0, +\infty), \tag{2.5}$$

which holds either for arbitrary $\phi \in C_0^\infty((0,+\infty))$ and h small enough, or for all $\phi \in C_0^\infty([E_0,+\infty))$ and $h = 1$, it is well known that (2.4) implies that, for all $s > 1/2$,

$$\int_{\mathbb{R}} \| \langle x \rangle^{-s} e^{-ithP} \phi(h^2P) u_0 \|^2_{L^2(\mathbb{R}^d)} dt \lesssim \| u_0 \|^2_{L^2(\mathbb{R}^d)}. \tag{2.6}$$

This is the semiclassical version of the global (in time) smoothing effect (see [9]). It basically follows from (2.4) with $k = 0$ by a Fourier transform $\mathcal{F}_{\lambda \to t}$.

We also have the following local energy decay. By the Stone formula, namely

$$e^{-ithP}\phi(h^2P) = \int e^{-it\lambda/h} \phi(\lambda) \left(R(\lambda + i0, h) - R(\lambda - i0, h) \right) \frac{d\lambda}{2i\pi},$$

the estimates (2.4) and integrations by part, namely

$$(it)^k e^{-ithP}\phi(h^2P) = \sum_{j \leq k} \frac{k! h^k}{j!} \int e^{-it\lambda/h} \phi^{(j)}(\lambda) \left(R^{k-j+1}(\lambda + i0, h) - R^{k-j+1}(\lambda - i0, h) \right) \frac{d\lambda}{2i\pi},$$

prove that, for all integer $N \geq 1$,

$$|| \langle x \rangle^{-N} e^{-ithP} \phi(h^2P) \langle x \rangle^{-N} ||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_N h^{-1} \langle t \rangle^{1-N}, \qquad t \in \mathbb{R}. \tag{2.7}$$

Let us note that (2.6) and (2.7) are uniform with respect to h such that (2.5) is satisfied. We also remark that (2.7) can be improved in the following way: for all $s > s' \geq 0$ and all $\epsilon > 0$

$$\big\| \langle x \rangle^{-s} \phi(h^2P) e^{-ithP} \langle x \rangle^{-s} \big\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \lesssim h^{-\epsilon} \langle t \rangle^{-s'}, \qquad t \in \mathbb{R}. \tag{2.8}$$

This follows from $|| \langle x \rangle^{-\theta N} e^{-ithP} \phi(h^2P) \langle x \rangle^{-\theta N} || \leq C_{N,\theta} h^{-\theta} \langle t \rangle^{\theta(1-N)}$ which is obtained by interpolation with $\theta \in (0,1]$ small enough such that $0 < \theta \leq \epsilon$, $s'/\theta \in \mathbb{N}$ and $N := s'/\theta + 1$ since, in that case, $N\theta \leq s$. We have to mention that the power $h^{-\epsilon}$ can actually be removed. This was proved by Wang for semiclassical Schrödinger operators in [33] and this proof can be adapted to the case of metrics using the propagation estimates displayed in Proposition 4.5 below. Therefore, removing the $h^{-\epsilon}$ in (2.8) is a byproduct of [33] and the analysis of Section 4 in this paper. Wang even showed that (1.6) was necessary to obtain the local energy decay (see also [24]).

However, we emphasize that we won't need (2.8) nor its version with $\epsilon = 0$ in this paper. The *a priori* estimates (2.7) are largely sufficient for our present purposes since they will appear only in remainder terms where we shall have arbitrary large powers of h.

3 Reduction of the problem

We first recall the classical frequency localization by the Littlewood-Paley theory. Consider the following dyadic partition of unity, with $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty((0,+\infty))$,

$$1 = \varphi_0(\lambda) + \sum_{k=0}^{\infty} \varphi(2^{-k}\lambda), \qquad \lambda \geq \inf \operatorname{spec}(P). \tag{3.1}$$

Lemma 3.1. *For all real number $q \geq 2$,*

$$\|u\|_{L^q(\mathbb{R}^d)} \lesssim \Big(\|\varphi_0(P)u\|^2_{L^q(\mathbb{R}^d)} + \sum_{k=0}^{\infty} \|\varphi(2^{-k}P)u\|^2_{L^q(\mathbb{R}^d)} \Big)^{1/2}.$$

This result is essentially standard and can be proved similarly to the case of the flat Laplacian using Proposition 2.1. Recall that (see e.g. [28]) the point (modulo the Minkowski inequality) is to control the action of the linear map

$$r_{-1}\varphi_0(P) + \sum_{k\geq 0} r_k \varphi(2^{-k}P)$$

on $L^q(\mathbb{R}^d)$, $q \in (1,\infty)$, with $(r_k)_{k\geq -1}$ the classical Rademacher sequence. The kernel of this operator splits into two parts. The principle term is explicit (enjoying the same bounds as in the flat case) and thus the corresponding operator satisfies the hypotheses of the Mikhlin-Hörmander Theorem (see e.g. [31]). The remainder term acts boundedly on $L^q(\mathbb{R}^d)$ for all $q \in [1,\infty]$, by Proposition 2.1. Notice that Lemma 3.1 is more precise than the "soft" version used in [7, 4] where an extra $\|u\|_{L^2}$ term was allowed on the right hand side. We also refer to [3] for results of this type in a more general context.

We next add a spatial localization. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ and set $u = u(t,x) = e^{-itP} f_{\mathrm{ac}}(P)u_0$. By Lemma 3.1 and the Minkowski inequality, we have

$$\|\chi u\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \lesssim \Big(\|\varphi_0(P)\chi u\|^2_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} + \sum_{k=0}^{\infty} \|\varphi(2^{-k}P)\chi u\|^2_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \Big)^{1/2}, \tag{3.2}$$

where (p,q) satisfies (1.12). Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$ such that $\tilde{\varphi} = 1$ near $\operatorname{supp}(\varphi)$. Then

$$\varphi(h^2P)\chi = \tilde{\varphi}(h^2P)\chi\varphi(h^2P) + [\varphi(h^2P),\chi]\tilde{\varphi}(h^2P) + \big[\tilde{\varphi}(h^2P),[\varphi(h^2P),\chi]\big]. \tag{3.3}$$

By Proposition 2.1 (see also [4]), we have, for all $s \geq 0$,

$$\begin{aligned} \|\tilde{\varphi}(h^2P)\|_{L^q(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} &\lesssim 1, \\ \|[\varphi(h^2P),\chi]\langle x\rangle^s\|_{L^2(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} &\lesssim 1, \\ \|\big[\tilde{\varphi}(h^2P)[\varphi(h^2P),\chi]\big]\langle x\rangle^s\|_{L^2(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} &\lesssim h, \end{aligned}$$

uniformly in $h \in (0,1]$. The same estimates hold for $h = 1$ with φ_0 instead of φ and some $\tilde{\varphi}_0 \in C_0^\infty(\mathbb{R})$ instead of $\tilde{\varphi}$. Using (3.2), we therefore obtain

$$\begin{aligned} \|\chi u\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \lesssim \; & \Big(\|\chi\varphi_0(P)u\|^2_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} + \sum_{k=0}^{\infty} \|\chi\varphi(2^{-k}P)u\|^2_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \Big)^{1/2} \\ & + \Big(\|\langle x\rangle^{-s}\tilde{\varphi}_0(P)u\|^2_{L^p(\mathbb{R};L^2(\mathbb{R}^d))} + \sum_{k=0}^{\infty} \|\langle x\rangle^{-s}\tilde{\varphi}(2^{-k}P)u\|^2_{L^p(\mathbb{R};L^2(\mathbb{R}^d))} \Big)^{1/2} \\ & + \big\|\langle x\rangle^{-s}u\big\|_{L^p(\mathbb{R};L^2(\mathbb{R}^d))}. \end{aligned}$$

Using (2.6), by interpolating the $L^p(\mathbb{R})$ norms between $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$, the second and third lines are bounded by $C\|u_0\|_{L^2(\mathbb{R}^d)}$, using also the fact $\sum_k \|\tilde{\varphi}(2^{-k}P)u_0\|^2 \lesssim \|u_0\|^2$ by almost orthogonality.

Note finally that the same estimates hold with χ replaced by $1-\chi$ (the commutators are the same as those of (3.3) up to the signs).

All this leads to the following reduction.

Proposition 3.2. *If, for some $\chi \in C_0^\infty(\mathbb{R}^d)$ and for all $\phi \in C_0^\infty((0,+\infty))$, we have*

$$\|\chi e^{-itP}\phi(h^2P)u_0\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}, \tag{3.4}$$

$$\|(1-\chi)e^{-itP}\phi(h^2P)u_0\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}, \tag{3.5}$$

uniformly with respect to h such that (2.5) holds, then Theorem 1.1 holds true.

This proposition is a direct consequence of the calculations above using the trivial remarks that $\varphi(2^{-k}P)f_{\rm ac}(P) = \phi_k(P)$ with $\mathrm{supp}(\phi_k) \in [E_0,+\infty)$ for all $k \geq 0$ and that $\varphi(2^{-k}P)f_{\rm ac}(P) = \varphi(2^{-k}P)$ for $k \gg 1$, and similar ones for $\tilde{\varphi}$, φ_0 and $\tilde{\varphi}_0$.

The crucial point to prove (3.4) is the following one.

Proposition 3.3. *For all $s \geq 0$, there exists $C>0$, such that, for all $T>0$ and $h \in (0,1]$ satisfying (2.5),*

$$\|\langle x\rangle^{-s}e^{-itP}\phi(h^2P)u_0\|_{L^p((-T,T);L^q(\mathbb{R}^d))} \leq Ch^{-1/2}\|\langle x\rangle^{-s}e^{-itP}\phi(h^2P)u_0\|_{L^2((-T,T);L^2(\mathbb{R}^d))} + ||u_0||_{L^2}.$$

This result follows from [29, 7] (see also [4]). Note that χ is arbitrary. On the other hand, (2.6) implies that

$$\|\chi e^{-itP}\phi(h^2P)u_0\|_{L^2(\mathbb{R};L^2(\mathbb{R}^d))} \lesssim h^{1/2}\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{3.6}$$

Therefore, Proposition 3.3 and (3.6) imply (3.4). This argument was first used in [29].

To treat the non compactly supported terms, namely to prove (3.5), we shall use the Isozaki-Kitada parametrix in a sharper version than in [4]. This is the purpose of the next section.

4 A review of the Isozaki-Kitada parametrix

If $R>0$, $I \Subset (0,+\infty)$ is an open relatively compact interval and $\sigma_\pm \in (-1,1)$, we set

$$\Gamma^\pm(R,I,\sigma_\pm) = \{(x,\xi) \in \mathbb{R}^{2d} \ ; \ |x|>R, \ |\xi|^2 \in I, \ \ \pm x\cdot\xi > \sigma_\pm |x||\xi|\}.$$

The area $\Gamma^+(R,I,\sigma_+)$ (resp. $\Gamma^-(R,I,\sigma_-)$) is said to be outgoing (resp. incoming). When I and $\sigma_\pm$ are fixed, we can find two families of smooth real valued functions $(S_R^\pm)_{R\gg1}$ satisfying the following stationary Hamilton-Jacobi equation

$$p_2(x,\partial_x S_R^\pm(x,\xi)) = |\xi|^2 \qquad (x,\xi) \in \Gamma^\pm(R,I,\sigma_\pm), \tag{4.1}$$

and the decay estimates

$$|\partial_x^\alpha\partial_\xi^\beta\left(S_R^\pm(x,\xi) - x\cdot\xi\right)| \leq C_{\alpha\beta}\min(R^{1-\nu-|\alpha|}, \langle x\rangle^{1-\nu-|\alpha|}), \qquad (x,\xi)\in\mathbb{R}^{2d}, \ R \gg 1. \tag{4.2}$$

See [16, 25] (and [4] for the R dependence).

Next, for all $a \in S_{\rm scat}(0,0)$, we can define the Fourier integral operator $J_h^{\pm}(a)$ by

$$J_h^{\pm}(a)u(x) = (2\pi h)^{-d} \iint e^{ih^{-1}(S_R^{\pm}(x,\xi) - y\cdot\xi)} a(x,\xi) u(y) dy d\xi.$$

By (4.2), these operators are bounded on $L^2(\mathbb{R}^d)$, uniformly with respect to $h \in (0,1]$ if R is large enough, using the standard Kuranishi argument [23]. More generally, this L^2 boundedness combined with iterations of the following elementary property

$$J_h^{\pm}(a)x_j = J_h^{\pm}\left(a \times \partial_{\xi_j} S_R^{\pm}\right) - hiJ_h^{\pm}\left(\partial_{\xi_j} a\right)$$

and the fact that $\langle x\rangle^{-1} a \times \partial_{\xi_j} S_R^{\pm} \in S_{\rm scat}(0,0)$ prove that, for all integer $M \geq 0$,

$$\left\| \langle x\rangle^{-M} J_h^{\pm}(a) \langle x\rangle^{M} \right\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim 1, \qquad h \in (0,1]. \tag{4.3}$$

The Isozaki-Kitada parametrix is basically an approximation of the form

$$e^{-ithP} Op_h(\chi_\pm) \approx J^{\pm}(a^{\pm}(h)) e^{ith\Delta} J^{\pm}(b^{\pm}(h))^*, \tag{4.4}$$

when χ_+ (resp. χ_-) is a symbol in $S_{\rm scat}(0,-\infty)$ supported in an outgoing (resp. incoming) area. The main purpose of this section is to give a precise sense to this approximation. In [4], we used this parametrix in a range of times of size h^{-1}. Note also that this parametrix has already been used globally in time, ie for $t \in [0, \pm\infty)$, for L^2 problems [16, 12, 24, 25, 2]. Here we want to prove $L^1 \to L^\infty$ estimates and control them globally in time. We therefore need to partially review its construction as well as the related propagation estimates required to control the associated remainder terms.

We can already point out that the interest of the Isozaki-Kitada parametrix for the present paper relies upon the following simple remark. If $a, b \in S_{\rm scat}(0,-\infty)$ with a or b compactly supported in ξ, then for each $R \gg 1$,

$$\| J_h^{\pm}(a) e^{ith\Delta} J_h^{\pm}(b)^* \|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq C_R \min\left(h^{-d}, |th|^{-d/2}\right), \qquad t \in \mathbb{R},\ h \in (0,1]. \tag{4.5}$$

Indeed, by writing the explicit oscillatory integrals giving the kernels of the operators, namely

$$(2\pi h)^{-d} \int e^{ih^{-1}\left(S_R^{\pm}(x,\xi) - t|\xi|^2 - S_R^{\pm}(y,\xi)\right)} a(x,\xi) \overline{b(y,\xi)} d\xi, \tag{4.6}$$

the h^{-d} bound is obvious and the $|th|^{-d/2}$ bound follows by a fairly standard stationary phase estimate (see [4] for the proof) which is valid provided R is large enough. Here we want to emphasize that such bounds hold for $t \in \mathbb{R}$ with no restriction on the sign of t and no other restriction on the supports that either a or b must be compactly supported in ξ.

The estimate (4.5) shows that operators of the form $J_h^{\pm}(a) e^{ith\Delta} J_h^{\pm}(b)^*$ enjoy the same global dispersion estimate as $e^{ith\Delta}$. We shall see below that they also satisfy microlocal propagation estimates similar to the ones of $e^{ith\Delta}$ by simple non stationary phase considerations. For these reasons and for further purposes, we state the following result.

Lemma 4.1. *The following statements hold true :*

- *For all $\sigma_+, \sigma_- \in (-1,1)$ and $x, y, \xi \in \mathbb{R}^d \setminus 0$, we have*

$$\pm \frac{x\cdot\xi}{|x||\xi|} > \sigma_\pm \quad and \quad \pm t \geq 0 \quad \Rightarrow \quad \pm \frac{(x+t\xi)\cdot\xi}{|x+t\xi||\xi|} > \sigma_\pm \quad and \quad |x+t\xi| \geq c_\pm(|x| + |t\xi|), \tag{4.7}$$

with $c_\pm = (1+\sigma_\pm)^{1/2}/2^{1/2}$.

- *If* $\sigma_- + \sigma_+ > 0$ *then there exists* $c = c(\sigma_+, \sigma_-) > 0$ *such that for all* $x, y, \xi \in \mathbb{R}^d \setminus 0$,

$$\frac{x\cdot\xi}{|x||\xi|} > \sigma_+ \quad and \quad -\frac{y\cdot\xi}{|y||\xi|} > \sigma_- \quad \Rightarrow \quad |x-y| \geq c\,(|x|+|y|)\,. \tag{4.8}$$

Proof. We prove (4.7) for $+$ since the $-$ case follows by changing ξ into $-\xi$ and t into $-t$. By possibly changing t into $t|\xi|$ and ξ into $\xi/|\xi|$ we may assume that $|\xi|=1$ and, by rotating the axis, we may choose coordinates on $\mathbb{R}^{2d}$ such that $\xi = (1,0,\dots,0)$. If $x=(x_1,x')$, then

$$\frac{d}{dt}\frac{(x+t\xi)\cdot\xi}{|x+t\xi||\xi|} = \frac{d}{dt}\left(\frac{x_1+t}{((x_1+t)^2+|x'|^2)^{1/2}}\right) = \frac{|x'|^2}{((x_1+t)^2+|x'|^2)^{3/2}} \geq 0$$

proves the first inequality in (4.7). The second one follows easily by computing the difference of the squares of each side.

Let us now prove (4.8). We still may assume that $\xi=(1,0,\dots,0)$. We remark that, on the compact set

$$K = \{(\omega,\omega') \in \mathbb{S}^{d-1}\times\mathbb{S}^{d-1} \ \text{ such that } \ \omega\cdot\xi \geq \sigma_+ \ \text{ and } \ \omega'\cdot\xi \leq -\sigma_-\},$$

we have $\omega\cdot\omega' < 1$. Indeed, if we suppose that $\omega=\omega'$ then $\omega\cdot\xi \leq -\sigma_- < \sigma_+ \leq \omega\cdot\xi$ yields a contradiction. Therefore, there exists $\epsilon > 0$ (depending only on σ_+, σ_-) such that $\omega\cdot\omega' \leq 1-\epsilon$ for $(\omega,\omega') \in K$. Under the assumption of (4.8), $(x/|x|, y/|y|) \in K$ and therefore

$$|x-y|^2 \geq |x|^2+|y|^2 - 2(1-\epsilon)|x||y| \geq \epsilon(|x|^2+|y|^2).$$

This completes the proof of Lemma 4.1. $\square$

Before stating Proposition 4.2 below, summarizing the algebraic relations between the symbols leading to the Isozaki-Kitada parametrix, we need to define special cutoffs. For arbitrary relatively compact open intervals $I_2 \Subset I_1 \Subset (0,+\infty)$ and arbitrary real numbers $-1<\sigma_1<\sigma_2<1$, we can find

$$\chi^\pm_{1\to2}(x,\xi) = \kappa(|x|/R^2)\varrho_{1\to2}(|\xi|^2)\theta_{1\to2}(\pm x\cdot\xi/|x||\xi|) \tag{4.9}$$

satisfying, for all $R \gg 1$,

$$\mathrm{supp}(\chi^\pm_{1\to2}) \subset \Gamma^\pm(R,I_1,\sigma_1), \qquad \chi^\pm_{1\to2} \equiv 1 \ \text{ near } \ \Gamma^\pm(R^2,I_2,\sigma_2).$$

This follows by choosing non decreasing $\kappa, \varrho_{1\to2} \in C^\infty(\mathbb{R})$ and $\theta_{1\to2} \in C_0^\infty(\mathbb{R})$ such that $\kappa(t)=0$ for $t<1/4$ and $\kappa(t)=1$ for $t>1/2$, $\varrho_{1\to2}\equiv1$ near I_2, supported in I_1, and

$$\theta_{1\to2}(t)=0 \ \text{ for } \ t<\sigma_1+\epsilon \qquad \text{and} \qquad \theta_{1\to2}(t)=1 \quad \text{for} \quad t>\sigma_2-\epsilon, \tag{4.10}$$

with $\epsilon \in (0,\sigma_2-\sigma_1)$. Note also that

$$\chi^\pm_{1\to2} \in S_{\mathrm{scat}}(0,-\infty).$$

Proposition 4.2. *Fix first* $I_4 \Subset (0,+\infty)$ *open interval and* $-1<\sigma_4<1$. *Choose arbitrary open intervals* I_1, I_2, I_3 *such that*

$$I_4 \Subset I_3 \Subset I_2 \Subset I_1 \Subset (0,+\infty)$$

and arbitrary real numbers $\sigma_1, \sigma_2, \sigma_3$ *such that*

$$-1 < \sigma_1 < \sigma_2 < \sigma_3 < \sigma_4 < 1.$$

Then, for all R large enough, we can find a sequence of symbols

$$a_j^{\pm} \in S_{\mathrm{scat}}(-j, -\infty), \qquad \mathrm{supp}(a_j^{\pm}) \subset \Gamma^{\pm}(R, I_1, \sigma_1),$$

such that for all

$$\chi_{\pm} \in S_{\mathrm{scat}}(0, -\infty), \qquad \mathrm{supp}(\chi_{\pm}) \subset \Gamma^{\pm}(R^4, I_4, \sigma_4),$$

there exist a second sequence of symbols

$$b_k^{\pm} \in S_{\mathrm{scat}}(-k, -\infty), \qquad \mathrm{supp}(b_k^{\pm}) \subset \Gamma^{\pm}(R^3, I_3, \sigma_3),$$

such that, for all $N \geq 0$, *the symbols*

$$a^{\pm}(h) = a_0^{\pm} + \cdots + h^{N-1} a_{N-1}^{\pm}, \qquad b^{\pm}(h) = b_0^{\pm} + \cdots + h^{N-1} b_{N-1}^{\pm},$$

satisfy:

$$(h^2 P) J_h^{\pm}(a^{\pm}(h)) - J_h^{\pm}(a^{\pm}(h))(-h^2 \Delta) = h^N J_h^{\pm}(r_N^{\pm}(h)) + J_h^{\pm}(\check{a}^{\pm}(h)),$$

where $J_h^{\pm}$ *is given by the phase* $S_R^{\pm}$ *associated to* I_1 *and* σ_1, *and*

$$(r_N^{\pm}(h))_{h \in (0,1]} \quad \textit{bounded in} \quad S_{\mathrm{scat}}(-N, -\infty),$$

and $(\check{a}^{\pm}(h))_{h \in (0,1]}$ *bounded in* $S_{\mathrm{scat}}(0, -\infty)$ *which is a finite sum of the form*

$$\check{a}^{\pm}(h) = \sum_{|\alpha+\beta| \geq 1} \check{a}_{\alpha\beta}^{\pm}(h) \partial_x^{\alpha} \partial_{\xi}^{\beta} \chi_{1 \to 2}^{\pm}, \qquad (\check{a}_{\alpha\beta}^{\pm}(h))_{h \in (0,1]} \ \textit{bounded in} \ \ S_{\mathrm{scat}}(0, -\infty), \tag{4.11}$$

with $\chi_{1 \to 2}^{\pm}$ *given by (4.9), and*

$$Op_h(\chi_{\pm}) = J_h^{\pm}(a^{\pm}(h)) J_h^{\pm}(b^{\pm}(h))^* + h^N Op_h(\tilde{r}_N^{\pm}(h)),$$

with

$$(\tilde{r}_N^{\pm}(h))_{h \in (0,1]} \quad \textit{bounded in} \quad S_{\mathrm{scat}}(-N, -\infty).$$

The proof of Proposition 4.2 follows from the considerations in [25, 1, 2]. By Proposition 4.2 and the Duhamel formula

$$\begin{aligned} e^{-ithP} J_h^{\pm}(a^{\pm}(h)) &- J_h^{\pm}(a^{\pm}(h)) e^{it\Delta} \\ &= ih^{-1} \int_0^t e^{-i(t-\tau)hP} \big((h^2 P) J_h^{\pm}(a^{\pm}(h)) - J_h^{\pm}(a^{\pm}(h))(-h^2 \Delta) \big) e^{i\tau h \Delta} d\tau, \end{aligned}$$

we obtain immediately

$$e^{-ithP} Op_h(\chi_{\pm}) = J_h^{\pm}(a^{\pm}(h)) e^{ith\Delta} J_h^{\pm}(b^{\pm}(h))^* + \sum_{k=1}^{3} \mathcal{R}_k^{\pm}(N, h, t) \tag{4.12}$$

where

$$\mathcal{R}_1^{\pm}(N,h,t) = h^N e^{-ithP} Op_h(\tilde{r}_N^{\pm}(h)), \tag{4.13}$$

$$\mathcal{R}_2^{\pm}(N,h,t) = ih^{N-1}\int_0^t e^{-i(t-\tau)hP} J_h^{\pm}(r_N^{\pm}(h)) e^{i\tau h\Delta} J_h^{\pm}(b^{\pm}(h))^* d\tau, \tag{4.14}$$

$$\mathcal{R}_3^{\pm}(N,h,t) = ih^{-1}\int_0^t e^{-i(t-\tau)hP} J_h^{\pm}(\check{a}^{\pm}(h)) e^{i\tau h\Delta} J_h^{\pm}(b^{\pm}(h))^* d\tau. \tag{4.15}$$

We emphasize that (4.12) is valid for any $t \in \mathbb{R}$ and $h \in (0,1]$ but it will become a parametrix only in regimes where the remainder terms $\mathcal{R}_k^{\pm}(N,h,t)$, $k = 1,2,3$, are "small". As long as this smallness is measured by powers of h, we see that $\mathcal{R}_1^{\pm}(N,h,t)$ and $\mathcal{R}_2^{\pm}(N,h,t)$ behave nicely, regardless the sense of the time (ie the sign of t) but, even locally in time, the sign of t plays a role in the analysis of $\mathcal{R}_3^{\pm}(N,h,t)$. For later purposes, we briefly review this fact.

Let $\chi \in C_0^{\infty}(\mathbb{R}^d)$ such that $\chi(x) = 1$ for $|x| \leq 1$. Then, if R is large enough, we have, for all $M \geq 0$,

$$\left\| \chi(x/R^2) J_h^{\pm}(\check{a}^{\pm}(h)) e^{i\tau h\Delta} J_h^{\pm}(b^{\pm}(h))^* \langle x\rangle^M \right\|_{H^{-M}(\mathbb{R}^d)\to H^M(\mathbb{R}^d)} \lesssim h^M \langle\tau\rangle^{-M}, \qquad \pm\tau \geq 0. \tag{4.16}$$

This is obtained by writing the kernel of this operator which is of the form (4.6) and by a non stationary phase argument using (4.7) which proves that the gradient of the corresponding phase

$$\nabla_{\xi}\left(S_R^{\pm}(x,\xi) - \tau|\xi|^2 - S_R^{\pm}(y,\xi)\right) = x - 2\tau\xi - y + \mathcal{O}(1) \tag{4.17}$$

is bounded from below by $|x|+|y|+|\tau|$ since $|x| \lesssim R^2$ and $|y+\tau\xi| \gtrsim |y|+|\tau| \gtrsim R^3+|\tau|$ for $\pm\tau \geq 0$ and $(y,\xi) \in \Gamma^{\pm}(R^3, I_3, \sigma_3)$.

Similarly, we also obtain that, for all $M \geq 0$,

$$\left\| \langle x\rangle^M (1-\chi)(x/R^2) J_h^{\pm}(\check{a}^{\pm}(h)) e^{i\tau h\Delta} J_h^{\pm}(b^{\pm}(h))^* \langle x\rangle^M \right\|_{H^{-M}(\mathbb{R}^d)\to H^M(\mathbb{R}^d)} \lesssim h^M\langle\tau\rangle^{-M}, \tag{4.18}$$

still for $\pm\tau \geq 0$ and $h \in (0,1]$. We proceed as above noting that, on the support of the amplitude, only the derivatives falling on $\theta_{1\to 2}$ will have a non zero contribution, using (4.10) and (4.11). Thus, on this support we have, $\mp x\cdot\xi \geq -\sigma_2|x||\xi|$ and $\pm y\cdot\xi > \sigma_3|y||\xi|$ with $\sigma_3 - \sigma_2 > 0$. This allows to use (4.8) and then (4.7) to prove that $|(4.17)| \gtrsim |x|+|y|+|\tau|$ which yields the result by integrations by parts.

We next state the following elementary propagation estimates.

Lemma 4.3. *For all $s \in \mathbb{N}$, all N large enough and all $c_N \in S_{\mathrm{scat}}(-N,-\infty)$*

$$\left\| \langle x\rangle^{N/8} J_h^{\pm}(c_N) e^{i\tau h\Delta} J_h^{\pm}(b^{\pm}(h))^* \langle x\rangle^{N/4} \right\|_{H^{-s}(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \lesssim h^{-2s}\langle\tau\rangle^{-N/8}, \qquad \pm\tau \geq 0.$$

Proof. We write the kernel of the operator under the form (4.6). The amplitude reads

$$\langle x\rangle^{N/8} c_N(x,\xi)\overline{b^{\pm}(y,\xi,h)}\langle y\rangle^{N/4} = \mathcal{O}(\langle x\rangle^{-7N/8}\langle y\rangle^{N/4})$$

and is compactly supported in ξ. Using $\chi \in C_0^{\infty}(\mathbb{R}^d)$ such that $\chi \equiv 1$ near 0, we write

$$1 = \chi\left(\partial_{\xi} S_R^{\pm}(x,\xi) - 2\tau\xi - \partial_{\xi} S_R^{\pm}(y,\xi)\right) + (1-\chi)\left(\partial_{\xi} S_R^{\pm}(x,\xi) - 2\tau\xi - \partial_{\xi} S_R^{\pm}(y,\xi)\right),$$

keeping (4.17) in mind. By Peetre's inequality, we have

$$|\chi\left(\partial_{\xi} S_R^{\pm}(x,\xi) - 2\tau\xi - \partial_{\xi} S_R^{\pm}(y,\xi)\right)| \lesssim \langle x\rangle^{3N/4}\langle y+2\tau\xi\rangle^{-3N/4} \lesssim \langle x\rangle^{3N/4}\langle y\rangle^{-N/2}\langle\tau\rangle^{-N/4}, \tag{4.19}$$

using, in the last estimate, that $|y+\tau\xi| \gtrsim |y| + |\tau|$ when $\pm\tau \geq 0$ and $(y,\xi) \in \Gamma^{\pm}(R^3, I_3, \sigma_3)$. Therefore, this kernel is bounded by $h^{-d}\langle x\rangle^{-N/8}\langle y\rangle^{-N/4}\langle \tau\rangle^{-N/4}$ and we can estimate the L^2 norm of the corresponding operator by $h^{-d}\langle\tau\rangle^{-N/4}$, for instance by its Hilbert-Schmidt norm. On the support of $(1-\chi)(\cdots)$, we can integrate by parts and get as many negative powers of $|\partial_\xi S_R^{\pm}(x,\xi) - 2\tau\xi - \partial_\xi S_R^{\pm}(y,\xi)|$ as we want and then estimate them similarly to (4.19). We can then estimate the Hilbert-Schmidt norm as above. This proves the result for $s=0$. In the general case $s \geq 0$, we apply first ∂_x^α, with $|\alpha| \leq s$, on both sides of the operator and repeat the same analysis. $\square$

Note that the last lemma holds in particular with $c_N = r_N^{\pm}(h)$ given by Proposition 4.2. We can summarize the results obtained so far on the remainder terms as follows.

Proposition 4.4. *Under the assumptions of Proposition 4.2, with $\mathcal{R}_k(N,h,t)$, $k=2,3$, defined by (4.14) and (4.15), and for all $0 \leq s \leq d+1$ and all N large enough, we can write*

$$\mathcal{R}_k^{\pm}(N,t,h) = h^{N/2}\int_0^t e^{-i(t-\tau)hP}\langle x\rangle^{-N/8} B_s^{\pm}(N,h,\tau)\langle x\rangle^{-N/4} d\tau, \tag{4.20}$$

$$\left\|B_s^{\pm}(N,h,\tau)\right\|_{H^{-s}(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \lesssim \langle\tau\rangle^{-N/8}, \qquad \pm\tau \geq 0,\ h \in (0,1]. \tag{4.21}$$

Combining this proposition with the local energy decay (2.7), we shall prove the next microlocal propagation estimates.

Proposition 4.5. *Let $\phi \in C_0^\infty((0,+\infty))$, let $I_4 \Subset (0,+\infty)$ an open interval and $-1 < \sigma_4 < 1$. For all R large enough and all $\chi^{\pm} \in S_{\mathrm{scat}}(0,-\infty)$ supported in $\Gamma^{\pm}(R^4, I_4, \sigma_4)$, we have the following estimates uniformly with respect to h such that (2.5) holds:*

- *for all $s \in \mathbb{N}$ and all integer M large enough,*

$$\left\|Op_h(\chi_\pm)^* e^{-ithP}\phi(h^2P)\langle x\rangle^{-M}\right\|_{L^2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \lesssim h^{-s}\langle t\rangle^{-3M/4}, \qquad \pm t \leq 0, \tag{4.22}$$

- *for all $s \in \mathbb{N}$, all $\chi \in C_0^\infty(\mathbb{R}^d)$ and all $M > 0$,*

$$\left\|Op_h(\chi_\pm)^* e^{-ithP}\phi(h^2P)\chi(x/R^2)\right\|_{L^2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \lesssim h^M\langle t\rangle^{-M}, \qquad \pm t \leq 0, \tag{4.23}$$

- *for all $\tilde{\chi}_\mp \in S_{\mathrm{scat}}(0,-\infty)$ supported in $\Gamma^{\mp}(R, I_1, \tilde{\sigma}_1)$, with $1 > \tilde{\sigma}_1 > -\sigma_4$ and $I_4 \Subset I_1$, and for all $M \geq 0$,*

$$\left\|Op_h(\chi_\pm)^* e^{-ithP}\phi(h^2P)Op_h(\tilde{\chi}_\mp)\right\|_{L^\infty(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim h^M\langle t\rangle^{-M}, \qquad \pm t \leq 0. \tag{4.24}$$

Let us point out that the estimates (4.22) and (4.24) are essentially well known. In the non semiclassical case ($h=1$), they follow from [21, 15, 17]. Here we give proofs in the semiclassical case $h \in (0,1]$ for metrics (the case of semiclassical Schrödinger operators being treated in [33]), using the remark that, once we have the Isozaki-Kitada parametrix, they follow rather quickly from (2.7) and elementary non stationary phase considerations.

We shall need a classical lemma describing the action of a pseudo-differential operator on a Fourier integral operator. We omit its proof which follows essentially from [23] (see [1, App.] for the proof in the present context).

Lemma 4.6. *Fix $I \Subset (0,+\infty)$, $\sigma \in (-1,1)$ and consider the associated family of phases $(S_R^{\pm})_{R\gg 1}$. Let $a, c \in S_{\rm scat}(0,-\infty)$. Then, for all $N \geq 0$,*

$$Op_h(c) J_h^{\pm}(a) = \sum_{j=0}^{N-1} h^j J_h^{\pm}(e_j) + h^N J_h^{\pm}(\tilde{e}_N(h)),$$

with $e_j \in S_{\rm scat}(-j,-\infty)$ supported in the intersection of $\mathrm{supp}(a)$ *and the support of*

$$c(x, \partial_x S_R^{\pm}(x,\xi)),$$

and $(\tilde{e}_N(h))_{h\in(0,1]}$ bounded in $S_{\rm scat}(-N,-\infty)$. In particular, for all $J \Subset (0,+\infty)$, $\sigma \in (-1,1)$ and $\epsilon > 0$ small enough, by choosing R large enough, we have

$$\mathrm{supp}(c) \subset \Gamma^{\pm}(R,J,\sigma) \quad \Rightarrow \quad \mathrm{supp}(e_j) \subset \Gamma^{\pm}(R, J + (-\epsilon,\epsilon), \sigma - \epsilon) \tag{4.25}$$

since $\partial_x S_R^{\pm}(x,\xi) = \xi + \mathcal{O}(R^{-\nu})$.

Proof of Proposition 4.5. For clarity, we consider χ_+ and $t \leq 0$. By taking the adjoint, (4.22) is equivalent to

$$\left\| \langle x \rangle^{-M} e^{-ithP} \phi(h^2 P) Op_h(\chi_+) \right\|_{H^{-s}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \lesssim h^{-s} \langle t \rangle^{-3M/4}, \qquad t \geq 0,$$

which we only prove for $s = 0$, the case of an arbitrary s being reduced to this one by noting that $Op_h(\chi_+) = Op_h(\chi_+)\psi(hD)$ for some $\psi \in C_0^\infty(\mathbb{R}^d \setminus 0)$. Using (4.12) with N large enough, (2.7) and Proposition 4.4, we may replace $e^{-ithP}\phi(h^2P)Op_h(\chi_+)$ by $J_h^+(a^+(h))e^{ith\Delta}J_h^+(b^+(h))^*$. The proof of the expected estimate follows similarly to the one of Lemma 4.3 by bounding the kernel of

$$\langle x \rangle^{[d/2]+1-M} J_h^+(a^+(h)) e^{ith\Delta} J_h^+(b^+(h))^* \langle x \rangle^{[d/2]+1}$$

by $\langle t \rangle^{-M+C_d}$ with $C_d = 2[d/2]+2$. Here $[d/2]$ is the integer part of $d/2$. Similarly, we obtain (4.23) by estimating $\chi(x/R^2)J_h^+(a^+(h))e^{ith\Delta}J_h^+(b^+(h))^*$ by non stationary phase estimates. This is due to the fact that one can replace $\check{a}^+(h)$ by $a^+(h)$ since the support of $\check{a}^+(h)$ plays no role in (4.16), the phase being non stationary only thanks to $b^+(h)$ and $\chi(x/R^2)$ (see (4.16) and (4.17)).

Let us now prove (4.24). We use the incoming Isozaki-Kitada parametrix for $\tilde{\chi}_-$, namely (4.12) for $e^{-ithP}Op_h(\tilde{\chi}_-)$ with $t \leq 0$. With obvious notation, by Proposition 4.2, we obtain related symbols $\tilde{a}^-(h)$ supported in $\Gamma^-(R^{1/4}, \tilde{I}_1, \tilde{\sigma}_{1/4})$ and $\tilde{b}^-(h)$ supported in $\Gamma^-(R^{3/4}, \tilde{I}_3, \tilde{\sigma}_{3/4})$ with $\tilde{I}_3 \Subset \tilde{I}_1$ being small neighborhoods of I_1 and $\tilde{\sigma}_{1/4}, \tilde{\sigma}_{3/4}$ that can be chosen so that

$$-1 < -\sigma_4 < \tilde{\sigma}_{1/4} < \tilde{\sigma}_{3/4} < \tilde{\sigma}_1 < 1. \tag{4.26}$$

Once multiplied by $Op_h(\chi_+)^*\phi(h^2P)$, the corresponding remainder terms $\tilde{\mathcal{R}}_1, \tilde{\mathcal{R}}_2, \tilde{\mathcal{R}}_3$ have the appropriate decay using (4.22), Proposition 4.4, standard Sobolev embeddings and the fact that $\langle x \rangle^{-N/2} L^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ if N is large enough. The estimate is therefore reduced to the study of the principal term, namely

$$\left\| Op_h(\chi_+)^* \phi(h^2P) J_h^-(\tilde{a}^-(h)) e^{ith\Delta} J_h^-(\tilde{b}^-(h))^* \right\|_{L^\infty \to L^\infty} \lesssim h^M \langle t \rangle^{-M}, \quad t \leq 0, \ h \in (0,1]. \tag{4.27}$$

By symbolic calculus and Proposition 2.1, we may replace $Op_h(\chi_+)^*\phi(h^2P)$ in (4.27) by $Op_h(c_+)$ with $\mathrm{supp}(c_+) \subset \mathrm{supp}(\chi_+)$. The remainder terms due to Proposition 2.1, which decay as fast as we want in x, will produce operators that we treat using Lemma 4.3. Expanding $Op_h(c_+)J_h^-(\tilde{a}^-(h))$ by Lemma 4.6, the remainder term can again be treated by Lemma 4.3 and we are thus left with the study of oscillatory integrals of the form (4.6) with amplitude supported in a region where

$$\frac{x\cdot\xi}{|x||\xi|} > \sigma_4 - \epsilon_R, \qquad -\frac{y\cdot\xi}{|y||\xi|} > \tilde{\sigma}_{3/4},$$

where $\epsilon_R \to 0$ as $R\to\infty$, using (4.25). For R large enough, we may ensure that $\sigma_4 - \epsilon_R + \tilde{\sigma}_{3/4} > 0$, by (4.26). Thus, by Lemma 4.1, the phase is non stationary and its gradient is bounded from below by $c(|x|+|y|+|t|)$ which allows to integrate by parts and the result follows. □

5 Proof of Theorem 1.1

By Proposition 3.2, Proposition 3.3 and (3.6), it remains to prove (3.5) for some $\chi \in C_0^\infty(\mathbb{R}^d)$. Choose first $\tilde{\phi} \in C_0^\infty((0,+\infty))$ such that $\tilde{\phi}\phi = \phi$. By Proposition 2.1, we can write, for all $s \geq 0$ and $N = N(s)$ large enough,

$$(1-\chi)\tilde{\phi}(h^2P) = \sum_{k=0}^{N} h^k Op_h(a_k)^* + h^{N+1}B_N(h)\langle x\rangle^{-s}$$

where, for each $q \geq 2$,

$$\|B_N(h)\|_{L^2(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \lesssim h^{-d/2}.$$

The contribution of $B_N(h)\langle x\rangle^{-s}$ is therefore easily deduced from (2.6). Choosing χ of the form $\chi(x) = \chi_0(x/R^4)$ with $\chi_0 \in C_0^\infty$ such that $\chi_0(x) = 1$ for $|x| \leq 2$, and using the energy localization of the symbols given by Proposition 2.1, it is therefore sufficient to prove the estimate for operators of the form

$$Op_h(a)^* e^{-ithP}\phi(h^2P)$$

with $a \in S_{\mathrm{scat}}(0,-\infty)$ such that

$$\mathrm{supp}(a) \subset \{(x,\xi)\in\mathbb{R}^{2d} \ ; \ |x| > R^4, \ |\xi|^2 \in I_4\} \tag{5.1}$$

where, by choosing R large enough and $\mathrm{supp}(\tilde{\phi})$ close enough to $\mathrm{supp}(\phi)$, $I_4 \Subset (0,+\infty)$ can be any relatively compact open interval containing $\mathrm{supp}(\phi)$. Choosing a suitable partition of unity, the operator above can be written as

$$\left(Op_h(\chi_-)^* + Op_h(\chi_+)^*\right) e^{-ithP}\phi(h^2P) \tag{5.2}$$

with $\chi_\pm \in S_{\mathrm{scat}}(0,-\infty)$ such that

$$\mathrm{supp}(\chi_+) \subset \Gamma^+(R^4, I_4, -1/2) \qquad \mathrm{supp}(\chi_-) \subset \Gamma^-(R^4, I_4, -1/2),$$

since the right hand side of (5.1) is contained in $\Gamma^+(R^4,I_4,-1/2)\cup\Gamma^-(R^4,I_4,-1/2)$.

Using the uniform boundedness of $Op_h(\chi_\pm)$ on $L^2(\mathbb{R}^d)$, for $h\in(0,1]$, and the usual $TT^\star$ argument of [18], (3.5) will follow from the following result.

Proposition 5.1. *Let $\phi \in C_0^\infty((0,+\infty))$. If R is large enough, then*

$$\left\|Op_h(\chi_\pm)^* e^{-ithP}|\phi|^2(h^2P)Op_h(\chi_\pm)\right\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |ht|^{-d/2}, \qquad \pm t \leq 0,$$

uniformly with respect to h such that (2.5) holds.

By the trick of [4], namely by considering the adjoint, this proposition imply that,

$$\big\|Op_h(\chi_\pm)^* e^{-ithP}|\phi|^2(h^2P)Op_h(\chi_\pm)\big\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |ht|^{-d/2}, \qquad \pm t\geq 0, \tag{5.3}$$

and hence we get the global dispersion estimates

$$\big\|Op_h(\chi_\pm)^* e^{-ithP}|\phi|^2(h^2P)Op_h(\chi_\pm)\big\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |ht|^{-d/2}, \qquad t\in\mathbb{R},$$

uniformly with respect to h such that (2.5) holds. This proves (3.5) and completes the proof of Theorem 1.1 assuming that Proposition 5.1 holds true.

Remark. In [4] we proved local (in time) Strichartz estimates by proving a result analogous to Proposition 5.1 for $0\leq \pm t \lesssim h^{-1}$. In particular we considered times with the opposite signs. Here we will take advantage of the microlocalizations $Op_h(\chi_\pm)^*$ to use Proposition 4.5 for $\pm t\leq 0$. In [4], we didn't assume (1.6) for these estimates and therefore couldn't use Proposition 4.5.

Proof of Proposition 5.1. Here again we only consider χ_+ with $t\leq 0$, the case of χ_- with $t\geq 0$ being completely similar. We write $e^{-ithP}Op_h(\chi_+)$ as (4.12), with N large enough to be chosen. In particular, by Proposition 4.2 with $\sigma_4=-1/2$ and I_4 defined above, we obtain the related symbols $a^+(h), b^+(h), \check{a}^+(h)$ with corresponding $\sigma_1,\sigma_2,\sigma_3$ and I_1,I_2,I_3. We first observe that

$$\big\|Op_h(\chi_+)^*|\phi|^2(h^2P)J_h^+(a^+(h))e^{ith\Delta}J_h^+(b^+(h))^*\big\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |ht|^{-d/2}, \tag{5.4}$$

for $t\in\mathbb{R}$ and $h\in(0,1]$, using (4.5) and the uniform boundedness of $Op_h(\chi_+)^*|\phi|^2(h^2P)$ on $L^\infty(\mathbb{R}^d)$. We are therefore left with the study of

$$Op_h(\chi_+)^*|\phi|^2(h^2P)\mathcal{R}_k(N,h,t), \qquad k=1,2,3,$$

with $\mathcal{R}_k(N,h,t)$ respectively defined by (4.13), (4.14) and (4.15) for $k=1,2,3$.

• $k=1$. If $s>d/2$ and N is large enough, (4.22), the fact that $||\langle x\rangle^N Op_h(\tilde{r}_N^+(h))||_{H^{-s}\to L^2}\lesssim h^{-s}$ and the fact that $Op_h(\chi_+)^* = \psi(hD)Op_h(\chi_+)^*$ for some $\psi\in C_0^\infty(\mathbb{R}^d\setminus 0)$ imply that

$$\big\|Op_h(\chi_+)^*|\phi|^2(h^2P)\mathcal{R}_1(N,h,t)\big\|_{H^{-s}(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \lesssim \langle t\rangle^{-d/2}\lesssim |ht|^{-d/2}.$$

By Sobolev imbeddings, we obtain

$$\big\|Op_h(\chi_+)^*|\phi|^2(h^2P)\mathcal{R}_1(N,h,t)\big\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |ht|^{-d/2}.$$

• $k=2$. Using (4.5) for $\langle x\rangle^N J_h^+(r_N^+(h))e^{i\tau h\Delta}J_h^+(b^+(h))$, (4.22) and Sobolev embeddings, we obtain, by choosing N large enough,

$$\begin{aligned}\big\|Op_h(\chi_+)^*|\phi|^2(h^2P)\mathcal{R}_2(N,h,t)\big\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} &\lesssim h^{N/2}\int_0^t \langle t-\tau\rangle^{-N/2}\min(h^{-d},|h\tau|^{-d/2})d\tau \\ &\lesssim |ht|^{-d/2}.\end{aligned}$$

• $k=3$. We choose $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x)\equiv 1$ for $|x|\leq 2$ and split $\mathcal{R}_3(N,h,t)$ into the following two terms

$$ih^{-1}\int_0^t e^{-i(t-\tau)hP}\chi(x/R^2)J_h^+(\check{a}^+(h))e^{i\tau h\Delta}J_h^+(b^+(h))^*d\tau, \tag{5.5}$$

$$ih^{-1}\int_0^t e^{-i(t-\tau)hP}(1-\chi)(x/R^2)J_h^+(\check{a}^+(h))e^{i\tau h\Delta}J_h^+(b^+(h))^*d\tau. \tag{5.6}$$

Once multiplied to the left by $Op_h(\chi_+)^*|\phi|^2(h^2P)$, (5.5) can be treated similarly to $\mathcal{R}_2(N,h,t)$ using (4.23) instead of (4.22). Note that the precise choice of χ plays no role for this term. It will be important in the analysis of (5.6). For the latter, we need the following lemma.

Lemma 5.2. *Choose $\tilde{\sigma}_1$ such that $-\sigma_2 > \tilde{\sigma}_1 > -\sigma_4$. If R is large enough, we may choose $\tilde{\chi}_- \in S_{\mathrm{scat}}(0,-\infty)$ satisfying $\mathrm{supp}(\tilde{\chi}_-) \subset \Gamma^-(R, I_1, \tilde{\sigma}_1)$ and such that, for all M large enough,*

$$\overline{\phi}(h^2P)(1-\chi)(x/R^2)J_h^+(\check{a}^+(h)) = Op_h(\tilde{\chi}_-)J_h^+(\tilde{e}_M(h)) + h^{M/2}\langle x\rangle^{-M/2}B_M(h) \tag{5.7}$$

with

$$(\tilde{e}_M(h))_{h\in(0,1]} \ \text{bounded in}\ S_{\mathrm{scat}}(0,-\infty) \qquad \text{and} \qquad \|B_M(h)\|_{L^\infty(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim 1.$$

Before proving this lemma, we complete the proof of Proposition 5.1. We rewrite

$$Op_h(\chi_+)^*|\phi|^2(h^2P) = Op_h(\chi_+)^*\phi(h^2P)\overline{\phi}(h^2P),$$

put it to the left of (5.6) and use Lemma 5.2. The term involving $h^{M/2}\langle x\rangle^{-M/2}B_M(h)$ is studied as $\mathcal{R}_2(N,h,t)$ using (4.22). The one involving $Op_h(\tilde{\chi}_-)J_h^+(\tilde{e}_M(h))$ is treated similarly using (4.24). □

Proof of Lemma 5.2. Using Proposition 2.1 and Lemma 4.6, the left hand side of (5.7) is the sum of $\sum_{j\le M-1} h^j J_h^+(e_j)$ with

$$\mathrm{supp}(e_j) \subset \left\{(x,\xi)\in\mathbb{R}^{2d} \ ; \ |x|\ge 2R^2,\ p_2(x,\partial_x S_R^+)\in\mathrm{supp}(\phi),\ (x,\xi)\in\mathrm{supp}(\check{a}^+(h))\right\},$$

and of a remainder term of the form

$$h^M J_h^+(\tilde{e}_M(h)) + h^M\langle x\rangle^{-M/2}R_M(h)\langle x\rangle^{-M/2}J_h^+(\check{a}^+(h)), \tag{5.8}$$

with $(\tilde{e}_M(h))_{h\in(0,1]}$ bounded in $S_{\mathrm{scat}}(-M,-\infty)$ and $||R_M(h)||_{L^\infty\to L^\infty} \lesssim 1$. Using (4.3), the fact that $\langle x\rangle^{-M/2}L^\infty(\mathbb{R}^d)\subset L^2(\mathbb{R}^d)$ and Sobolev imbeddings, we see that if M is large enough $||\langle x\rangle^{M/2}(5.8)||_{L^\infty\to L^\infty}\lesssim h^{M/2}$.

By (4.9) and (4.11), $\check{a}^+(h)$ is a sum of terms vanishing either for $|x|\ge R^2$ or $|\xi|^2\notin I_2$ or $x\cdot\xi/|x||\xi|\ge\sigma_2-\epsilon$, where ϵ is introduced in (4.10). Notice that we do not impose any further assumption on ϵ than $\epsilon\in(0,\sigma_2-\sigma_1)$. By choosing R large enough, we necessarily have $|\xi|^2\in I_2$ since $\partial_x S_R^+ = \xi + \mathcal{O}(R^{-\nu})$ implies that $p_2(x,\partial_x S_R^+(x,\xi)) = |\xi|^2 + \mathcal{O}(R^{-\nu})$ for $|x|\gtrsim R$ and $|\xi|\lesssim 1$. Therefore, on the support of $\check{a}^+(h)$, only the derivatives falling on $\theta_{1\to 2}$ will contribute (see (4.9) and (4.11)) and we have necessarily $x\cdot\xi/|x||\xi|\le\sigma_2-\epsilon$ on $\mathrm{supp}(e_j)$. Thus

$$\mathrm{supp}(e_j)\subset\Gamma^-\left(R^2, I_2, -\sigma_2+\frac{\epsilon}{2}\right). \tag{5.9}$$

Next, choose $\tilde{\sigma}_{3/2}$ and $\tilde{I}_{3/2}$ such that $-\sigma_2>\tilde{\sigma}_{3/2}>\tilde{\sigma}_1>-\sigma_4$ and $I_2\Subset\tilde{I}_{3/2}\Subset I_1$. We now can find $\tilde{\chi}_-$ such that

$$\mathrm{supp}(\tilde{\chi}_-)\subset\Gamma^-(R,I_1,\tilde{\sigma}_1), \qquad \tilde{\chi}_-=1 \ \text{near}\ \Gamma^-(R^{3/2},\tilde{I}_{3/2},\tilde{\sigma}_{3/2}).$$

If R is large enough, by Lemma 4.6 and (4.25) (with $a=e_j$ and $c=1-\tilde{\chi}_-$), all the terms of the expansion of $Op_h(1-\tilde{\chi}_-)J_h^+(e_j)$ vanish so that we only have remainder terms which are of the same form as (5.8). This completes the proof of Lemma 5.2. □

References

[1] J.-M. Bouclet, *Distributions spectrales pour des opérateurs perturbés*, PhD Thesis, Nantes University (2000).

[2] ________, *Spectral distributions for long range perturbations*, J. Funct. Anal. 212, 431-471 (2004).

[3] ________, *Littlewood-Paley decompositions on manifolds with ends*, preprint.

[4] J.-M. Bouclet, N. Tzvetkov, *Strichartz estimates for long range perturbations*, Amer. J. Math., to appear.

[5] N. Burq, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta. Math. 1-29 (1998).

[6] ________, *Estimations de Strichartz pour des perturbations à longue portée de l'opérateur de Schrödinger*, Séminaire X-EDP, Ecole Polytechnique (2002).

[7] N. Burq, P. Gérard, N. Tzvetkov, *Strichartz inequalities and the non linear Schrödinger equation on compact manifolds*, Amer. J. Math. 126, 569-605 (2004).

[8] P. Cotta-Ramusino, W. Krüger, R. Schrader, Ann. I.H.P., Phys. Théor. 31, 43-71 (1979).

[9] S. Doi, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds*, Duke Math. J. 82, 679-706 (1996).

[10] B. Erdogan, M. Goldberg, W. Schlag, *Strichartz and smoothing estimates for Schroedinger operators with large magnetic potentials in* $\mathbb{R}^3$, available at http://arxiv.org/pdf/math.AP/0608699

[11] C. Gérard, A. Martinez, *Principe d'absorption limite pour les opérateurs de Schrödinger à longue portée*, C. R. Acad. Sci. Paris t. 306, sér. I, 121-123 (1988).

[12] ________, *Prolongement méromorphe de la matrice de scattering pour des problèmes à deux corps à longue portée*, Ann. I.H.P., Phys. Théor. 51, 81-110 (1989).

[13] A. Hassel, T. Tao, J. Wunsch, *Sharp Strichartz estimates on non-trapping asymptotically conic manifolds*, Amer. J. Math. 128, 963-1024 (2006).

[14] L. Hörmander, *The analysis of linear partial differential operators III*, Springer Verlag, 1985.

[15] H. Isozaki, H. Kitada, *Micro-local resolvent estimates for 2-body Schrödinger operators*, J. Func. Anal. 57, 270-300 (1984) and *Erratum*, J. Funct. Anal. 62, 336 (1985).

[16] ________ *Modified wave operators with time independent modifiers*, J. Fac. Sci. University of Tokyo, Section I A 32, 77-104 (1985).

[17] A. Jensen, E. Mourre, P. Perry, *Multiple commutator estimates and resolvent smoothness*, Ann. I.H.P., Phys. Théor. 41, 207-225 (1984).

[18] M. Keel, T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120, 955-980 (1998).

[19] C. Kerler, *Perturbations of the Laplacian with variable coefficients in exterior domains and differentiability of the resolvent*, Asymp. Anal. 19, 209-232 (1999).

[20] E. Mourre, *Absence of singular continuous spectrum for certain selfadjoint operators*, Comm. Math. Phys. 78 (1981), no. 3, 391-408.

[21] ________, *Opérateurs conjugués et propriétés de propagation*, Comm. Math. Phys. 91, 279-300 (1983).

[22] L. Robbiano, C. Zuily, *Strichartz estimates for Schrödinger equations with variable coefficients*, Mém. Soc. Math. Fr. (N.S.) No. 101-102, (2005).

[23] D. Robert, *Autour de l'approximation semi-classique*, Progress in mathematics, **68**, Birkhaüser (1987).

[24] ________, *Asymptotique de la phase de diffusion à haute énergie pour les perturbations du second ordre du Laplacien*, An. scient. Éc. Norm. Sup. 25, 107-134 (1992).

[25] ________, *Relative time delay for perturbations of elliptic operators and semi-clasical asymptotics*, J. Funct. Anal. 126, 36-82 (1994).

[26] I. Rodnianski, T. Tao, *Long-time decay estimates for Schrödinger equations on manifolds*, IAS conference proceedings, to appear. (See http://arxiv.org/abs/math.AP/0412416).

[27] H. Smith, C. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. P.D.E. 25, 2171-2183 (2000).

[28] C. Sogge, *Fourier integrals in classical analysis*, Cambridge tracts in Mathematics, (1993).

[29] G. Staffilani, D. Tataru, *Strichartz estimates for a Schrödinger operator with non smooth coefficients*, Comm. PDE 27, 1337-1372 (2002).

[30] D. Tataru, *Parametrices and dispersive estimates for Schroedinger operators with variable coefficients*, preprint. (See http://math.berkeley.edu/ tataru/nlw.html).

[31] M. E. Taylor, *Partial differential equation III, Nonlinear equations*, Springer Verlag (1996).

[32] G. Vodev, *Dispersive estimates of solutions to the Schrödinger equation*, Ann. H.P. 6, 1179-1196 (2005).

[33] X. P. Wang, *Time decay of scattering solutions and Classical trajectories*, Ann. I.H.P. Phys. Théor. 47, 25-37 (1987).