Strichartz inequalities on non compact manifolds

Jean-Marc Bouclet
Institut de Mathématiques de Toulouse

Rencontre Nosevol #3, 9 avril 2014, Rennes

What are Strichartz inequalities ?

Schrödinger-Strichartz estimates

$$i\partial_t u = \Delta u \qquad \Longrightarrow \qquad ||u||_{L^p([0,T],L^q)} \lesssim ||u(0)||_{L^2}$$

if $p, q \geq 2$ satisfy the admissibilty condition

$$p, q \geq 2, \qquad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}.$$

Wave-Strichartz estimates

$$\partial_t^2 u = \Delta u \quad \Longrightarrow \quad ||u||_{L^p([0,T],L^q)} \lesssim ||u(0)||_{H^\gamma} + ||\partial_t u(0)||_{H^{\gamma-1}}$$

under the (sufficient) condition on $p, q \geq 2$ that

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \qquad \gamma = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$$

[Strichartz,Ginibre-Velo]

An explicit example

Consider a wave packet centered at (y, ζ)

$$G_h(x) = \pi^{-n/4} h^{-\frac{\kappa n}{2}} \exp\left(\frac{i}{h}\zeta \cdot (x - y) - \frac{|x - y|^2}{2h^{2\kappa}}\right)$$

By explicit computation:

$$\left|e^{it\Delta} G_h\right| = \pi^{-\frac{n}{4}} h^{-\frac{n\kappa}{2}} \frac{h^{\kappa n}}{(h^{4\kappa} + 4t^2)^{\frac{n}{4}}} \exp\left(-\frac{|x - y - 2t\zeta/h|^2}{2(h^{2\kappa} + 4t^2 h^{-2\kappa})}\right)$$

and

$$\left|\left|e^{it\Delta} G_h\right|\right|_{L^q(\mathbb{R}^n_x)} = c_{qn}\left(h^{2\kappa} + 4t^2 h^{-2\kappa}\right)^{\frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right)}$$

where $c_{qn} = \pi^{\frac{n}{2q} - \frac{n}{4}} (2/q)^{\frac{n}{2q}}$. Using the admissibility condition:

$$\int_0^T \left|\left|e^{it\Delta} G_h\right|\right|^p_{L^q} dt = c^p_{qn} \int_0^{2Th^{-2\kappa}} \frac{1}{1 + \tau^2} d\tau.$$

Why are they useful ?

Non linear Cauchy problem at low regularity, e.g.

$$i\partial_t u + \Delta u = \pm |u|^{\nu-1} u, \qquad u_{|t=0} = u_0 \in L^2(\mathbb{R}^2), \qquad 1 < \nu < 3.$$

Rewrite it as an integral equation

$$u(t) = e^{it\Delta} u_0 \mp i \int_0^t e^{i(t-s)\Delta} |u(s)|^{\nu-1} u(s) ds$$

and use a fixed point argument in a suitable closed ball of

$$X_T := C([0,T], L^2) \cap L^p([0,T], L^q), \qquad p = \frac{2\nu+2}{\nu-1}, \ \ q = \nu + 1.$$

Strichartz inequalities allow to show that $e^{it\Delta} u_0 \in X_T$, and that

$$v \mapsto \int_0^t e^{i(t-s)\Delta} |v(s)|^{\nu-1} v(s) ds \ \text{ is a contraction}$$

for T small enough (this uses *inhomogeneous* inequalities).

Estimates in non Euclidean geometries

Wave equation: weaker dispersion but finite propagation speed

1. M **smooth with positive injectivity radius**: same estimates (local in time) as on $\mathbb{R}^n$ [Kapitanski]
2. M **with boundary**: Additional losses in general [Ivanovici-Lebeau-Planchon]. Unavoidable at least if if $q > 4$ and $n \in \{2, 3, 4\}$ (additional loss of $\frac{1}{6}\left(\frac{1}{4} - \frac{1}{q}\right)$ [Ivanovici])
3. **low regularity metrics**: additional losses in general below C^2 regularity [Bahouri-Chemin, Tataru, Smith-Tataru]

Estimates in non Euclidean geometries (continued)

Schrödinger equation: one expects possible losses

$$||u||_{L^p([0,T],L^q(M))} \lesssim ||u(0)||_{H^\sigma(M)} := ||(1-\Delta)^{\sigma/2}u(0)||_{L^2(M)}$$

(infinite propagation speed!)

1. *M* **closed**: $\sigma = \frac{1}{p}$ [Burq-Gérard-Tzvetkov] (optimal on $\mathbb{S}^3$), but for $M = \mathbb{T}^2$ and $p = q = 4$, any $\sigma > 0$ [Bourgain]!
2. *M* **compact with boundary**: Additional losses in general ($\sigma = \frac{3}{2p}$ [Anton], $\frac{4}{3p}$ [Blair,Smith,Sogge])
3. *M* **non compact with large ends**: No loss if no (or little) trapping; either for *M* asymp. flat or hyperbolic (including: outside a convex [Ivanovici] or polygonal obstacles [Baskin-Marzuola-Wunsch])

About the proof of Strichartz estimates

The classical strategy is to prove $L^1 \to L^\infty$ estimates for the evolution and use the following type of abstract result.

Proposition. Assume

$$\begin{aligned} ||U_h(t)||_{L^2 \to L^2} &\leq B_h, \qquad |t| \leq T \\ ||U_h(t)U_h(s)^*||_{L^1 \to L^\infty} &\leq \frac{D_h}{|t-s|^\delta}, \qquad |t|, |s| \leq T \end{aligned}$$

Then, if $p > 2$, $q \geq 2$ and

$$\delta\left(\frac{1}{2} - \frac{1}{q}\right) = \frac{1}{p},$$

we have

$$||U_h(\cdot)f||_{L^p([0,T],L^q)} \lesssim B_h^{\frac{2}{q}} D_h^{\frac{1}{2}-\frac{1}{q_1}} ||f||_{L^2}$$

About the proof of Strichartz estimates (continued)

Up to a Littlewood-Paley argument, to localize spectrally the problem (with $\varphi \in C_0^\infty(0, +\infty)$), the usual estimates follow from:

Schrödinger

$$\left|\left|\varphi(-h^2\Delta)e^{i(t-s)\Delta}\right|\right|_{L^1(M)\to L^\infty(M)} \lesssim |t-s|^{-\frac{n}{2}}$$

Wave

$$\left|\left|\varphi(-h^2\Delta)e^{i(t-s)\sqrt{-\Delta}}\right|\right|_{L^1(M)\to L^\infty(M)} \lesssim h^{-\frac{n+1}{2}}|t-s|^{-\frac{n-1}{2}}$$

on suitable time scales. Typically, if $\varrho_{\mathrm{inj}} =$ injectivity radius,

$$|t|, |s| \lesssim \varrho_{\mathrm{inj}} \quad \text{(Wave)} \qquad |t|, |s| \lesssim h \times \varrho_{\mathrm{inj}} \quad \text{(Schrödinger)}$$

Problem: what happens if $\varrho_{\rm inj}$ vanishes ?

- are there still Strichartz estimates ?
- if yes, are there additional losses ?
- if yes, are they unavoidable ?

We address these questions for (smooth) **surfaces with cusps.**

Surfaces with cusps

- Model for the cusp:

$$\mathcal{S}_0 = [r_0, \infty) \times \mathcal{A}, \qquad G_0 = dr^2 + e^{-2\phi(r)} d\theta^2,$$

$\mathcal{A}$ = a union of circles and

$$\int_{r_0}^{\infty} e^{-\phi(r)} dr < \infty \qquad i.e. \qquad \text{aera}(\mathcal{S}_0) < \infty$$

We also assume that $\phi^{(j)}$ is bounded for all $j \geq 1$.

- More generally, we can consider $(\mathcal{S}, G)$ with

$$\mathcal{S} = \mathcal{K} \sqcup \mathring{\mathcal{S}}_0, \qquad \text{with } \mathcal{K} \text{ compact and } G = G_0 \text{ on } \mathring{\mathcal{S}}_0 \,.$$

Example: $\mathcal{S} = \mathbb{R} \times \mathbb{S}^1$ with $G = dr^2 + d\theta^2 / \cosh^2(r)$

Operators and measures on $\mathcal{S}_0$

$$\Delta_0 = \frac{\partial^2}{\partial r^2} - \phi'(r)\frac{\partial}{\partial r} + e^{2\phi(r)}\Delta_{\mathcal{A}}, \qquad d\mathrm{vol}_0 = e^{-\phi(r)} dr d\mathcal{A}$$

Δ_0 is symmetric on $L^2_{G_0} := L^2(\mathcal{S}_0, d\mathrm{vol}_0)$. We also let

$$||\psi||_{H^\sigma_{G_0}} = ||(1-\Delta_0)^{\sigma/2}\psi||_{L^2_{G_0}}$$

To use the standard Lebesgue measure, it is useful to consider

$$\mathcal{U} : L^2_{G_0} \ni \psi \mapsto u := \mathcal{U}\psi = e^{-\phi(r)/2}\psi \in L^2 := L^2(\mathcal{S}_0, dr d\mathcal{A}).$$

$$P := \mathcal{U}(-\Delta_0)\mathcal{U}^* = -\frac{\partial^2}{\partial r^2} - e^{2\phi(r)}\Delta_{\mathcal{A}} + w(r),$$

where $w = (\phi'^2 - 2\phi'')/4$. P is symmetric on L^2. Note also that

$$||\psi||_{L^q_{G_0}} = ||e^{\phi(r)\left(\frac{1}{2}-\frac{1}{q}\right)} u||_{L^q}$$

Projection away from zero modes

We let

$$\pi_0 = \text{ orthogonal projection on } \mathrm{Ker}_{L^2(\mathcal{A})}(\Delta_{\mathcal{A}})$$

and define

$$\Pi = I \otimes \pi_0, \qquad \Pi^c = I \otimes (I - \pi_0)$$

seen as operators (orthogonal projections) on both

$$\begin{aligned} L^2((r_0,\infty), dr) \otimes L^2(\mathcal{A}, d\mathcal{A}) &\approx L^2 \\ L^2((r_0,\infty), e^{-\phi(r)}dr) \otimes L^2(\mathcal{A}, d\mathcal{A}) &\approx L^2_{G_0} \end{aligned}$$

If $e_0, \ldots, e_{k_0-1}$ is an orthonormal basis of $\mathrm{Ker}_{L^2(\mathcal{A})}(\Delta_{\mathcal{A}})$,

$$\Pi\psi = \sum_{k<k_0} \left(\int_{\mathcal{A}} \overline{e_k(\alpha)}\psi(r,\alpha) d\mathcal{A} \right) \otimes e_k$$

Zero angular modes ⇒ No Strichartz estimates

Theorem 1 Let $p \geq 1$, $q > 2$ and $\sigma \geq 0$.

1. There is a sequence $(\psi_n)_{n\geq 0}$ in $H^\sigma_{G_0} \cap Ran(\Pi)$ such that

$$\sup_{n\geq 0} \frac{||\psi_n||_{L^q_{G_0}}}{||\psi_n||_{H^\sigma_{G_0}}} = +\infty.$$

2. There is a sequence $(\psi_n)_{n\geq 0}$ of in $H^\sigma_{G_0} \cap Ran(\Pi)$ such that

$$\sup_{n\geq 0} \frac{||\cos(t\sqrt{-\Delta_0})\psi_n||_{L^p([0,1]_t;L^q_{G_0})}}{||\psi_n||_{H^\sigma_{G_0}}} = +\infty.$$

3. Consider $e^{\phi(r)} = e^r$ and $r_0 = 0$. There is a sequence $(\psi_n)_{n\geq 0}$ in $H^\sigma_{G_0} \cap Ran(\Pi)$ such that

$$\sup_{n\geq 0} \frac{||e^{it\Delta}\psi_n||_{L^p([0,1]_t;L^q_{G_0})}}{||\psi_n||_{H^\sigma_{G_0}}} = +\infty.$$

Wave-Strichartz estimates at infinity away from zero angular modes

Let $r_1 > r_0$ and $\mathbb{1}_{[r_1,\infty)}(r)$ be a localization inside the cusp.

Theorem 2 Let (p, q) be sharp wave admissible in dimension two

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$$

and set

$$\sigma_{\mathrm{w}} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right).$$

Then, if we set

$$\Psi(t) = \cos(t\sqrt{-\Delta})\psi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\psi_1,$$

we have

$$\left|\left|\Pi^c \mathbb{1}_{[r_1,\infty)}(r)\Psi\right|\right|_{L^p([0,1];L^q_{G_0})} \lesssim ||\psi_0||_{H_G^{\sigma_{\mathrm{w}}}} + ||\psi_1||_{H_G^{\sigma_{\mathrm{w}}-1}}$$

Schrödinger-Strichartz estimates at infinity away from zero angular modes

Theorem 3 Let (p, q) be Schrödinger admissible

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \qquad \sigma_S = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) = \frac{1}{2p}$$

Fix $\varphi \in C_0^\infty(\mathbb{R})$. Then, if we set

$$\Psi_h(t) = e^{it\Delta}\varphi(-h^2\Delta)\psi$$

we have

$$\left|\left|\Pi^c \mathbb{1}_{[r_1,\infty)}(r)\Psi_h\right|\right|_{L^p([0,h];L^q_{G_0})} \lesssim ||\psi||_{H_G^{\sigma_S}}$$

Corollary Let (p, q) be a Schrödinger admissible pair. If we set

$$\Psi(t) = e^{it\Delta}\psi$$

we have

$$\left|\left|\Pi^c \mathbb{1}_{[r_1,\infty)}(r)\Psi\right|\right|_{L^p([0,1];L^q_{G_0})} \lesssim ||\psi||_{H_G^{\frac{3}{2p}}}$$

Separation of variables

Using an orthonormal eigenbasis $(e_k)_{k\geq 0}$ of $\Delta_{\mathcal{A}}$,

$$\Delta_{\mathcal{A}} e_k = -\mu_k^2 e_k$$

we have a unitary equivalence

$$L^2 \ni u \mapsto (u_k)_k \in \bigoplus_{k\geq 0} L^2\big((r_0,\infty), dr\big), \quad u_k(r) = \int \overline{e_k(\alpha)} u(r,\alpha) d\mathcal{A}$$

Through this mapping, for any bounded Borel function f, we have

$$f(P)u = \sum_k f(\mathfrak{p}_k) u_k \otimes e_k$$

where

$$\mathfrak{p}_k = -\partial_r^2 + \mu_k^2 e^{2\phi(r)} + w(r).$$

Elliptic estimates away from zero angular modes

Proposition Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ near r_0. Then for any $N > 0$

$$\left\|(e^{2\phi(r)}\Delta_{\mathcal{A}})^{N_1}\partial_r^{N_2}\Pi^c(1-\chi(r))(1-\Delta_0)^{-N}\right\|_{L^2_{G_0}\to L^2_{G_0}} < \infty$$

provided that $2N_1 + N_2 \leq 2N$. In particular, for N large enough

$$\left\|e^{N\phi(r)}\Pi^c(1-\Delta_0)^{-N}\right\|_{L^2_{G_0}\to L^\infty_{G_0}} < \infty$$

Localization in frequency: Littlewood-Paley decomposition

Consider a dyadic partition of unity

$$I = \varphi_0(-\Delta_0) + \sum_{h^2=2^{-n}} \varphi(-h^2\Delta_0)$$

with $\varphi_0 \in C_0^\infty(\mathbb{R})$, $\varphi \in C_0^\infty(0,+\infty)$

Proposition. For all $q \in [2,\infty)$ and $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ near r_0,

$$||\Pi^c(1-\chi)\psi||_{L^q_{G_0}} \lesssim \left(\sum_h ||\Pi^c(1-\chi)\varphi(-h^2\Delta_0)\psi||^2_{L^q_{G_0}}\right)^{\frac{1}{2}} + ||\psi||_{L^2_{G_0}}$$

Localization in space

For $r_1 > r_0 + \delta$ with $\delta > 0$, define

$$\mathbb{1}_L = \mathbb{1}_{[r_1+L, r_1+L+1)}, \qquad \widetilde{\mathbb{1}}_L = \mathbb{1}_{[r_1-\delta+L, r_1+1+\delta+L)}$$

Proposition. Let $q \in [2, \infty)$ and $\nu \in \{1, \frac{1}{2}\}$.

$$\left|\left|\Pi^c \mathbb{1}_{[r_1,\infty)}(r)\varphi(-h^2\Delta_0)\psi\right|\right|_{L^q_{G_0}} \leq \left(\sum_L \left|\left|\Pi^c \mathbb{1}_L(r)\varphi(-h^2\Delta_0)\psi\right|\right|^2_{L^q_{G_0}}\right)^{\frac{1}{2}}$$

For $|t| \leq t_0$ small enough independent of L and h,

$$\left|\left|\Pi^c \mathbb{1}_L \varphi(-h^2\Delta_0) e^{i\frac{t}{h}(-h^2\Delta_0)^\nu}(1 - \widetilde{\mathbb{1}}_L)\right|\right|_{L^2_{G_0}\to L^q_{G_0}} = O((he^{-\phi(L)})^\infty)$$

Angular decomposition

The first two localizations reduce the problem to prove Strichartz inequalities for

$$\Psi^{\nu}_{h,L}(t) := \Pi^c \mathbb{1}_L(r) e^{i\frac{t}{h}(-h^2\Delta_0)^{\nu}} \varphi(-h^2\Delta_0)\psi, \qquad \nu \in \{1, \frac{1}{2}\}$$

Using $1D$ Sobolev inequalities

$$\begin{aligned}
\left|\left|\Pi^c \Psi^{(\nu)}_{h,L}(t)\right|\right|_{L^q_{G_0}} &= \left|\left| ||\Pi^c \Psi^{(\nu)}_{h,L}(t,r,.)||_{L^q(\mathcal{A})}\right|\right|_{L^q((r_0,\infty),e^{-\phi(r)}dr)} \\
&\leq C_{\mathcal{A}} \left|\left| ||\Pi^c \sqrt{|\Delta_{\mathcal{A}}|}^{\frac{1}{2}-\frac{1}{q}} \Psi^{(\nu)}_{h,L}(t,r,.)||_{L^2(\mathcal{A})}\right|\right|_{L^q((r_0,\infty),e^{-\phi}} \\
&\leq C_{\mathcal{A}} \left(\sum_{k \geq k_0} \left|\left|\mu_k^{\frac{1}{2}-\frac{1}{q}} \Psi^{(\nu)}_{h,L,k}(t)\right|\right|^2_{L^q((r_0,\infty),e^{-\phi(r)}dr)} \right)^{1/2},
\end{aligned}$$

where $\Psi^{(\nu)}_{h,L,k}(t) = \mathbb{1}_L(r) e^{\phi(r)/2} e^{i\frac{t}{h}(h^2\mathfrak{p}_k)^{\nu}} \varphi(h^2\mathfrak{p}_k) e^{-\phi(r)/2}\psi$

Dispersion estimates

We have eventually to estimate

$$\left|\left|e^{\frac{\phi(r)}{2}}\mathbb{1}_L(r)\varphi(h^2\mathfrak{p}_k)^2 e^{i\frac{(t-s)}{h}(h^2\mathfrak{p}_k)^\nu}\mathbb{1}_L(r)e^{\frac{\phi(r)}{2}}\right|\right|_{L^1(\mathbb{R})\to L^\infty(\mathbb{R})}$$
$$\lesssim e^{\phi(L)}\left|\left|\mathbb{1}_L(r)\varphi(h^2\mathfrak{p}_k)^2 e^{i\frac{(t-s)}{h}(h^2\mathfrak{p}_k)^\nu}\mathbb{1}_L(r)\right|\right|_{L^1(\mathbb{R})\to L^\infty(\mathbb{R})}$$

where

$$\varphi(h^2\mathfrak{p}_k)^2 \approx Op_h\big(\varphi(\rho^2 + h^2\mu_k^2 e^{2\phi(r)})\big).$$

We approximate the operators by FIOs with phases

$$\partial_t S_{h,L}^{(\nu)} = \Big((\partial_r S_{h,L}^{(\nu)})^2 + h^2\mu_k^2 e^{2\phi(r)}\Big)^\nu, \qquad S_{h,L}^{(\nu)}(0,r,\rho) = r\rho$$

and argue by Stationary Phase/Van der Corput estimates using

$$\partial_\rho^2 S_{h,L}^{(1)} \gtrsim |t|, \qquad \partial_\rho^2 S_{h,L}^{(1/2)} \gtrsim |t| h^2\mu_k^2 e^{2\phi(L)}$$

Optimality of the semiclassical Schrödinger-Strichartz inequality

We consider $\phi(r) = r$, e_{k_1} an eigenfunction of $\Delta_{\mathcal{A}}$ with non zero eigenvalue $-\mu_{k_1}^2$, and set

$$\psi_0^h(r,\alpha) := e^{\frac{r}{2}} u_0^h(r) e_{k_1}(\alpha),$$

where, for a given $\chi \in C_0^\infty(\mathbb{R})$ which is equal to 1 near 0,

$$u_0^h(r) = (\pi h)^{-1/4} \chi(r + \log h) \exp\left(\frac{-(r+\log h)^2}{2h}\right).$$

Then

$$e^{it\Delta}\psi_0^h = e^{r/2}\left(e^{-i\frac{s}{h}h^2\mathfrak{p}_1} u_0^h\right) \otimes e_{k_1}, \qquad t = hs$$

where

$$\mathfrak{p}_1 = D_r^2 + \mu_{k_1}^2 e^{2r} + \frac{1}{4}$$

Fact1: ψ_0^h is localized at frequency $1/h$ (mod a h^∞ remainder)
Fact2: By coherent states propagation ([Combescure-Robert])

$$e^{-i\frac{s}{h}h^2\mathfrak{p}_1}u_0^h \approx \text{ wave packet centered at } (-\log(h), 0) + O(1)$$

Therefore

$$\left|\left|\mathbb{1}_{[r_1,\infty)}(r)e^{-i\frac{s}{h}h^2\mathfrak{p}_1}u_0^h\right|\right|_{L^q(\mathbb{R})} \gtrsim h^{-\left(\frac{1}{4}-\frac{1}{2q}\right)} = h^{-\frac{1}{2p}}$$

and

$$\begin{aligned}\|\|\Pi^c\mathbb{1}_{[r_1,\infty)}(r)e^{it\Delta}\psi_0^h\|\|_{L^q_{G_0}} &\gtrsim \left|\left|e^{r\left(\frac{1}{2}-\frac{1}{q}\right)}\mathbb{1}_{[r_1,\infty)}(r)e^{-i\frac{s}{h}h^2\mathfrak{p}_1}u_0^h\right|\right|_{L^q(\mathbb{R})} \\ &\gtrsim h^{-\left(\frac{1}{2}-\frac{1}{q}\right)}\left|\left|\mathbb{1}_{[r_1,\infty)}(r)e^{-i\frac{s}{h}h^2\mathfrak{p}_1}u_0^h\right|\right|_{L^q(\mathbb{R})} \\ &\gtrsim h^{-\frac{3}{2p}}\end{aligned}$$