Low energy behaviour of powers of the resolvent of long range perturbations of the Laplacian

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1 Introduction and main result

Consider a differential operator in divergence form on $\mathbb{R}^d$, $d \geq 3$,

$$P = -\mathrm{div}\,(G(x)\nabla)\,,$$

with $G(x)$ a real, symmetric, positive definite matrix, such that,

$$c \leq G(x) \leq C, \qquad x \in \mathbb{R}^d,$$

for some $C, c > 0$. When $G(x) = Id$, the identity matrix, P is of course the usual flat Laplacian $-\Delta$. Here, we will assume that P is asymptotically flat, and more precisely a long range perturbation of the flat Laplacian, in the sense that, for some real number $\rho > 0$,

$$|\partial^\alpha \left(G(x) - Id\right)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \tag{1.1}$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$ is the usual japanese bracket. Let us point out that (1.1) is a condition at infinity. It only states that P is close to $-\Delta$ when x is large, but P is arbitrary in a compact set.

Equipped with the domain $H^2(\mathbb{R}^d)$ (the usual Sobolev space), P is self-adjoint on $L^2(\mathbb{R}^d)$. Its spectrum is $[0, \infty)$ and is purely absolutely continuous: there are no singular continuous spectrum [10] nor embedded eigenvalues [9]. In this paper, we are interested in the resolvent $(P - z)^{-1}$, defined for $z \in \mathbb{C} \setminus [0, \infty)$, and more specifically to its powers $(P - z)^{-k}$ as z approaches 0.

Let us briefly recall that the justification of the *limiting absorption principle*, namely the uniform boundedness of $(P-\lambda-i\epsilon)^{-1}$ with respect to $\epsilon > 0$ (in weighted L^2 spaces, see below) and, even better, the existence of the limits

$$(P - \lambda \mp i0)^{-1} = \lim_{\epsilon \to 0,\ \epsilon > 0} (P - \lambda \mp i\epsilon)^{-1}, \tag{1.2}$$

is a basic question in scattering theory. For the stationary problem, this allows to construct the scattering matrix and the scattering amplitude as well as the generalized eigenfunctions. For the time dependent problems, ie the Schrödinger or wave equations associated to P, it is a key tool in the proof of asymptotic completeness. Without refering to scattering theory, one may even see the interest of the limiting absorption principle in the so called Stone formula

$$dE_\lambda = \lim_{\epsilon \downarrow 0} \frac{1}{2i\pi} \left((P - \lambda - i\epsilon)^{-1} - (P - \lambda + i\epsilon)^{-1} \right) d\lambda, \tag{1.3}$$

which relates the spectral familly E_λ of P to its resolvent. See Reed-Simon [15].

The existence of the limits (1.2) has been proved a long time ago for the operators considered in this paper; since this is not the present purpose to trace back the history of such estimates, which hold for a very large class of operators, we only quote the historical paper [10] from which they all follow (see also [14]). To state those estimates precisely, we introduce the following notation. Given any real number ν, we denote by L^2_ν the space of functions $u \in L^2_{\rm loc}(\mathbb{R}^d)$ such that $\langle x \rangle^\nu u \in L^2$, with norm

$$\left|\left| u \right|\right|_{L^2_\nu} = \left|\left| \langle x \rangle^\nu u \right|\right|_{L^2}.$$

Then, for any $\lambda > 0$, $(P - \lambda \mp i0)^{-1}$ are bounded from L^2_ν to $L^2_{-\nu}$ for any $\nu > 1/2$. Furthermore, by choosing appropriate weights, the resolvents (1.2) are smooth with respect to λ in the sense that

$$(P - \lambda \mp i0)^{-k} = \frac{1}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} (P - \lambda \mp i0)^{-1}, \tag{1.4}$$

are well defined and bounded from L^2_ν to $L^2_{-\nu}$, if $\nu > k - 1/2$. See [8].

These are local results. The next natural question is the asymptotic behaviour of such estimates as λ approaches the boundary (or thresholds) of the spectrum, namely $\lambda \to +\infty$ and $\lambda \to 0$. The high energy asymptotic $\lambda \to +\infty$ is fairly well understood through its dependence on the classical flow, here the geodesic flow associated to G^{-1}. If this flow is non trapping, ie if all geodesics escape to infinity, then $||(P - \lambda \mp i0)^{-k}||_{L^2_\nu \to L^2_{-\nu}}$ is of order $\lambda^{-k/2}$ (see [16] and references therein). In the general case, there is an exponential upper bound [3, 4], but there are intermediate situations where the classical flow exhibit some trapping and the resolvent decays as $\lambda \to \infty$, but not as fast as in the non trapping case, see for instance [11, 13] in the semiclassical setting.

These results can be used to prove local energy decay estimates for evolution equations, but only for initial data spectrally localized away from 0. For instance, if $u(t)$ denotes the solution to the Schrödinger equation

$$i\partial_t u = Pu, \qquad u_{|t=0} = u_0,$$

namely $u(t) = e^{-itP}u_0$, it follows from formal integrations by parts in the Fourier transform,

$$e^{-itP}u_0 = \int_{[0,\infty)} e^{-it\lambda} dE_\lambda u_0,$$

using the Stone formula (1.3) and (1.4), that

$$||u(t)||_{L^2_{-\nu}} \leq C\langle t\rangle^{-k}||u_0||_{L^2_\nu},$$

if $\nu > k + 1/2$, and provided that

$$u_0 = \chi(P)u_0,$$

with either $\chi \in C_0^\infty(0, +\infty)$ in the general case, or possibly $\chi \equiv 1$ near infinity and $\chi \equiv 0$ near 0 if the behaviour of $(P - \lambda \mp i0)^{-k-1}$ is nice as $\lambda \to \infty$.

To relax this spectral localization, one must study the behaviour of powers of the resolvent as $\lambda \to 0$. This is the first motivation of the results below, though, in a wider perspective, the study of the behaviour of the resolvent at low energies is anyway a natural problem. Recently there has been some interest and progress on the low energy estimates of the resolvent itself for metric perturbations [7, 2, 1, 17], but here we are interested in powers of the resolvent. The latter question has been treated for potentials V in [12] (for V sufficiently positive) and in [6] (for V sufficiently small or negative at infinity). For very short range perturbations of exact conical metrics, this question has been considered in [5, 18]. Our purpose here is to deal with the long range metric case.

To state Theorem 1.2, and to simplify certain statements afterwards, we introduce the following notation. Denote by $k(d)$ the largest even integer $< \frac{d}{2} + 1$, namely

$$k(d) = 2n, \qquad \text{if} \quad 4n - 1 \leq d \leq 4n + 2.$$

Note in particular that

$$k(d) \geq 2,$$

and for instance that

$$k(3) = 2.$$

Definition 1.1. *Given $1 \leq k \leq k(d)$ and a non negative function defined near 0, $f : (0, \lambda_0] \to [0, +\infty)$, one writes*

$$f(\lambda) = \mathcal{R}_{k,d}(\lambda)$$

to state that,

- *when* $1 \leq k \leq k(d) - 1$,
$$f \quad \text{is bounded},$$
- *when* $k = k(d)$,
  1. *when* $d \equiv 1$ *or* $d \equiv 2$ *mod* 4,
  $$f \quad \text{is bounded},$$
  2. *when* $d \equiv 3$ *mod* 4,
  $$f(\lambda) \leq C\lambda^{-1/2},$$
  3. *when* $d \equiv 0$ *mod* 4,
  $$f(\lambda) \leq C_s\lambda^{-s},$$
  for all $s > 0$.

The typical example to illustrate this definition and the theorem below is the following. Consider the kernel of the resolvent of the Laplacian $(-\Delta - z)^{-1}$ in $\mathbb{R}^3$, namely

$$G_z(x,y) = \frac{1}{4\pi}\frac{e^{iz^{1/2}|x-y|}}{|x-y|}, \qquad \mathrm{Im}(z) > 0.$$

From this expression and the fact that $(-\Delta - z)^{-2} = \frac{d}{dz}(-\Delta - z)^{-1}$, one easily checks that, for any $\nu > 3/2$,

$$\begin{aligned} \left|\left|(-\Delta - \lambda - i\epsilon)^{-1}\right|\right|_{L^2_\nu \to L^2_{-\nu}} &\leq C, \\ \left|\left|(-\Delta - \lambda - i\epsilon)^{-2}\right|\right|_{L^2_\nu \to L^2_{-\nu}} &\leq C\lambda^{-1/2} \end{aligned}$$

which reads, in a more compact form,

$$\sup_{\epsilon > 0} \left|\left|(-\Delta - \lambda - i\epsilon)^{-k}\right|\right|_{L^2_\nu \to L^2_{-\nu}} = \mathcal{R}_{k,3}(\lambda), \qquad 1 \leq k \leq 2.$$

Our result is the following.

Theorem 1.2. *For all* $d \geq 3$, *there exists* $\nu > 0$ *such that, for all* $1 \leq k \leq k(d)$,

$$\sup_{\epsilon > 0} \left|\left|(P - \lambda - i\epsilon)^{-k}\right|\right|_{L^2_\nu \to L^2_{-\nu}} = \mathcal{R}_{k,d}(\lambda),$$

for $\lambda \in (0,1]$.

To give a concrete value, we can take any $\nu > k(d) + \frac{d}{2}$ in this theorem. This is certainly not sharp, but our point here is the behaviour with respect to λ. To the latter extent, our result is sharp in dimension 3, as shown by the above example of the flat Laplacian. Indeed, if $d = 3$, Theorem 1.2 reads more explicitly

$$\begin{aligned} \left|\left|(P-\lambda-i\epsilon)^{-1}\right|\right|_{L^2_\nu \to L^2_{-\nu}} &\leq C, \\ \left|\left|(P-\lambda-i\epsilon)^{-2}\right|\right|_{L^2_\nu \to L^2_{-\nu}} &\leq C\lambda^{-1/2}, \end{aligned}$$

for all $\epsilon > 0$ and $\lambda \in (0, 1]$. In higher dimensions, one may notice that, at least in odd dimensions, the kernel of the resolvent of the flat Laplacian is an analytic function of $z^{1/2}$. One may thus fear that the $|z|^{-1/2}$ singularity already shows up for the first derivative (ie for $(-\Delta - z)^{-2}$). Our result shows that the latter does not happen, and more precisely that the resolvent does not blow up as long as one does not differentiate more than $d/2 - 1$ times. This can be seen directly from Bessel functions expansions in the flat case, but it is a consequence of our estimates in the non flat case. We also mention that this $d/2$ order is a natural threshold for it is related to the (expected) dispersion decay rate of the Schrödinger group e^{-itP} (ie the bound $||e^{it\Delta}||_{L^1 \to L^\infty} = c_d|t|^{-d/2}$ in the flat case).

The full proof (and the applications) of Theorem 1.2 will appear in a more detailed paper. However we hope to give in the following sections the main points of the analysis.

The proof is divided into two steps. The first one, which is the purpose of Section 2, is to prove the result when $G - Id$ is uniformly small on $\mathbb{R}^d$. By uniformly small we mean not only that it satisfies (1.1), which guarantees the smallness at infinity, but also that it is small on every compact set. The proof in this case is based on a scaling argument, although the condition (1.1) is not scale invariant. More precisely, for $\mu \in \mathbb{R}$, let us recall that S^μ is the space of functions a such that

$$\sup_{x \in \mathbb{R}^d} |\langle x \rangle^{-\mu + |\alpha|} \partial^\alpha a(x)| < \infty, \tag{1.5}$$

where the left hand side define seminorms which give the topology of S^μ and make it a Fréchet space. For instance, the condition (1.1) states that (the coefficients of) $G - Id$ belongs to $S^{-\rho}$. Theses spaces are commonly used in scattering theory. They are natural and convenient for microlocal techniques but have the drawback of not being scale invariant, in sense that, given $a \in S^\mu$, the family $(a_t)_{t>0}$, $a_t(x) = a(x/t)$, is in general not bounded in S^μ. However, our analysis is based on the following observations: for all k,

$$a \in S^0 \qquad \Rightarrow \qquad (x \cdot \nabla)^k a \in L^\infty, \tag{1.6}$$

and, if $\mu < 0$,

$$a \in S^{\mu} \qquad \Rightarrow \qquad \partial^{\alpha} a \in L^{d/|\alpha|}, \tag{1.7}$$

and the properties in the right hand sides are scale invariant.

To guarantee the smallness of $G - Id$, we will require that it is small in $S^{-\rho/2}$. Notice that, if $G - Id \in S^{-\rho}$, then, if $\mu = -\rho' > -\rho$, the quantities obtained by restricting the sup of (1.5) outside a ball of radius R go to 0 as $R \to \infty$[1]. Thus, by possibly replacing ρ by a smaller value, e.g. $\rho/2$, requiring the smallness of $G - Id$ in $S^{-\rho/2}$ is equivalent to require its smallness on any compact set.

In the second step, which is the purpose of Section 3, we will consider the general case by combining a compactness argument with the result of Section 2. This will be obtained by writing G as the sum of a compactly supported term and of a one uniformly close to Id.

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2 Proof of Theorem 1.2 when $G - Id$ is small

The proof in this case follows from a scaling argument, using the usual unitary group $e^{i\tau A}$ of L^2 dilations, ie

$$e^{i\tau A}\psi(x) = e^{\tau d/2}\psi(e^{\tau}x),$$

whose generator is

$$A = \frac{x \cdot \nabla + \nabla \cdot x}{2i}.$$

The proof starts as follows. For $\lambda > 0$ and $\epsilon \neq 0$ real, we can write

$$P - \lambda - i\epsilon = \lambda \left(\frac{P}{\lambda} - 1 - \delta \right), \tag{2.1}$$

with of course $\delta = \epsilon/\lambda$. Using the rescaling,

$$G_{\lambda}(x) \;\; = \;\; G\left(\frac{x}{\lambda^{1/2}} \right),$$

and then by setting

$$P_{\lambda} \;\; = \;\; -\mathrm{div}\left(G_{\lambda}(x)\nabla \right),$$

[1]This implies the standard fact that compactly supported functions are dense in $S^{-\rho}$ for the topology of S^{μ} for all $\mu > -\rho$

it is elementary to check that

$$\frac{P}{\lambda} = e^{i\tau A} P_\lambda e^{-i\tau A}, \tag{2.2}$$

with $\tau \in \mathbb{R}$ such that

$$e^{\pm i\tau A}\psi(x) = \lambda^{\pm d/4}\psi\big(\lambda^{\pm 1/2}x\big), \tag{2.3}$$

ie $\tau = \ln(\lambda^{1/2})$. Therefore, (2.1) and (2.2) give

$$\big(P - \lambda - i\epsilon\big)^{-1} = e^{i\tau A}\left(\lambda^{-1}\big(P_\lambda - 1 - i\delta\big)^{-1}\right)e^{-i\tau A}, \tag{2.4}$$

which reduces the problem to get estimates on the resolvent of P_λ.

Proposition 2.1. *Let $k \geq 1$ be an integer. If the norm*

$$\sum_{j\leq k+1} \big|\big|(x\cdot\nabla)^j(G - Id)\big|\big|_{L^\infty}, \tag{2.5}$$

is small enough, then there exists $C > 0$ such that,

$$\big|\big|\langle A\rangle^{-k}\big(P_\lambda - 1 - i\delta\big)^{-k}\langle A\rangle^{-k}\big|\big|_{L^2\to L^2} \leq C,$$

for all real numbers $\delta \neq 0$ and $\lambda > 0$.

This result follows basically from the multiple commutators estimates of Jensen-Mourre-Perry [8]. The smallness of $G - Id$ guarantees the *strong* positive commutator estimate

$$\varphi(P_\lambda)i\big[P_\lambda, A\big]\varphi(P_\lambda) \geq \varphi(P_\lambda)^2,$$

with $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ supported close to 1. By strong we mean that there is no compact remainder on the right hand side (see [8] for the details). We simply recall that this estimate follows from the elementary identity

$$i[\Delta, A] = 2\Delta,$$

for this identity implies that $i[P_\lambda, A]$ is close to $2P_\lambda$ ($i[P_\lambda, A]$ and P_λ are respectively close to -2Δ and $-\Delta$) if $G - Id$ is small. More precisely, one actually only needs $G - Id$ and $x\cdot\nabla G$ to be small in L^∞ to get a positive commutator estimate, but $(x\cdot\nabla)^j G$ don't need to be small for $j \geq 2$. To this extent, our assumption may seem too strong. However, since we shall require (2.5) to be small in Proposition 2.2 too, there is actually no real loss of generality for our final purpose.

The uniformity with respect to λ in Proposition 2.1 relies on the easy to check observation that

$$\big|\big|(x\cdot\nabla)^j(G_\lambda - Id)\big|\big|_{L^\infty} = \big|\big|(x\cdot\nabla)^j(G - Id)\big|\big|_{L^\infty}.$$

We finally note that the weights $\langle A\rangle^{-k}$ can be replaced by any non vanishing function of A of the same order as $\langle\cdot\rangle^{-k}$ near infinity: in the sequel we will for instance use $(hA+i)^{-k}$ for some fixed number h. This is related to the following result.

Proposition 2.2. *1- Assume that n is an integer such that*

$$2n < \frac{d}{2} + 1. \tag{2.6}$$

If the norm

$$\sum_{|\alpha|\leq 2n-1} \big|\big|\partial^\alpha\big(G - Id\big)\big|\big|_{L^{\frac{d}{|\alpha|}}}, \tag{2.7}$$

is small enough, then there exists $C > 0$ such that

$$C^{-1}||u||_{H^{2n}} \leq \big|\big|(P_\lambda + 1)^n u\big|\big|_{L^2} \leq C||u||_{H^{2n}}, \tag{2.8}$$

for all $\lambda > 0$ and $u \in \mathcal{S}(\mathbb{R}^d)$.
2- More generally, fix an integer $N \geq 0$ and assume (2.6): if the norm

$$\big|\big|\big|G - Id\big|\big|\big|_{2n-1,N} := \sum_{j\leq N}\sum_{|\alpha|\leq 2n-1} \big|\big|\partial^\alpha(x\cdot\nabla)^j\big(G - Id\big)\big|\big|_{L^{\frac{d}{|\alpha|}}}, \tag{2.9}$$

is small enough and h is small enough, then

$$C^{-1}||u||_{H^{2n}} \leq \big|\big|(hA+i)^{-N}(P_\lambda + 1)^n(hA+i)^N u\big|\big|_{L^2} \leq C||u||_{H^{2n}}, \tag{2.10}$$

for all $\lambda > 0$ and $u \in \mathcal{S}(\mathbb{R}^d)$.

The proof of this proposition is fairly elementary. We only emphasize the following points.

1. The norms (2.5) and (2.7) are special cases of the norm (2.9). Their interest for our approach is that they are scale invariant, ie that

   $$\big|\big|\big|G - Id\big|\big|\big|_{2n-1,N} = \big|\big|\big|G_\lambda - Id\big|\big|\big|_{2n-1,N}.$$

   The uniformity of the estimates of Proposition 2.2 with respect to λ follows from this scale invariance.

2. We have the following continuity property, which follow from (1.6) and (1.7). If $\mu < 0$,

$$G - Id \to 0 \ \text{ in } S^{\mu} \qquad \Longrightarrow \qquad \big|\big|\big|G - Id\big|\big|\big|_{n,N} \to 0. \tag{2.11}$$

Thus, even though the spaces S^{μ} have not scale invariant seminorms, they can be embedded into a space with scale invariant norms and adapted to the study of resolvent estimates .

3. The interest of considering the resolvent of A rather than $\langle A\rangle^{-1}$ is that we have the simple formula

$$(hA + i)^{-1} = \frac{1}{i}\int_0^{+\infty} e^{-\tau} e^{i\tau hA} d\tau, \tag{2.12}$$

which is convenient to study the behaviour of the resolvent of A on Sobolev spaces. The smallness of h is required to show the boundedness of $(hA+i)^{-1}$ on Sobolev spaces: using (2.12) and the explicit form of $e^{i\tau hA}$ it is easy to show that, for all real number $s \geq 0$,

$$\big|\big||D|^s (hA + i)^{-1}\psi\big|\big|_{L^2} \leq \frac{1}{1 - hs}\big|\big||D|^s\psi\big|\big|_{L^2}, \tag{2.13}$$

from which the boundedness of $(hA + i)^{-1}$ on H^s follows easily.

We next use Proposition 2.2 to turn the $L^2 \to L^2$ estimates of Proposition 2.1 into $H^{-k} \to H^k$ estimates.

Proposition 2.3. *Fix an even integer k such that*

$$k = 2n < \frac{d}{2} + 1. \tag{2.14}$$

Then, if $G - Id$ is small enough in $S^{-\rho/2}$, there exist $h > 0$ and $C > 0$ such that

$$\big|\big|(hA + i)^{-k}(P_\lambda - 1 - i\delta)^{-k}(hA - i)^{-k}\big|\big|_{H^{-k}\to H^k} \leq C, \tag{2.15}$$

for all real numbers $\delta \neq 0$ and $\lambda > 0$.

Notice that the constant C depends on h here (and blows up as $h \to 0$). Although the estimates of Proposition 2.2 and Lemma 2.4 below are uniform with respect to h small, we shall use at some point Proposition 2.1 in which we will replace $\langle A\rangle^{-1}$ by $(hA \pm i)^{-1}$ which is the reason of the h dependence of C.

We will need one more lemma.

Lemma 2.4. *For all $\varphi \in C_0^\infty(\mathbb{R})$, there exists $C > 0$ such that*

$$\big|\big|(hA+i)^{-k}\varphi(P_\lambda)(hA-i)^k\big|\big|_{L^2\to L^2} \leq C, \tag{2.16}$$

for all $\lambda > 0$.

Slightly more precisely, the lemma states implicitly that the operator in (2.16), defined on $\mathrm{Dom}(A^k)$, has a bounded closure on L^2. The latter result is fairly standard. The operator $(hA-i)^k$ on the right hand side can be replaced by $(hA+i)^k$ for $(hA+i)(hA-i)^{-1}$ is unitary. The result follows then basically from (iterations of) the identity

$$(hA+i)^{-1}(P_\lambda-\zeta)^{-1}(hA+i) = (P_\lambda-\zeta)^{-1} - (hA+i)^{-1}(P_\lambda-\zeta)^{-1}[P_\lambda, hA](P_\lambda-\zeta)^{-1},$$

and the Helffer-Sjöstrand formula to pass from the resolvent $(P_\lambda-\zeta)^{-1}$ to bump functions of P_λ.

Proof of Proposition 2.3. Pick $\phi \in C_0^\infty(\mathbb{R})$, real valued, such that $\phi \equiv 1$ near 1 and let $\Phi = 1-\phi^2$. We then have the following spectral partition of unity

$$I_{L^2} = \phi(P_\lambda)^2 + \Phi(P_\lambda)$$

which we use to decompose

$$\begin{aligned}\big(P_\lambda - z\big)^{-k} &= \phi(P_\lambda)\big(P_\lambda-z\big)^{-k}\phi(P_\lambda) + \Phi(P_\lambda)(P_\lambda-z)^{-k}, \\ &= \mathrm{I} + \mathrm{II},\end{aligned}$$

where, for simplicity,

$$z = 1 + i\delta.$$

Contribution of II. Using that $1 \notin \mathrm{supp}(\Phi)$, and that $(p-z)^{-k}$ decays as $p^{-k} \sim p^{-2n}$ at infinity, the spectral theorem yields

$$\big|\big|(P_\lambda+1)^n\mathrm{II}(P_\lambda+1)^n\big|\big|_{L^2\to L^2} \leq C,$$

for all $\lambda > 0$ and $\delta \in \mathbb{R}$. By (2.11), we can use (2.8) and we get

$$\big|\big|\mathrm{II}\big|\big|_{H^{-k}\to H^k} \leq C. \tag{2.17}$$

By the boundedness of $(hA+i)^{-k}$ on H^k (ie essentially (2.13)), we conclude that

$$\big|\big|(hA+i)^{-k}\mathrm{II}(hA-i)^{-k}\big|\big|_{H^{-k}\to H^k} \leq C,$$

for all $\lambda > 0$ and $\delta \in \mathbb{R}$.

Contribution of I. We write first

$$(hA+i)^{-k}\mathrm{I}(hA-i)^{-k} \quad = \quad B(hA+i)^{-k}(P_\lambda - z)^{-k}(hA-i)^{-k}B^*, \qquad (2.18)$$

where

$$B = (hA+i)^{-k}\phi(P_\lambda)(hA+i)^k.$$

If we define $\widetilde{\phi} \in C_0^\infty(\mathbb{R})$ by

$$\widetilde{\phi}(p) = (p+1)^n\phi(p), \qquad p \in \mathbb{R},$$

we obtain

$$B = (hA+i)^{-k}(P_\lambda+1)^{-n}(hA+i)^k\left((hA+i)^{-k}\widetilde{\phi}(P_\lambda)(hA-i)^k\right).$$

Using (2.11), we can assume that (2.10) holds. By Lemma 2.4 , this yields

$$\big|\big|B\big|\big|_{L^2\to H^k} \le C, \qquad \lambda > 0. \qquad (2.19)$$

Using Proposition 2.1 and (2.18), we conclude that

$$\big|\big|(hA+i)^{-k}\mathrm{I}(hA-i)^{-k}\big|\big|_{H^{-k}\to H^k} \le C, \qquad \lambda > 0,$$

for all $\delta \neq 0$ and $\lambda > 0$. This completes the proof. □

Proof of Theorem 1.2. Recall first the Sobolev embeddings

$$\begin{aligned} H^k &\subset L^{\frac{2d}{d-2k}}, && \text{if } k < \frac{d}{2}, \\ H^{\frac{d}{2}} &\subset L^p, && \text{for all } p \in [2,\infty), \\ H^k &\subset L^\infty, && \text{if } k > \frac{d}{2}, \end{aligned}$$

and then define

$$p(k) = \begin{cases} \frac{2d}{d-2k} & \text{if } \ k < \frac{d}{2}, \\ p & \text{if } \ k = \frac{d}{2}, \\ \infty & \text{if } \ k > \frac{d}{2}, \end{cases} \qquad (2.20)$$

with an arbitrary $p \in [2,\infty)$ (large in the application below) in the second case. Proposition 2.3 implies that

$$\big|\big|(hA+i)^{-k}(P_\lambda - 1 - i\delta)^{-k}(hA-i)^{-k}\big|\big|_{L^{p(k)'}\to L^{p(k)}} \le C, \qquad (2.21)$$

for all $\lambda > 0$ and $\delta \neq 0$. On the other hand, using (2.3), we have

$$\left|\left|e^{i\tau A}\right|\right|_{L^{p(k)}\to L^{p(k)}} = \left|\left|e^{-i\tau A}\right|\right|_{L^{p(k)'}\to L^{p(k)'}} = \lambda^{\sigma(k)}, \tag{2.22}$$

with

$$\sigma(k) = \begin{cases} \frac{k}{2} & \text{if } \ k < \frac{d}{2}, \\ \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & \text{if } \ k = \frac{d}{2}, \\ \frac{d}{4} & \text{if } \ k > \frac{d}{2}. \end{cases}$$

Therefore, using (2.4), (2.21) and (2.22) yield

$$\left|\left|(hA+i)^{-k}(P-\lambda-i\epsilon)^{-k}(hA-i)^{-k}\right|\right|_{L^{p(k)'}\to L^{p(k)}} \leq C\lambda^{2\sigma(k)-k},$$

for all $\lambda > 0$ and $\epsilon \neq 0$. Recall the assumption (2.14) which gives

$$2\sigma(k) - k \quad = \quad \begin{cases} 0 & \text{if } \ k < \frac{d}{2}, \\ -\frac{d}{p} & \text{if } \ k = \frac{d}{2}, \\ \frac{d}{2} - k & \text{if } \ \frac{d}{2} < k < 1 + \frac{d}{2}. \end{cases} \tag{2.23}$$

A elementary examination of the values of $2\sigma(k) - k$ implies that, for all $1 \leq k \leq k(d)$,

$$\sup_{\epsilon>0} \left|\left|(hA+i)^{-k}(P-\lambda-i\epsilon)^{-k}(hA-i)^{-k}\right|\right|_{L^{(p(k))'}\to L^{p(k)}} = \mathcal{R}_{k,d}(\lambda).$$

The replacement of $(hA \pm i)^{-k}$ by the weight $\langle x\rangle^{-k}$ follows easily from the following proposition by splitting

$$(P-\lambda-i\epsilon)^{-k} = \varphi(P)(P-\lambda-i\epsilon)^{-k}\varphi(P) + (1-\varphi(P)^2)(P-\lambda-i\epsilon)^{-k},$$

with $\varphi \equiv 1$ on a neighborhood of the interval where λ lives.

Proposition 2.5. *For all $\varphi \in C_0^\infty(\mathbb{R})$,*

$$\langle x\rangle^{-k}\varphi(P)(hA+i)^k \quad \textit{is bounded on } L^{p(k)}.$$

By choosing $\nu > \frac{d}{2} + k$, we finally obtain $L^2_\nu \to L^2_{-\nu}$ estimates using the boundedness of the multiplication operator

$$\langle x\rangle^{k-\nu} : L^{p(k)} \to L^2,$$

and its adjoint. □

3 Non small perturbations

To treat the general case, we will consider long range perturbations as *compactly supported perturbations* of *small* long range perturbations, for which we already have resolvent estimates by Section 2.

Throughout this section, we shall thus consider G_0 such that

$$G - G_0 \text{ is compactly supported,} \tag{3.1}$$

and such that,

$$G_0 - Id \text{ is small in } S^{-\rho/2}. \tag{3.2}$$

More precisely, this means that we may consider a family of matrices $G_0 = G_{0,\varepsilon}$ satisfying (3.1) for each ε, such that $G_{0,\varepsilon} - Id \to 0$ in $S^{-\rho/2}$ as $\varepsilon \to 0$, and then by choosing ε small enough when necessary.

By (3.2), we may in particular assume that

$$P_0 = -\mathrm{div}\left(G_0(x)\nabla\right),$$

is uniformly elliptic. We then define V by

$$P = P_0 + V. \tag{3.3}$$

The condition (3.2) will be used in a couple of places. First, by Section 2, we have the following result.

Proposition 3.1. *Theorem 1.2 holds for* P_0.

Our strategy is then based on the resolvent identity

$$\begin{aligned} (P-z)^{-1} &= (P_0-z)^{-1} - (P_0-z)^{-1}V(P-z)^{-1}, &(3.4)\\ &= (P_0-z)^{-1} - (P-z)^{-1}V(P_0-z)^{-1}, &(3.5)\end{aligned}$$

and the fact that

$$(P-z)^{-k} = \frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1}(P-z)^{-1}. \tag{3.6}$$

More precisely, since we want to study the resolvent when z is small, it is sufficient (and convenient) to consider

$$R_\phi(z) = \phi(P)(P-z)^{-1},$$

with $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ near 0 (the support of ϕ will have to be chosen small enough below). If $\widetilde{\phi} \in C_0^\infty(\mathbb{R})$ is real valued and such that

$$\widetilde{\phi} \equiv 1 \ \text{ near the support of } \phi, \tag{3.7}$$

and if we set

$$S_{\widetilde{\phi}}(z) = V(P_0 - z)^{-1}\widetilde{\phi}(P),$$

we then get

$$R_\phi(z) = B_1(z) + S_{\widetilde{\phi}}(\bar{z})^* R_\phi(z) S_{\widetilde{\phi}}(z), \tag{3.8}$$

with

$$B_1(z) = \widetilde{\phi}(P)(P_0 - z)^{-1}\phi(P) - \widetilde{\phi}(P)(P_0 - z)^{-1}V\phi(P)(P_0 - z)^{-1}\widetilde{\phi}(P).$$

This is easily obtained by composing (3.4) on the right hand side with $\phi(P)$, then by replacing $(P - z)^{-1}$ by (3.5) therein and finally by composing on both sides by $\widetilde{\phi}(P)$.

The idea is then to show that, if the support of $\widetilde{\phi}$ is small enough and z close enough to 0, then

$$\big|\big|S_{\widetilde{\phi}}(z)\big|\big|_{L^2_\nu \to L^2_\nu} \leq 1/2, \tag{3.9}$$

(or smaller than a fixed real number < 1). Using (3.8), this easily implies that

$$\big|\big|R_\phi(z)\big|\big|_{L^2_\nu \to L^2_{-\nu}} \leq \frac{4}{3}\big|\big|B_1(z)\big|\big|_{L^2_\nu \to L^2_{-\nu}},$$

where the right hand side is uniformly bounded as $\mathrm{Im}(z) \to 0$ by Proposition 3.1. Note that the support of ϕ has to be small enough too, due to the condition (3.7). For higher powers of $(P - z)^{-1}$, we use (3.6) to deduce, after a finite induction on $k \leq k(d)$, that

$$R^k_\phi(z) := (P - z)^{-k}\phi(P),$$

satisfies

$$R^k_\phi(z) = B_k(z) + S_{\widetilde{\phi}}(\bar{z})^* R^k_\phi(z) S_{\widetilde{\phi}}(z),$$

where

$$\sup_{\epsilon > 0} \big|\big|B_k(\lambda + i\epsilon)\big|\big|_{L^2_\nu \to L^2_{-\nu}} = \mathcal{R}_{k,d}(\lambda)$$

for λ close to 0. The latter uses the induction assumption and the resolvent estimates for P_0 guaranteed by Proposition 3.1. Since

$$\big|\big|R^k_\phi(z)\big|\big|_{L^2_\nu \to L^2_{-\nu}} \leq \frac{4}{3}\big|\big|B_k(z)\big|\big|_{L^2_\nu \to L^2_{-\nu}},$$

Theorem 1.2 follows.

Granted this discussion, the proof of Theorem 1.2 is a consequence of the estimate (3.9). The rest of the section is devoted to the proof of this estimate.

The first step is the following proposition.

Proposition 3.2. *We have*

$$\big|\big|V\left((P_0-\lambda-i\epsilon)^{-1}-(P_0-i\epsilon)^{-1}\right)\big|\big|_{L^2_\nu\to L^2_\nu}\to 0, \qquad as\ \ |\lambda|+|\epsilon|\to 0. \quad (3.10)$$

Up to minor technical details (such as writing $V(P_0+1)^{-1}(P_0+1)$ to replace the unbounded operator V by the bounded one $V(P_0+1)^{-1}$ which has still a strong spatial decay), this proposition follows from the fast decay of (the coefficients of) V and the fact that

$$\big|\big|(P_0-\lambda-i\epsilon)^{-1}-(P_0-i\epsilon)^{-1}\big|\big|_{L^2_\nu\to L^2_{-\nu}}\leq\int_0^\lambda\big|\big|(P_0-\gamma-i\epsilon)^{-2}\big|\big|_{L^2_\nu\to L^2_{-\nu}}d\gamma\to 0,$$

as $\lambda\downarrow 0$, uniformly with respect to $\epsilon\neq 0$ real. Here we use the fact that the largest power of the resolvent that we can consider in Theorem 1.2 is at least 2 and that the upper bounds are integrable at 0.

The second place where we need G_0-Id to be small is the following.

Proposition 3.3. *If the norm*

$$\sum_{|\alpha|\leq 1}\big|\big|\partial^\alpha(G_0-Id)\big|\big|_{L^{\frac{d}{|\alpha|}}},$$

is small enough, then there exists $C>0$ such that,

$$\big|\big|\Delta(P_0-i\epsilon)^{-1}\big|\big|_{L^2\to L^2}\leq C, \quad (3.11)$$

for all real number $\epsilon\neq 0$.

This result is an easy consequence of the estimate

$$||\Delta u||_{L^2}\leq C||P_0u||_{L^2}, \qquad u\in H^2(\mathbb{R}^d),$$

which is obtained as follows. If we denote by h_{jk} the coefficients of G_0-Id, we can write

$$\begin{aligned} P_0 &= -\Delta-\sum_{j,k}\partial_j(h_{jk}(x)\partial_k) \\ &= -\left(1-\sum_{j,k}h_{jk}(x)\frac{\partial_j\partial_k}{|D|^2}+(\partial_jh_{jk}(x))\frac{1}{|D|}\frac{\partial_k}{|D|}\right)\Delta, \end{aligned}$$

where the bracket is invertible on L^2 (as a small perturbation of identity) using the smallness of $||h_{jk}||_{L^\infty}$ and

$$\left|\left|(\partial_j h_{jk})\frac{1}{|D|}\psi\right|\right|_{L^2} \leq ||(\partial_j h_{jk})||_{L^d}\left|\left|\frac{1}{|D|}\psi\right|\right|_{\frac{2d}{d-2}} \leq C\big|\big|\nabla(G_0 - Id)\big|\big|_{L^d}||\psi||_{L^2}.$$

The last tool we need is the following.

Proposition 3.4. *For all $N \geq 0$, there exists a bounded operator*

$$B_N : L^2 \to L^2,$$

such that

$$V \;=\; \langle x\rangle^{-N} B_N \Delta. \tag{3.12}$$

The weight $\langle x\rangle^{-N}$ in (3.12) could even be replaced by a compactly supported function but this is useless for our purpose. This proposition is based on the same kind of estimate as the one used to prove Proposition 3.3, namely, if $V = \sum_{j,k}\partial_j(v_{jk}(x)\partial_k)$,

$$\begin{aligned} V &= \sum_{j,k} v_{jk}(x)\partial_j\partial_k + (\partial_j v_{jk}(x))\partial_k \\ &= -\left(\sum_{j,k} v_{jk}(x)\frac{\partial_j\partial_k}{|D|^2} + (\partial_j v_{jk}(x))\frac{1}{|D|}\frac{\partial_k}{|D|}\right)\Delta \end{aligned}$$

where the bracket maps L^2 to L^2_N.

Before proving (3.9), we record two more propositions which are completely standard but whose roles are crucial.

Proposition 3.5. *For any $\varphi \in C_0^\infty(\mathbb{R})$ and $\mu > 0$,*

$$\big|\big|\varphi(P/\eta)\langle x\rangle^{-\mu}\big|\big|_{L^2\to L^2} \to 0, \qquad \eta \to 0.$$

This one follows from the fact that 0 is not an eigenvalue of P and thus that $\varphi(P/\eta) \to 0$ in the weak sense.

Proposition 3.6. *For any $\phi \in C_0^\infty(\mathbb{R})$, L^2_ν is stable by $\phi(P)$ and*

$$\langle x\rangle^\nu \phi(P)\langle x\rangle^{-\nu} \quad \textit{is bounded on } L^2.$$

The proof of this proposition is similar to the one of Lemma 2.4.

Proof of (3.9). We want to show that (3.9) holds if z is close enough to 0 and $\widetilde{\phi}$ has a small enough support. We write

$$\begin{aligned} S_{\widetilde{\phi}}(\lambda + i\epsilon) &= V(P_0 - i\epsilon)^{-1}\widetilde{\phi}(P) + V\left((P_0 - \lambda - i\epsilon)^{-1} - (P_0 - i\epsilon)^{-1}\right)\widetilde{\phi}(P), \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

By (3.11) and (3.12),

$$||V(P_0 - i\epsilon)^{-1}||_{L^2 \to L^2_\nu} \le C, \qquad \epsilon \neq 0. \tag{3.13}$$

On the other hand, if we choose

$$\widetilde{\phi}(\cdot) = \varphi(\cdot/\eta),$$

Proposition (3.5) shows that

$$||\widetilde{\phi}(P)||_{L^2_\nu \to L^2} \to 0, \qquad \eta \to 0. \tag{3.14}$$

Hence (3.13) and (3.14) imply that, if η is small enough,

$$\big|\big|\mathrm{I}\big|\big|_{L^2_\nu \to L^2_\nu} \le 1/4, \qquad \epsilon \neq 0. \tag{3.15}$$

Once η is fixed, Proposition 3.6 shows that $\widetilde{\phi}(P)$ is bounded on L^2_ν and, using Proposition 3.2, we have

$$\big|\big|\mathrm{II}\big|\big|_{L^2_\nu \to L^2_\nu} \le 1/4, \tag{3.16}$$

for λ and ϵ small enough (which is sufficient for the result is trivial if ϵ is outside a neighborhood of 0). Then (3.15) and (3.16) give (3.9). □

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